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INVERSE SPECTRAL BOUNDARY PROBLEM STURM LIOUVILLE TYPE WITH CONSTANT DELAY AND NON-ZERO INITIAL FUNCTION

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Summary. This paper is dedicated to solving of the direct and inverse spectral problem for Sturm Liouville type of operator with constant delay from $\frac{\pi}{2}$ to π, non-zero initial function and Robin's boundary conditions. It has been proved that two series of eigenvalues unambiguously define the following parameters: delay, coefficients of delay within boundary conditions, the potential on the segment from the point of delay to the right-hand side of the distance and the product of the starting function and potential from the left end of the distance to the delay point.

1 INTRODUCTION

Spectral theory represents a part of mathematical analysis which studies the spectrum, i.e. series of eigenvalues and vectors associated with linear operators defined on infinite dimensional functional spaces. Spectral problems can be divided into direct and inverse problems. Direct problems imply the constructions of characteristic functions, decomposition of functions from the domain of the operator according to the eigenfunctions of the operator, studying the asymptotics of their zeros as well as the asymptotics of eigenvalues of the operator. Inverse problems imply the construction of linear operator based on some of its known spectral characteristics. Fort he inverse problem of classic operator Sturm Liouville type, first results are obtained in papers [1], [2].
A current problematics that has been developing since the 1990s, (see [3],[4]) is exactly the one dealing with inverse spectral problems of Sturm Liouville type with removed argument.

The great number of published papers and significant results speak in favour of the fact that this field of mathemathics has developed intensly. Here are some of the results related to various types of delay. (see [3-24])

This paper studies the direct and inverse spectral problem given with the following:

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$$y''(x) + q(x)y(x-\tau) = \lambda y(x), \quad \lambda = z^2, \ \ \tau \in \left[\frac{\pi}{2}, \pi\right) \qquad (1)$$
$$y(x-\tau) = \varphi(x-\tau), \ \ x \in [0,\tau], \quad \varphi(0) = 1 \qquad (2)$$
$$y'(0) - hy(0) = 0, \quad h \in R \qquad (3)$$
$$y'(\pi) + Hy(\pi) = 0, \quad H \in R \qquad (4)$$

In papers [5-19] instead of the condition (2) we use the condition

$$y(x-\tau) \equiv 0, \ \ x \in [0,\tau).$$

Let us have $q \in L_2[0,\pi]$, $\varphi \in L_2[-\tau,0]$.

The equation (1) with boundary condition (2) is equivalent to the integral equation

$$y(x,z) = \cos xz + \frac{h}{z}\sin xz + \frac{1}{z}\int_0^x q(t_1)\sin z(x-t_1)y(t_1-\tau,z)\,dt_1 \qquad (5)$$

Let us define the function

$$\tilde{q}(t_1) = \begin{cases} q(t_1)\varphi(t_1-\tau), & t_1 \in [0,\tau] \\ 0, & t_1 \in (\tau,\pi] \end{cases} \qquad (6)$$

The solution of the equation (5) at distance $(0,\tau]$ is given with

$$y(x,z) = \cos xz + \frac{h}{z}\sin xz + \frac{1}{z}\int_0^x \tilde{q}(t_1)\sin z(x-t_1)\,dt_1 \qquad (7)$$

Next, we use the following functions

$$a_{sc}(x,z) = \int_\tau^x q(t_1)\sin z(x-t_1)\cos z\,(t_1-\tau)dt_1$$
$$a_{s^2}(x,z) = \int_\tau^x q(t_1)\sin z(x-t_1)\sin z\,(t_1-\tau)dt_1$$
$$a_s^{(\tilde{1})}(x,z) = \int_0^\tau \tilde{q}(t_1)\sin z(x-t_1)\,dt_1$$
$$a_{s^2}^{(1,\tilde{1})}(x,z) = \int_\tau^x q(t_1)\int_0^{t_1-\tau}\tilde{q}(t_2)\sin z(x-t_1)\sin z(t_1-\tau-t_2)dt_2dt_1$$

At distance $(\tau,\pi]$ the solution of the equation (5) is given with

$$y(x,z) = \cos xz + \frac{h}{z}\sin xz + \frac{1}{z}a_s^{(\tilde{1})}(x,z) + \frac{1}{z}a_{sc}(x,z) + $$
$$+\frac{h}{z^2}a_{s^2}(x,z) + \frac{1}{z^2}a_{s^2}^{(1,\tilde{1})}(x,z) \qquad (8)$$

2 THE CONSTRUCTION OF CHARCTERISTIC FUNCTIONS OF D^2 OPERATOR

The boundary problem given in (1-4) will be denoted shorter as $D^2y = \lambda y$. Let us construct a characteristic function of the operator D^2.

If we vary the condition (4) with H to H_j, $j = 1,2$ we obtain two characteristic equations $F_j(z) = 0$.

Let us also introduce the following functions

$$a_{c^2}(x,z) = \int_\tau^x q(t_1)\cos z(x-t_1)\cos z\,(t_1-\tau)dt_1$$

$$a_{cs}(x,z) = \int_{\tau}^{x} q(t_1)\cos z(x-t_1)\sin z\,(t_1-\tau)dt_1$$
$$a_c^{(\tilde{1})}(x,z) = \int_0^{\tau} \tilde{q}(t_1)\cos z(x-t_1)dt_1$$
$$a_{cs}^{(1,\tilde{1})}(x,z) = \int_{\tau}^{x} q(t_1)\cos z(x-t_1)\int_0^{t_1-\tau}\tilde{q}(t_2)\sin z(t_1-\tau-t_2)dt_2dt_1$$

From (8) we obtain the following

$$\frac{d}{dx}y(x,z) = -z\sin xz + h\cos xz + a_c^{(\tilde{1})}(x,z) + a_{c^2}(x,z) + \frac{h}{z}a_{cs}(x,z) + \frac{1}{z}a_{cs}^{(1,\tilde{1})}(x,z) \quad (9)$$

Putting $x=\pi$ in (8) and (9), and then omit π in the labels, for example $a_{c^2}(\pi,x) = a_{c^2}(x)$, based on the condition (4) the following is obtained

$$F_j(z) = (-z+\frac{hH_j}{z})\sin\pi z + \left(h+H_j\right)\cos\pi z + a_c^{(\tilde{1})}(z) + a_{c^2}(z) + \frac{hH_j}{z^2} + \\ + a_{s^2}(z) + \frac{H_j}{z}a_s^{(\tilde{1})}(z) + \frac{h}{z}a_{cs}(z) + \frac{H_j}{z}a_{sc}(z) + \frac{1}{z}a_{cs}^{(1,\tilde{1})}(z) + \frac{H_j}{z^2}a_{s^2}^{(1,\tilde{1})}(z) \quad (10)$$

Let us transform the functions F_j given in (10).

If we have

$$\check{q}(\theta) = \tilde{q}(2\theta),\ \ \hat{q}(\theta) = q\left(\theta+\frac{\tau}{2}\right),\ \ I_1 = \int_{\tau}^{\pi} q(t_1)dt_1 = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\hat{q}\,(\theta)d\theta$$
$$\hat{a}(z) = \int_{\frac{\tau}{2}}^{\pi}\hat{q}\,(\theta)\cos(\pi-2\theta)zd\theta,\ \ \hat{b}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\hat{q}\,(\theta)\sin(\pi-2\theta)zd\theta$$

Then the following relations are valid

$$a_{c^2}(z) = \frac{1}{2}\hat{a}(z) + \frac{I_1}{2}\cos(\pi-\tau)z,\ \ a_{s^2}(z) = \frac{1}{2}\hat{a}(z) - \frac{I_1}{2}\cos(\pi-\tau)z$$
$$a_{cs}(z) = -\frac{1}{2}\hat{b}(z) + \frac{I_1}{2}\sin(\pi-\tau)z,\ \ a_{sc}(z) = \frac{1}{2}\hat{b}(z) + \frac{I_1}{2}\sin(\pi-\tau)\,z \quad (11_1)$$

Next, let us have

$$\check{a}(z) = \int_0^{\frac{\tau}{2}}\check{q}\,(\theta)\cos(\pi-2\theta)zd\theta,\ \ \check{b}(z) = \int_0^{\frac{\tau}{2}}\check{q}\,(\theta)\sin(\pi-2\theta)zd\theta \quad (11_2)$$

Then we get

$$a_c^{(\tilde{1})}(z) = 2\check{a}(z),\ \ a_s^{(\tilde{1})}(z) = 2\check{b}(z) \quad (11_3)$$

Then we transform the functions $a_{s^2}^{(1,\tilde{1})}(z)$ and $a_{cs}^{(1,\tilde{1})}(z)$.
By translating the product of trigonometric functions into sums, and then changing the order of integration, we obtain the following

$$a_{s^2}^{(1,\tilde{1})}(z) = \\ = \int_{\frac{\tau}{2}}^{\frac{\pi}{2}}\left[\int_{\theta+\frac{\tau}{2}}^{2\theta} q(t_1)\tilde{q}(2t_1-2\theta-\tau)dt_1 - \tilde{q}(2\theta-\tau)\int_{2\theta}^{\pi} q(t_1)\,dt_1\right]\cos(\pi-2\theta)zd\theta$$

$$+\int_{\frac{\pi}{2}}^{\pi-\frac{\tau}{2}}\int_{\theta+\frac{\tau}{2}}^{\pi} q(t_1)\tilde{q}(2t_1-2\theta-\tau)dt_1\cos(\pi-2\theta)zd\theta$$

Let us define the function $Q^{(1,\tilde{1})}(\theta)$ with

$$Q^{(1,\tilde{1})}(\theta)=\begin{cases}\int_{\theta+\frac{\tau}{2}}^{2\theta} q(t_1)\tilde{q}(2t_1-2\theta-\tau)dt_1-\tilde{q}(2\theta-\tau)\int_{2\theta}^{\pi} q(t_1)dt_1, & \theta\in\left[\frac{\tau}{2},\frac{\pi}{2}\right) \\ \int_{\theta+\frac{\tau}{2}}^{\pi} q(t_1)\tilde{q}(2t_1-2\theta-\tau)dt_1, & \theta\in\left[\frac{\pi}{2},\pi-\frac{\tau}{2}\right]\end{cases}$$

and put

$$a^{(1,\tilde{1})}(z)=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} Q^{(1,\tilde{1})}(\theta)\cos(\pi-2\theta)zd\theta,$$

$$b^{(1,\tilde{1})}(z)=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} Q^{(1,\tilde{1})}(\theta)\sin(\pi-2\theta)zd\theta$$

Then the following relation is valid

$$a_{s^2}^{(1,\tilde{1})}(z)=a^{(1,\tilde{1})}(z) \tag{11_4}$$

Quite analogously, we obtain the following equation

$$a_{cs}^{(1,\tilde{1})}(z)=-b^{(1,\tilde{1})}(z) \tag{11_5}$$

Using the relations (11_l), $l=\overline{1,5}$, the functions in (10) obtain the following form

$$\begin{aligned}F_j(z)=(-z+\tfrac{hH_j}{z})\sin\pi z+\left(h+H_j\right)\cos\pi z+2\check{a}(z)+\tfrac{2H_j}{z}\check{b}(z)+\\+\tfrac{I_1}{2}\left(1-\tfrac{hH_j}{z^2}\right)\cos(\pi-\tau)z+\tfrac{1}{2}\left(1+\tfrac{hH_j}{z^2}\right)\hat{a}(z)+\tfrac{I_1}{2z}(h+H_j)\sin(\pi-\tau)z+\\+\tfrac{1}{2z}\left(H_j-h\right)\hat{b}(z)-\tfrac{b^{(1,\tilde{1})}(z)}{z}+\tfrac{H_j}{z^2}a^{(1,\tilde{1})}(z)\end{aligned} \tag{12}$$

Functions F_j are entire functions of exponential type and unit increase rate at variable z.

3 ASYMPTOTICS OF ZERO FUNCTION *F*

Let us find the asymptotic of zero functions F_j.
From [3], the following is known to be valid

$$z_{nj}=n+\frac{C_{1j}(n)}{n}+\frac{C_{2j}(n)}{n^2}+o\left(\frac{C_{2j}(n)}{n^2}\right),\ n\to\infty,\ j=1,2 \tag{13_1}$$

Therefore, we have

$$\sin\pi z_{nj}=(-1)^n\left[\frac{\pi C_{1j}(n)}{n}+\frac{\pi C_{2j}(n)}{n^2}+o(\frac{C_{2j}(n)}{n^2})\right] \tag{13_2}$$

$$-z_{nj}\sin\pi z_{nj}=(-1)^{n+1}\left[\pi C_{1j}(n)+\frac{\pi C_{2j}(n)}{n}+o(\frac{C_{2j}(n)}{n})\right] \tag{13_3}$$

$$\cos \pi z_{nj} = (-1)^n \left[1 + O(\frac{(C_{1j}(n))^2}{n^2})\right] \quad (13_4)$$

We extend the function $\check{q}$ from the segment $\left[0, \frac{\tau}{2}\right]$ to zero, by the space $[0, \pi]$, and if we have

$$\check{a}_{2n} = \frac{2}{\pi} \int_0^{\frac{\tau}{2}} \check{q}\,(\theta) \cos 2n\theta d\theta$$

then the following is valid

$$\check{a}(z_{nj}) = (-1)^n \frac{\pi}{2} \check{a}_{2n} + o(\frac{C_{1j}(n)}{n}) \quad (13_5)$$

Similarly, if we have

$$\check{b}_{2n} = \frac{2}{\pi} \int_0^{\frac{\tau}{2}} \check{q}\,(\theta) \sin 2n\theta d\theta$$

then we obtain

$$\check{b}(z_{nj}) = (-1)^{n+1} \frac{\pi}{2} \check{b}_{2n} + O(\frac{C_{1j}(n)}{n}) \quad (13_6)$$

Next, let us have

$$\hat{a}_{2n} = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi - \frac{\tau}{2}} \hat{q}\,(\theta) \cos 2n\theta d\theta,$$

$$\hat{b}_{2n} = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi - \frac{\tau}{2}} \hat{q}\,(\theta) \sin 2n\theta d\theta$$

then we obtain

$$\hat{a}(z_{nj}) = (-1)^n \frac{\pi}{2} \hat{a}_{2n} + o\left(\frac{1}{n}\right), \quad \hat{b}(z_{nj}) = (-1)^{n+1} \frac{\pi}{2} \hat{b}_{2n} + o\left(\frac{C_{1j}(n)}{n}\right) \quad (13_7)$$

Next, the following is valid

$$\cos(\pi - \tau) z_{nj} = (-1)^n \cos n\tau + (-1)^{n+1} (\pi - \tau) C_{1j}(n) \frac{\sin n\tau}{n} + O(\frac{1}{n^2}) \quad (13_8)$$

$$\sin(\pi - \tau) z_{nj} = (-1)^{n+1} \sin n\tau + O(\frac{1}{n}) \quad (13_9)$$

Also, if we have

$$b_{2n}^{(1,\tilde{1})} = \frac{2}{\pi} \int_{\frac{\tau}{2}}^{\pi - \frac{\tau}{2}} a^{(1,\tilde{1})}(\theta) \sin 2n\theta d\theta$$

then we obtain

$$b^{(1,\tilde{1})}(z_{nj}) = (-1)^{n+1} \frac{\pi}{2} b_{2n}^{(1,\tilde{1})} + o(\frac{1}{n}) \quad (13_{10})$$

By using (13_l), $l = \overline{1,10}$ we come to the following estimates

$$F_j(z_{nj}) = (-1)^{n+1} \pi C_{1j}(n) + (-1)^n (h + H_j) + (-1)^n \frac{I_1}{2} \cos n\tau + (-1)^n \pi \check{a}_{2n} +$$

$$+(-1)^n \frac{\pi}{4} \hat{a}_{2n} + \frac{1}{n}\Big[(-1)^{n+1} \pi C_{2j}(n) + (-1)^{n+1} \pi \check{b}_{2n} + (-1)^{n+1} \frac{I_1(\pi - \tau)}{2} \cdot \frac{\sin n\tau}{n} C_{1j}(n)$$

$$+(-1)^{n+1}\frac{I_1(h+H_j)}{2}\sin n\tau+(-1)^{n+1}\pi\frac{H_j-h}{2}\hat{b}_{2n}+(-1)^n\frac{\pi}{2}b_{2n}^{(1,\tilde{1})}\Bigg]+o\left(\frac{1}{n^2}\right)=0$$

Next,

$$C_{1j}(n)=\frac{h+H_j}{\pi}+\frac{I_1}{2\pi}\cos n\tau+\check{a}_{2n}+\frac{1}{4}\hat{a}_{2n};\quad \check{a}_{2n}+\frac{1}{4}\hat{a}_{2n}\in l_2 \qquad (14_1)$$

$$C_{2j}(n)=\frac{I_1(h+H_j)}{2\pi^2}(\tau-2\pi)\sin n\tau-\frac{I_1^2(\pi-\tau)}{8\pi^2}\sin 2n\tau+\frac{1}{2\pi}b_{2n}^{(1,\tilde{1})}-$$
$$-\frac{H_j-h}{2}\hat{b}_{2n}-\check{b}_{2n}-\frac{I_1(\pi-\tau)}{2\pi}\sin n\tau(\check{a}_{2n}+\frac{1}{4}\hat{a}_{2n}) \qquad (14_2)$$

If we put

$$\zeta_{0j}=\frac{h+H_j}{\pi},\quad \zeta_1=\frac{I_1}{2\pi},\ \ \zeta_{1n}=\check{a}_{2n}+\frac{1}{4}\hat{a}_{2n},\ \ \eta_{1j}=\frac{I_1(h+H_j)}{2\pi^2}(\pi-2\tau),$$
$$\eta_2=\frac{I_1^2(\pi-\tau)}{8\pi^2},\ \ \eta_{2n}=\frac{1}{2\pi}b_{2n}^{(1,\tilde{1})}-\frac{H_j-h}{2}\hat{b}_{2n}-\check{b}_{2n}-\frac{I_1(\pi-\tau)}{2\pi}\zeta_{1n}$$

then we get

$$z_{nj}=n+\frac{1}{n}\left(\zeta_{0j}+\zeta_1\cos n\tau+\zeta_{1n}\right)+\frac{1}{n^2}\left(\eta_{1j}\sin n\tau+\eta_2\sin 2n\tau+\eta_{2n}\right)+o\left(\frac{\eta_{2n}}{n^2}\right) \qquad (14_3)$$

In this way we have proved the result:

Theorem 3.1. *Eigenvalues λ_{nj} of boundary problems $(1,2,3,4_j)$, $j=1,2$ have the asymptotic given with*

$$\lambda_{nj}=n^2+2\zeta_{0j}+2\zeta_1\cos n\tau+2\zeta_{1n}+\frac{1}{n}\left(2\eta_{1j}\sin n\tau+2\eta_2\sin 2n\tau+2\eta_{2n}\right)+o\left(\frac{\eta_{2n}}{n}\right) \quad (15)$$

where we have $\eta_{2n}\in l_2$, $\zeta_{1n}\in l_2$.

4 SETTING THE INVERSE TASK

Let us have two series λ_{nj}, $n\in N_0$, $j=1,2$ of eigenvalues with asymptotics (15). If the series $\lambda_{nj}-n^2$ oscillate, that is if $I_1\neq 0$, it is well known that the values τ, I_1, h, H_j, $j=1,2$ are uniquely determined.

Next, we study the possibility of determining functions $\check{q}$ and $\hat{q}$.

Characteristic functions are constructed through Hadamard's products, so the following applies:

$$F_j(z)=\pi\lambda_{0j}\prod_{n=1}^{\infty}\frac{\lambda_{nj}}{n^2}(1-\frac{z^2}{\lambda_{0j}})\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_{nj}}\right),\ \ j=1,2,\ \ z\in C \qquad (16)$$

The process of constructing integral equations by functions $\check{q}$ i $\hat{q}$ is based on the following identities

$$(-z+\frac{hH_j}{z})\sin\pi z+\left(h+H_j\right)\cos\pi z+2\check{a}(z)+\frac{hH_j}{z}\check{b}(z)+$$
$$+\frac{I_1}{2}(1-\frac{hH_j}{z^2})\cos(\pi-\tau)z+\frac{1}{2}\left(1+\frac{hH_j}{z^2}\right)\hat{a}(z)+\frac{I_1}{2z}\left(h+H_j\right)\sin(\pi-\tau)z+ \quad (17)$$

$$+\frac{1}{2z}\left(H_j-h\right)\hat{b}(z)-\frac{b^{(1,\tilde{1})}(z)}{z}+\frac{H_j}{z^2}a^{(1,\tilde{1})}(z)=F_j(z),\ \ j=1,2$$

Let us have

$$A(z)=2\frac{H_2F_1(z)-H_1F_2(z)}{H_2-H_1}+2z\sin\pi z-2h\cos\pi z$$
$$B(z)=2z\frac{F_2(z)-F_1(z)}{H_2-H_1}-2h\sin\pi z-2z\cos\pi z$$

Identities (17) are equivalent with

$$4\breve{a}(z)+\hat{a}(z)-h\frac{\hat{b}(z)}{z}-2\frac{b^{(1,\tilde{1})}(z)}{z}+I_1\cos(\pi-\tau)z+\frac{I_1h}{z}\sin(\pi-\tau)z=A(z) \qquad (18_1)$$
$$4\breve{b}(z)+\hat{b}(z)-h\frac{\hat{a}(z)}{z}+2\frac{a^{(1,\tilde{1})}(z)}{z}+I_1\sin(\pi-\tau)z-\frac{I_1h}{z}\cos(\pi-\tau)z=B(z) \qquad (18_2)$$

Then we carry out partial integration over quotients $\frac{\hat{b}(z)}{z}$, $\frac{b^{(1,\tilde{1})}(z)}{z}$, $\frac{\hat{a}(z)}{z}$ and $\frac{a^{(1,\tilde{1})}(z)}{z}$.
First, we put

$$a^{I^1\hat{q}}(z)=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int_{\frac{\tau}{2}}^{\theta}\hat{q}\,(\theta_1)d\theta_1\right)\cos(\pi-2\theta)zd\theta$$
$$b^{I^1\hat{q}}(z)=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int_{\frac{\tau}{2}}^{\theta}\hat{q}\,(\theta_1)d\theta_1\right)\sin(\pi-2\theta)zd\theta$$
$$a^{I^1Q^{(1,\tilde{1})}}(z)=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int_{\frac{\tau}{2}}^{\theta}Q^{(1,\tilde{1})}\,(\theta_1)d\theta_1\right)\cos(\pi-2\theta)zd\theta$$
$$b^{I^1Q^{(1,\tilde{1})}}(z)=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int_{\frac{\tau}{2}}^{\theta}Q^{(1,\tilde{1})}\,(\theta_1)d\theta_1\right)\sin(\pi-2\theta)zd\theta$$

The following relations are valid

$$\frac{\hat{b}(z)}{z}=-I_1\frac{\sin(\pi-\tau)z}{z}+2a^{I^1\hat{q}}(z)$$
$$\frac{\hat{a}(z)}{z}=I_1\frac{\cos(\pi-\tau)z}{z}-2b^{I^1\hat{q}}(z)$$
$$\frac{b^{(1,\tilde{1})}(z)}{z}=-\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}Q^{(1,\tilde{1})}(\theta)\,d\theta\,\frac{\sin(\pi-\tau)z}{z}+2a^{I^1Q^{(1,\tilde{1})}}(z)$$
$$\frac{a^{(1,\tilde{1})}(z)}{z}=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}Q^{(1,\tilde{1})}(\theta)\,d\theta\,\frac{\cos(\pi-\tau)z}{z}+2b^{I^1Q^{(1,\tilde{1})}}(z)$$

It is clearly seen that the following is valid $\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}Q^{(1,\tilde{1})}(\theta)d\theta=0$.
Thus, the identities $(18_l), l=1,2$ obtain the following form:

$$4\breve{a}(z)+\hat{a}(z)-2ha^{I^1\hat{q}}(z)-2a^{I^1Q^{(1,\tilde{1})}}(z)=C(z) \qquad (19_1)$$
$$4\breve{b}(z)+\hat{b}(z)-2hb^{I^1\hat{q}}(z)-2b^{I^1Q^{(1,\tilde{1})}}(z)=S(z) \qquad (19_2)$$

Here we have $C(z)=A(z)-2hI_1\frac{\sin(\pi-\tau)z}{z}$, $S(z)=B(z)$

Let us have the set $E = \{m + iy,\ m \in N_0\ ,\ y \in R\ \} \subset C.$
Since the set E has the finite accumulation points, according to Vitali's theorem the system of identity $(19_l), l = 1,2$ at C is equivalent to the system of identity at E.

Therefore,

$$\breve{a}(m+iy) = \int_0^{\frac{\tau}{2}} \breve{q}(\theta)\cos(m+iy)(\pi-2\theta)d\theta =$$

$$= (-1)^m\left[\int_0^{\frac{\tau}{2}}(\breve{q}(\theta)\text{ch}(\pi-2\theta)y)\cos 2m\theta d\theta + i\int_0^{\frac{\tau}{2}}(\breve{q}(\theta)\text{sh}(\pi-2\theta)y)\sin 2m\theta d\theta\right]$$

$$\hat{a}(m+iy) = (-1)^m\left[\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}(\hat{q}(\theta)\text{ch}(\pi-2\theta)y)\cos 2m\theta d\theta + \right.$$

$$\left. + i\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}(\hat{q}(\theta)\text{sh}(\pi-2\theta)y)\sin 2m\theta d\theta\right]$$

$$\breve{b}(m+iy) = (-1)^m\left[-\int_0^{\frac{\tau}{2}}(\breve{q}(\theta)\text{ch}(\pi-2\theta)y)\sin 2m\theta d\theta + \right.$$

$$\left. + i\int_0^{\frac{\tau}{2}}(\breve{q}(\theta)\text{sh}(\pi-2\theta)y)\cos 2m\theta d\theta\right]$$

$$\hat{b}(m+iy) = (-1)^m\left[-\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}(\hat{q}(\theta)\text{ch}(\pi-2\theta)y)\sin 2m\theta d\theta + \right.$$

$$\left. + i\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}(\hat{q}(\theta)\text{sh}(\pi-2\theta)y)\cos 2m\theta d\theta\right]$$

Besides,

$$C(m+iy) = \alpha(m,y) + i\beta(m,y),\ \ S(m+iy) = \gamma(m,y) + i\delta(m,y)$$

Next, the following tags are used

$$\breve{a}(m+iy) = (-1)^m\left[\breve{a}_{2m}^{ch}(y) + i\breve{b}_{2m}^{sh}(y)\right], \qquad \breve{b}(m+iy) = (-1)^m\left[-\breve{b}_{2m}^{ch}(y) + i\breve{a}_{2m}^{sh}(y)\right]$$
$$\hat{a}(m+iy) = (-1)^m\left[\hat{a}_{2m}^{ch}(y) + i\hat{b}_{2m}^{sh}(y)\right], \qquad \hat{b}(m+iy) = (-1)^m\left[-\hat{b}_{2m}^{ch}(y) + i\hat{a}_{2m}^{sh}(y)\right]$$

The identities $(19_l), l = 1,2$ are equivalent to the following system of identities

$$4\breve{a}_{2m}^{ch}(y) + \hat{a}_{2m}^{ch}(y) - 2ha_{2m}^{I^1\hat{q}ch}(y) - 2a_{2m}^{I^1Q^{(1,\bar{1})}ch}(y) = (-1)^m\alpha(m,y) \qquad (20_1)$$
$$4\breve{b}_{2m}^{sh}(y) + \hat{b}_{2m}^{sh}(y) - 2hb_{2m}^{I^1\hat{q}sh}(y) - 2b_{2m}^{I^1Q^{(1,\bar{1})}sh}(y) = (-1)^m\beta(m,y) \qquad (20_2)$$
$$4\breve{b}_{2m}^{ch}(y) + \hat{b}_{2m}^{ch}(y) - 2hb_{2m}^{I^1\hat{q}ch}(y) - 2hb_{2m}^{I^1Q^{(1,\bar{1})}ch}(y) = (-1)^{m+1}\gamma(m,y) \qquad (20_3)$$
$$4\breve{a}_{2m}^{sh}(y) + \hat{a}_{2m}^{sh}(y) - 2ha_{2m}^{I^1\hat{q}sh}(y) - 2ha_{2m}^{I^1Q^{(1,\bar{1})}sh}(y) = (-1)^m\delta(m,y) \qquad (20_4)$$

According to the asymptotics (15) it is easy to conclude that the right-hand sides of $(20_l),\ \ l = \overline{1,4}$ are from the space l_2, in fact that Fourier's coefficients of some functions from $L_2[0,\tau]$ are known. From (20_1) and (20_3) we come to the following equation

$$4\breve{q}(\theta) + \hat{q}(\theta) - 2h\int_{\frac{\tau}{2}}^{\theta}\hat{q}(\theta_1)d\theta_1 - 2\int_{\frac{\tau}{2}}^{\theta}Q^{(1,\bar{1})}(\theta_1)d\theta_1 = \frac{f_1(\theta,y)}{ch(\pi-2\theta)y} \qquad (21_1)$$

Accordingly, from (20_4) and (20_2) at $y \neq 0$ we also have

$$4\check{q}(\theta)+\hat{q}(\theta)-2h\int_{\frac{\tau}{2}}^{\theta}\hat{q}(\theta_1)d\theta_1-2\int_{\frac{\tau}{2}}^{\theta}Q^{(1,\tilde{1})}(\theta_1)d\theta_1=\frac{f_2(\theta,y)}{sh(\pi-2\theta)y} \qquad (21_2)$$

From (21_1) and (21_2) we obtain the following

$$\frac{f_1(\theta,y)}{ch(\pi-2\theta)y}=\frac{f_2(\theta,y)}{sh(\pi-2\theta)y}=f(\theta),\ \ \theta\in[0,\pi],$$

where equality is implied in terms of L_2.

In this way we have proved and important result:

Theorem 4.1. *In order for functions* $\check{q}$, $\hat{q}$ *to be parameters of operators* D_j^2 $(j=1,2)$ *whose eigenvalues are given, it is necessary and sufficient for them to be the solutions of the equation*

$$4\check{q}(\theta)+\hat{q}(\theta)-2h\int_{\frac{\tau}{2}}^{\theta}\hat{q}(\theta_1)d\theta_1-2\int_{\frac{\tau}{2}}^{\theta}Q^{(1,\tilde{1})}(\theta_1)d\theta_1=f(\theta) \qquad (22)$$

5 SOLVING THE EQUATION (22)

For $\theta\in\left[0,\frac{\tau}{2}\right]$ the equation (22) becomes the identity

$$4\check{q}(\theta)=f(\theta),$$

that is

$$q(x)\varphi(x-\tau)=\frac{1}{4}f\left(\frac{x}{2}\right),\ \ x\in[0,\tau] \qquad (23)$$

For $\theta\in\left(\frac{\tau}{2},\pi-\frac{\tau}{2}\right]$ the equation (22) reduces to the following equation

$$\hat{q}(\theta)=f(\theta)+\int_{\frac{\tau}{2}}^{\theta}\left[2h\hat{q}(\theta_1)+2Q^{(1,\tilde{1})}(\theta_1)\right]d\theta_1$$

or

$$q\left(\theta+\frac{\tau}{2}\right)=f(\theta)+\int_{\frac{\tau}{2}}^{\theta}\left[2hq\left(\theta_1+\frac{\tau}{2}\right)+2Q^{(1,\tilde{1})}(\theta_1)\right]d\theta_1$$

i.e.

$$q(x)=f\left(x-\frac{\tau}{2}\right)+\int_{\tau}^{x}\left[2hq(x_1)+2Q^{(1,\tilde{1})}\left(x_1-\frac{\tau}{2}\right)\right]dx_1 \qquad (24)$$

Since we have

$$Q^{(1,\tilde{1})}\left(x_1-\frac{\tau}{2}\right)=\begin{cases}\displaystyle\int_{x_1}^{2x_1-\tau}q(t_1)\tilde{q}(2t_1-2x_1)dt_1-\tilde{q}(2x_1-2\tau)\int_{2x_1-\tau}^{\pi}q(t_1)dt_1, & x_1\in\left[\tau,\frac{\pi+\tau}{2}\right]\\ \displaystyle\int_{x_1}^{\pi}q(t_1)\tilde{q}(2t_1-2x_1)dt_1, & x_1\in\left(\frac{\pi+\tau}{2},\pi\right]\end{cases}$$

the equation (24) is divided into two equations

$$q(x) = f\left(x - \frac{\tau}{2}\right) + \int_{\tau}^{x} [2hq(x_1) +$$

$$+ \int_{x_1}^{2x_1-\tau} q(t_1)\tilde{q}(2t_1 - 2x_1) - \tilde{q}(2x_1 - 2\tau) \int_{2x_1-\tau}^{\pi} q(t_1)\, dt_1 \Big] dx_1, \ \ x \in \left(\tau, \frac{\pi+\tau}{2}\right] \tag{25}$$

$$q(x) = f\left(x - \frac{\tau}{2}\right) + \int_{\tau}^{x} \left[2hq(x_1) + Q^{(1,\tilde{1})}\left(x_1 - \frac{\tau}{2}\right)\right] dx_1\,, \ \ x \in \left(\frac{\pi+\tau}{2}, \pi\right] \tag{26}$$

or

$$q(x) = f\left(x - \frac{\tau}{2}\right) + \int_{\tau}^{\pi} 2hq(x_1)dx_1 +$$

$$+ \int_{\tau}^{\pi} Q^{(1,\tilde{1})}\left(x_1 - \frac{\tau}{2}\right) dx_1 - 2h \int_{x}^{\pi} q(x_1)dx_1 - \int_{x}^{\pi} Q^{(1,\tilde{1})}\left(x_1 - \frac{\tau}{2}\right) dx_1$$

Since we have $\int_{\tau}^{\pi} q(x_1)dx_1 = I_1$ and $\int_{\tau}^{\pi} Q^{(1,\tilde{1})}\left(x_1 - \frac{\tau}{2}\right) dx_1 = 0$ if we put $f_1(x) = f\left(x - \frac{\tau}{2}\right) + 2hI_1$ the equation (26) is equivalent to the equation

$$q(x) = f_1(x) + \int_{x}^{\pi} \left[2hq(x_1) + \int_{x_1}^{\pi} q(t_1)\tilde{q}(2t_1 - 2x_1)dt_1\right] dx_1 \tag{27}$$

The equation (27) is Volterra's non homogenous linear integral equation of q function, since $\tilde{q}$ is a known function.
Let $q_1(x)$ be the unique solution of the equation (27).
Let us now return to the equation (25).
We can write the following

$$q(x) = f\left(x - \frac{\tau}{2}\right) +$$

$$\int_{\tau}^{\frac{\pi+\tau}{2}} \left[2hq(x_1) + \int_{x_1}^{2x_1-\tau} q(t_1)\tilde{q}(2t_1 - 2x_1)dt_1 - \tilde{q}(2x_1 - 2\tau) \int_{2x_1-\tau}^{\pi} q(t_1)dt_1\right] dx_1 -$$

$$- \int_{x}^{\frac{\pi+\tau}{2}} \left[2hq(x_1) + \int_{x_1}^{2x_1-\tau} q(t_1)\tilde{q}(2t_1 - 2x_1)dt_1 - \tilde{q}(2x_1 - 2\tau) \int_{2x_1-\tau}^{\pi} q(t_1)dt_1\right] dx_1$$

Since we have $\int_{\tau}^{\frac{\pi+\tau}{2}} 2hq(x_1)dx_1 = 2h\left(I_1 - \int_{\frac{\pi+\tau}{2}}^{\pi} q_1(x_1)dx_1\right)$
and

$$\int_{\tau}^{\frac{\pi+\tau}{2}} \left[\int_{x_1}^{2x_1-\tau} q(t_1)\tilde{q}(2t_1 - 2x_1)dt_1 - \tilde{q}(2x_1 - 2\tau) \int_{2x_1-\tau}^{\pi} q(t_1)dt_1\right] dx_1 =$$

$$= - \int_{\frac{\pi+\tau}{2}}^{\pi} \int_{x_1}^{\pi} q_1(t_1)\tilde{q}(2t_1 - 2x_1)dt_1$$

therefore, we get

$$q(x) = f\left(x - \frac{\tau}{2}\right) +$$

$$+2h\left(I_1-\int_{\frac{\pi+\tau}{2}}^{\pi}q_1(x_1)dx_1\right)-\int_{\frac{\pi+\tau}{2}}^{\pi}\int_{x_1}^{\pi}q_1(t_1)\tilde{q}(2t_1-2x_1)dt_1dx_1-$$

$$-\int_{x}^{\frac{\pi+\tau}{2}}\left[2hq(x_1)+\int_{x_1}^{2x_1-\tau}q(t_1)\tilde{q}(2t_1-2x_1)dt_1-\tilde{q}(2x_1-2\tau)\int_{2x_1-\tau}^{\pi}q(t_1)dt_1\right]dx_1$$

From $2x_1-\tau\le\frac{\pi+\tau}{2}\Leftrightarrow x_1\le\frac{\pi+3\tau}{4}$ we can write

$$\int_{x_1}^{2x_1-\tau}q(t_1)\tilde{q}(2t_1-2x_1)dt_1=$$

$$=\int_{x_1}^{\frac{\pi+3\tau}{4}}q(t_1)\tilde{q}(2t_1-2x_1)dt_1+\int_{\frac{\pi+3\tau}{4}}^{2x_1-\tau}q_1(t_1)\tilde{q}(2t_1-2x_1)dt_1$$

and

$$\tilde{q}(2x_1-2\tau)\int_{2x_1-\tau}^{\pi}q(t_1)dt_1=\tilde{q}(2x_1-2\tau)\int_{2x_1-\tau}^{\frac{\pi+\tau}{2}}q(t_1)\,dt_1+\tilde{q}(2x_1-2\tau)\int_{\frac{\pi+\tau}{2}}^{\pi}q_1(t_1)dt_1$$

Let us put

$$f_2(x)=f\left(x-\frac{\tau}{2}\right)+$$

$$+2h\left(I_1-\int_{\frac{\pi+\tau}{2}}^{\pi}q_1(x_1)dx_1\right)-\int_{\frac{\pi+\tau}{2}}^{\pi}\int_{x_1}^{\pi}q_1(t_1)\tilde{q}(2t_1-2x_1)dt_1\,dx_1-$$

$$-\int_{x}^{\frac{\pi+\tau}{2}}\left[\int_{\frac{\pi+\tau}{2}}^{2x_1-\tau}q_1(t_1)\tilde{q}(2t_1-2x_1)dt_1+\tilde{q}(2x_1-2\tau)\int_{\frac{\pi+\tau}{2}}^{\pi}q_1(t_1)dt_1\right]dx_1$$

The equation (25) is equivalent to the following equation

$$q(x)=f_2(x)-$$

$$-\int_{x}^{\frac{\pi+\tau}{2}}\left[2hq(x_1)+\int_{x_1}^{\frac{\pi+\tau}{2}}q(t_1)\tilde{q}(2t_1-2x_1)dt_1-\tilde{q}(2x_1-2\tau)\int_{2x_1-\tau}^{\frac{\pi+\tau}{2}}q(t_1)dt_1\right]dx_1 \quad (28)$$

Thus, we obtain a linear integral equation of Volterra's type.
If we write (28) in the operator form

$$q=Aq,\ q\in L_2\left[\tau,\frac{\pi+\tau}{2}\right]$$

then for

$$q^{(1)},q^{(2)}\in L_2\left[\tau,\frac{\pi+\tau}{2}\right]$$

we get

$$\left|\left(Aq^{(2)}-Aq^{(1)}\right)(x)\right|\le 3M(\gamma-x)^{\frac{1}{2}}\left\|q^{(2)}-q^{(1)}\right\|,$$
$$M=max\{2|h|,\|\tilde{q}\|\},\quad \gamma=\frac{\pi+\tau}{2},\ \gamma-x<1$$

$$\left|\left(A^2q^{(2)} - A^2q^{(1)}\right)(x)\right| \le (3M)^2 \frac{(\gamma - x)^{\frac{3}{2}}}{\frac{3}{2}} \left\|q^{(2)} - q^{(1)}\right\|$$

$$\left|\left(A^3q^{(2)} - A^3q^{(1)}\right)(x)\right| \le (3M)^3 \frac{(\gamma - x)^{\frac{5}{2}}}{\frac{3}{2}\cdot\frac{5}{2}} \left\|q^{(2)} - q^{(1)}\right\|$$

Induction easily proves that the following is valid

$$\left|\left(A^mq^{(2)} - A^mq^{(1)}\right)(x)\right| \le (6M)^m \frac{1}{(2m+1)!!} (\gamma - x)^{\frac{2m+1}{2}} \left\|q^{(2)} - q^{(1)}\right\|$$

Therefore,

$$\left\|A^mq^{(2)} - A^mq^{(1)}\right\| \le \frac{(6M)^m}{(2m+1)!!} \left(\frac{\pi - \tau}{2}\right)^{m+2} \left\|q^{(2)} - q^{(1)}\right\|,$$

which means that A^m is the clamp operator , wherever m is large enough. This means that there is a unique solution for $q_2(x)$ of the equation (28) at space $\left[\tau, \frac{\pi+\tau}{2}\right]$.

This is how we have proved the basic result:

Theorem 4.1. *Using two sets of eigenvalues λ_{nj} obtained by varying boundary conditions at the righthand end of the space $[0, \pi]$ under the condition $I_1 \neq 0$, the parameters $\tau, h, H_1, H_2, q(x)\varphi(x - \tau),\ x \in [0, \tau]$ and $q(x),\ x \in [\tau, \pi]$ are uniquely determined.*

6 CONCLUSION

The aim of the research but also the motivation of this paper is to contribute to the development of inverse spectral theory for operators with deviation. Some results concerning the inverse spectral problems for the classical operator are given in the literature, as well as the results related to different types of delay.

Here, the direct and inverse boundary value problem of the Sturm-Liouville type with constant delay and non - zero initial function is observed and studied. Using the condition, we solved the inverse problem and took one step in the development of this theory.

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