TOWARD THE CONSTRUCTION OF RELATIVISTIC THERMO-HYDRODYNAMICS OF AN IDEAL FLUID BY THE METHOD OF EXTENDED IRREVERSIBLE THERMODYNAMICS

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Summary. In this paper we constructed thermo-hydrodynamics for relativistic fluid (taking into account the second order of deviation from equilibrium for dissipative heat and viscosity flows) on the basis of extended irreversible thermodynamics. EIT formalism, providing adequate modeling of systems close to the equilibrium state, goes beyond the local equilibrium hypothesis by expanding the number of basic independent variables (including dissipative flows), as well as by modifying such conceptual concepts as entropy, temperature and pressure. The evolutionary laws for the main nonequilibrium field quantities of the relativistic system are postulated: 4-vector particle flux, 4-vector energy-momentum and 4-vector entropy flux. In order to derive the constitutive equations, a nonlocal Gibbs covariance relation and a nonlocal form of the second principle of thermodynamics with a source of entropy due to additional variables-dissipative flows-were obtained. The defining equations of the hyperbolic type, forbidding superluminal velocities, modified by relaxation terms, have been obtained. The construction of relativistic thermodynamics is carried out using the hydrodynamic 4-speed defined by Eckart. The constructed relativistic hydrodynamics has its applications in such important fields of science as nuclear physics, astrophysics and cosmology.

1 INTRODUCTION

The standard formulation of the thermal and viscous fluid transport laws in irreversible thermodynamics (both classical and relativistic) leads to parabolic differential equations for thermal and viscous flows and thus to infinite velocities of thermal and viscous perturbations. Since heat and momentum are carried by molecules, it would be natural to expect that these waves should propagate at about the average molecular velocity and certainly not faster than the speed of light, which is completely forbidden by the consistent relativistic theory. It has long been clear that the origins of this paradox lie in the well-known shortcoming of class irreversible thermodynamics (CIT), which belongs to the class of first-order approximation theories, which makes its application in describing relaxation transient phenomena incorrect.

Historically, this problem was considered first within the framework of kinetic theory and only later within the framework of phenomenological thermodynamics. The solution of the transport equation by the Chapman-Enskog method is based on the assumption that the mean free path of particles is much smaller than the characteristic macro-peak length. Then macroscopic behavior of the system can be described by hydrodynamic variables, i.e. using the first five moments of the distribution function - particle number density, hydrodynamic velocity, energy density (or temperature). If the mean free path length is not small compared to the macro-

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scopic distances, the five hydrodynamic variables are not sufficient, and the kinetic transport equation should be solved using other methods.

Grad [1] showed how transient effects can be effectively explained within the framework of classical kinetic theory by using the method of moments instead of the Chapman-Enskog method. According to Grad's method of moments, the nonequilibrium distribution function $f(\boldsymbol{r},\boldsymbol{v},t)$ is expressed in terms of its molecular velocity moments (particle number density, hydrodynamic velocity, energy density and their spatial gradients), with the truncation procedure for the infinite chain of coupled momentum equations obtained being limited to second order moments, correlated with heat flow, and some third order moments. The resulting equations for the thirteen moments allow us to obtain a closed system of generalized hydrodynamic equations, which, being hyperbolic, leads to thermal and viscous perturbations propagation velocities of the order of the speed of sound. Various relativistic variants of the Grad momentum method, taking into account transient effects, have been developed by Stewart [2], Israel and Stewart [3] and independently by Malle [4] and Kranyš [5, 6]. At the same time, detailed calculations carried out within these theories showed that $\sqrt{3/5}\,c$ - this is the upper limit of the velocity of the thermal disturbance wave front in a relativistic gas at high temperature.

The thermodynamic approach to the analysis of relativistic fluids has also been applied repeatedly in the special-relativistic field (see, for example, [7-17]. However, the standard linear theory of relativistic thermodynamics developed by Eckart [7] and Landau and Lifshitz [9] on the principle of local equilibrium predicts an infinite rate of thermal and viscous signals, and, in addition, leads to linear defining relations characterized by general instability - in fact, in the presence of small perturbations, the solutions based on them diverge exponentially from the equilibrium state [18].

In classical phenomenological theory, the problem of instantaneous propagation of, for example, thermal disturbances was solved by formally adding relaxation terms to the Fourier law. Thus, Cattaneo [19] proposed the following version of the Fourier law with a damper $c\,\tau\,\partial\boldsymbol{q}/\partial t = -(\boldsymbol{q}+\nabla T)$, which generalizes the approximate Fourier law for thermal conductivity to the case of accounting for long relaxation times τ of heat flux or accounting for high-frequency high-frequency (high-speed) heat transfer processes at pulses with a steep front. It is important to emphasize that the hyperbolic Maxwell-Cattaneo equation, incompatible with the assumption of a local equilibrium state, has found effective use in the simulation of many thermodynamic experiments with heat waves for some astrophysical processes, in particular, related to calculations of nuclear fusion in accretion shells of neutron stars [20,21].

The theoretical justification for the legitimacy of using this kind of relaxation equations was given in [22] on nonrelativistic thermodynamics, in which the author showed that the source of paradoxes in phenomenological theory is related to the fact that traditional transport theory neglects the second order terms in heat flux and viscosity in the entropy expression. By reconstructing these terms, Muller [22] derived a modified system of phenomenological relaxation equations, which agreed with the linearized form of Grad's kinetic equations. Miller's theory was rediscovered and extended to relativistic fluids by Israel [16]. Since then, a number of publications on the subject have appeared (see, for example, [3, 23, 24]).

At the same time, in recent years intensive studies have been carried out in the field of the so-called extended irreversible thermodynamics that goes beyond the local equilibrium hypothesis by expanding the number of basic independent variables when considering systems close to the equilibrium state and by modifying such conceptual concepts as entropy, temperature, pressure and chemical potentials [25-30]. This theory introduces dissipative thermodynamic flows, which appear in the equations of mass, momentum and energy balance, as additional structural parameters. These include, in particular: hydrodynamic velocity, stress tensor (minus its hydrostatic part), heat flow (total energy flow minus flows associated with advection and mechanical energy), etc.

According to EIT theory, entropy also depends on dissipative flows, and the expression for the entropy flow may contain additional terms other than $T^{-1}\boldsymbol{q}$. Note that the EIT theory and the method of thirteen moments of Grad use the same independent variables. The use of these new state parameters allows us to thermodynamically obtain relaxation-determining relations for a highly nonequilibrium system that cannot be obtained in the CIT framework, such as closure relations for various dissipative transport fluxes in a turbulent fluid [31]. Thus, the EIT formalism is designed to describe phenomena with relatively long relaxation times and long correlation lengths, as well as high-frequency and short-wave phenomena [32].

The aim of the present paper is to construct in the framework of EIT relativistic hydrodynamics taking into account the second-order terms for dissipative flows. Such construction, based on a set of basic macroscopic quantities N^{α}, $T^{\alpha\beta}$, S^{α}, describing a no equilibrium state of the relativistic system, is connected with obtaining 14 equations, 5 of which are provided by the postulated conservation laws $\partial_{\alpha}N^{\alpha}=0$ and $\partial_{\alpha}T^{\alpha\beta}=0$ for 4-vector of particle flow and 4-vector of energy-momentum tensor, respectively, and the covariant Gibbs relation, which is the basis of the phenomenological approach (arising from the postulated law of increasing $\partial_{\alpha}S^{\alpha}=\sigma\geq 0$ no equilibrium 4-vector entropy flux), gives exactly 9 additional equations required. Thus, the restriction imposed by the local equilibrium principle on the speed of propagation of thermal and viscous perturbations is completely removed in EIT, since this assumption is too rough for a rather extensive class of nonequilibrium relativistic systems (for example, astrophysical high-energy systems associated with steep gradients or fast changes).

The extension of EIT to general relativistic systems is relatively easy if one uses the path of simple replacement of ordinary derivatives by covariant ones and replacement of the Minkowski metric by its Riemann analog to derive covariant hydrodynamic equations. However, in this case, there is known to be some ambiguity associated with the possible choice of the hydrodynamic 4-speed U^{α}. In Eckart's formulation [7] U^{α} is a particle transfer speed, so in the accompanying coordinate system the value N^{α}, disappears, while in Landau and Lifshitz [9] the speed U^{α} is related to the energy flux; then in the moving system the energy flux cT^{0i} disappears. In principle, the methodology of construction of relativistic irreversible thermodynamics should consider any of these fully equivalent variants.

In this paper, in contrast to the Landau and Lifshitz [9] theory, the more convenient Eckart definition of the hydrodynamic speed will be used. In addition, relativistic hydromechanics and irreversible relativistic thermodynamics for a relaxing cosmological fluid (located in a weak gravitational field) are considered anew from a unified covariant point of view. In contrast to many publications cited above, the author, when constructing defining relations for dissipative flows, takes the view that these relations should be formulated by means of explicit expression for generalized entropy production associated with additional variables. In this approach, each dissipative flow (heat flow, viscous pressure, and particle flow) is defined by its own evolution equation (the relaxation), with the kinetic coefficients taking the form that is most natural in the context of relativistic EIT [33].

The results presented in this synopsis are of practical interest both in nuclear physics and in cosmological and astrophysical situations involving the effects of thermal conduction and neutrino gas viscosity, as well as in studies of the collapse of stars, accretion of black holes, and the early Universe.

2 INITIAL DEFINITIONS OF BASIC MACROSCOPIC QUANTITIES

For an inhomogeneous relativistic system, the macroscopic quantities characterizing it are functions of space-time coordinates $x := x^{\alpha} := (ct, \boldsymbol{x}),$ where the index α takes 4 values: $\alpha = 0,1,2,3$; t – time, c – speed of light. Next, we will use the metric $g^{\alpha\beta}$= diag $(1,-1,-1,-1)$ and denote the covariant differentiation operator as [i)]

$$\partial_{\alpha} := \frac{\partial}{\partial x^{\alpha}} = \left(c^{-1} \frac{\partial}{\partial t}, \frac{\partial}{\partial \boldsymbol{x}} \right) =: (\partial_0, \nabla) \tag{1}$$

The nonequilibrium macroscopic state of a relativistic liquid in the thermodynamic theory will be characterized, as well as in relativistic kinetics, by a 4-vector of particle $N^{\alpha}(x)$, a symmetric 4-vector of energy-momentum $T^{\alpha\beta}(x)$ and a 4-vector of entropy stream S^{α} $(\alpha, \beta = 0,1,2,3)$.

The hydrodynamic 4-velocity $U^{\alpha}(x),$ is defined in this case as a time-like vector with a modulus c at each spatio-temporal point

$$U^{\alpha}(x) U_{\alpha}(x) = c^2 . \tag{2}$$

If we differentiate expression (2) by space-time coordinates, we obtain the following relation $U^{\alpha} \partial_{\nu} U_{\alpha} = 0$. With the help of the velocity $U^{\alpha}(x)$ we can determine the tensor-projector

$$\Delta^{\alpha\beta}(x) := g^{\alpha\beta} - c^2 U^{\alpha}(x) U^{\beta}(x) , \tag{3}$$

which, when convolved with an arbitrary 4-vector, acts as a projection operator, since it destroys the part of the 4-vector parallel to the velocity $U^{\alpha}(x)$

$$\Delta^{\alpha\beta}(x) U_{\beta}(x) = 0 . \tag{4}$$

The projection operator $\Delta^{\alpha\beta}$ is characterized by the properties:

$$\Delta^{\alpha\beta} = \Delta^{\beta\alpha} , \quad \Delta^{\alpha\beta} \Delta_{\beta\sigma} = \Delta^{\alpha}_{\sigma} , \quad \Delta^{\alpha}_{\alpha} = 3 . \tag{5}$$

With the help of the fundamental field quantities $N^{\alpha}(x)$, $T^{\alpha\beta}(x)$, $S^{\alpha}(x)$, and the hydrodynamic velocity $U^{\alpha}(x)$ we can determine macroscopic parameters of a relativistic system, such

[i)] This definition is valid only in the absence of gravity [17]. In the case of a cosmological fluid in the presence of substantial gravitational fields, this definition becomes significantly more complicated (see, e.g., [34]).

as particle density $n(x)$, energy density $\varepsilon(x) := en$, heat flux $J_q^\alpha(x)$, pressure tensor $P^{\alpha\beta}(x)$, and entropy density $S(x) := sn$. At that

(i) particle density $n(x)$ is given by the covariant expression

$$n(x) := N^\alpha U_\alpha / c^2; \tag{6}$$

(ii) energy density $\varepsilon(x)$ is defined as

$$\varepsilon(x) := en := U_\alpha T^{\alpha\sigma} U_\sigma / c^2, \tag{7}$$

where, $e(x)$ is the average energy per particle;

(iii) heat flux $J_q^\alpha(x)$ is given by the expression

$$J_q^\alpha(x) := (U^\nu T_{\nu\sigma} - h N_\sigma)\Delta^{\sigma\alpha}, \tag{8}$$

where,

$$h(x) := e + p / n \tag{9}$$

– enthalpy (or heat function) per particle; $p(x)$ – local hydrostatic pressure; from (5) follows the orthogonality condition

$$J_q^\alpha U_\alpha = 0; \tag{10}$$

(iv) the symmetric pressure tensor $P^{\alpha\beta}$ is defined by the formula

$$P^{\alpha\beta}(x) := \Delta_\sigma^\alpha T^{\sigma\tau} \Delta_\tau^\beta, \tag{11}$$

and the symmetry of the tensor $P^{\alpha\beta}$ follows from the symmetry of the energy-momentum tensor; the pressure tensor is usually split into «reversible» and «irreversible» parts:

$$P^{\alpha\beta} = -p\Delta^{\alpha\beta} + \Pi^{\alpha\beta}, \tag{12}$$

where, the value $\Pi^{\alpha\beta}(x)$ is called the viscous pressure tensor;

(v) entropy density $S(x)$ is defined as a scalar

$$S(x) := sn := S^\alpha U_\alpha / c^2, \tag{13}$$

where, $s(x)$ is entropy per particle.

Given the definitions of energy density (7), heat flux (8) and pressure tensor (12), we can write down the following relations:

$$\varepsilon = U_\alpha T^{\alpha\nu} U_\nu / c^2, \quad J_q^\alpha + h\Delta^{\alpha\nu} N_\nu = U_\nu T^{\nu\sigma}\Delta_\sigma^\alpha, \quad -p\Delta^{\alpha\nu} + \Pi^{\alpha\nu} = \Delta_\sigma^\alpha T^{\sigma\lambda} \Delta_\lambda^\nu. \tag{14}$$

Decomposition of the energy-momentum tensor. Using definition (4) for the projective operator $\Delta^{\alpha\nu}$ we can obtain the identity

$$T^{\alpha\nu} = T^{(0)\alpha\nu} + T^{(1)\alpha\nu}, \tag{15}$$

where, $T^{(0)\alpha\nu}$ is the «reversible» part:

$$T^{(0)\alpha\nu} := \varepsilon U^{\alpha} U^{\nu} / c^2 - p\Delta^{\alpha\nu}; \tag{16}$$

$T^{(1)\alpha\nu}$ is the «irreversible» part:

$$T^{(1)\alpha\nu} := c^{-2}\left[\left(\mathsf{J}_q^{\alpha} + h\,\Delta^{\alpha\sigma} N_{\sigma}\right)U^{\nu} + \left(\mathsf{J}_q^{\nu} + h\,\Delta^{\nu\sigma} N_{\sigma}\right)U^{\alpha}\right] + \Pi^{\alpha\nu}. \tag{17}$$

These two forms play, as will be shown later, an important role in the derivation of macroscopic conservation laws.

Hydrodynamic speed selection. The cosmological literature uses two equivalent ways of determining the hydrodynamic velocity U^{α}.

In the approach of Landau and Lifshitz (1988) the velocity U^{α} is defined as the energy transfer rate, while in the Eckart approach the velocity U^{α} is the particle transfer rate (for this reason the value N^{i} ($i = 1, 2, 3$) disappears in the accompanying coordinate system). Further we will use Eckart's approach, in which the velocity U^{α} is defined through a 4-vector of particle flux N^{α} as follows:

$$U^{\alpha} := cN^{\alpha} / \sqrt{N^{\nu} N_{\nu}}\,. \tag{18}$$

Taking into account normalization (2) and the definition of the projector tensor (3), definition (18) is equivocal to the two forms:

$$U^{\alpha} := c^2 N^{\alpha} / N^{\nu} U_{\nu}, \quad \Delta^{\alpha\nu} N_{\nu} = 0\,. \tag{19}$$

Substituting (19) into (8), we obtain an expression for the total heat flux in the Eckart approach:

$$\mathsf{J}_q^{\alpha} := U_{\nu} T^{\nu\sigma} \Delta_{\sigma}^{\alpha}, \tag{20}$$

and by substituting (19) into (17) we obtain for the energy-momentum tensor the representation

$$T^{(1)\alpha\nu} = c^{-2}\left(\mathsf{J}_q^{\alpha} U^{\nu} + \mathsf{J}_q^{\nu} U^{\alpha}\right) + \Pi^{\alpha\nu}. \qquad (21$$

The time derivative and the gradient. Using the hydrodynamic 4-velocity it is convenient to decompose the covariant derivative (1) on space-time coordinates into time-like and space-like parts. Using the tensor-projector (3), one can obtain the identity

$$\partial^{\alpha} = c^{-2} U^{\alpha} U^{\nu} \partial_{\nu} + \Delta^{\alpha\nu} \partial_{\nu} = c^{-2} U^{\alpha}\, \mathrm{D}_u + \nabla^{\alpha}, \tag{22}$$

where the following designations are used

$$\mathrm{D}_u := U^{\nu} \partial_{\nu}, \quad \nabla^{\alpha} := \Delta^{\alpha\nu} \partial_{\nu}\,. \tag{23}$$

The operator D_u is a convectional time derivative (in the accompanying reference frame (local rest) it is a purely temporal differentiation, $\mathrm{D}_{u\mathrm{LR}} = \partial / \partial t)$, and the gradient operator ∇^α is purely spatial, since in this case it has the form :

$$\nabla^0_{\mathrm{LR}} = 0, \quad \nabla^i_{\mathrm{LR}} = -\nabla_{\mathrm{LR}\,i} = -\partial / \partial x^i , \quad \text{or} \quad U^\alpha \nabla_\alpha = 0 .$$

In addition to the concept of convectional time derivative, let us introduce additionally the substantive time derivative $\mathrm{D} := N^\alpha \partial_\alpha$, which describes the change of medium parameters when moving them together with the particle flow. When determining the hydrodynamic velocity U^α according to Eckart, these two operators D_u and D differ only by a multiplier:

$$\mathrm{D} = n\,\mathrm{D}_u = nU^\nu \partial_\nu . \tag{24}$$

3 BALANCE EQUATIONS OF MECHANICS AND THERMODYNAMICS

Within relativistic kinetic theory, the balance equations are derived from the corresponding conservation laws that hold at the microscopic level [10, 35]. In purely macroscopic theory, these conservation laws are postulated.

In a cosmological fluid, in which the number of particles is conserved, the law of conservation of the 4-vector of particle flux $N^\alpha(x)$ has the form:

$$\partial_\alpha N^\alpha(x) = 0 . \tag{25}$$

The macroscopic law of conservation of energy-momentum in the case where there is no external field takes the form:

$$\partial_\nu T^{\alpha\nu}(x) = 0 , \tag{26}$$

where the energy-momentum tensor , defined by formulae (16) and (21), is given by the formula

$$T^{\alpha\nu} := \varepsilon U^\alpha U^\nu / c^2 - p\Delta^{\alpha\nu} + c^{-2}\left(\mathrm{J}^\alpha_q U^\nu + \mathrm{J}^\nu_q U^\alpha\right) + \Pi^{\alpha\nu} \tag{27}$$

At $\alpha = 0$ this is the law of conservation of energy and at $\alpha = 1,2,3$ this is the law of conservation of momentum of the system.

Continuity equations. The total numerical density $n(x)$ is given by the covariant expression (6). Hence, and from the rate normalization (2), it follows that $nU^\alpha := N^\alpha$. Using the conservation law (25) and the time convection derivative operator we obtain the continuity equation for the density $n(x)$

$$\mathrm{D}_u\, n = U^\alpha \partial_\alpha n = \left[\partial_\alpha (nU^\alpha) - n\partial_\alpha U^\alpha = \partial_\alpha N^\alpha - n\partial_\alpha U^\alpha\right] = -n\,\partial_\alpha U^\alpha ,$$

which taking into account identity (22) and auxiliary relation (2), can also be written in the form:

$$\mathrm{D}_u\, n = -n\,\partial_\alpha U^\alpha = -n\nabla_\alpha U^\alpha + nc^{-2} U_\alpha\, \mathrm{D}_u\, U^\alpha =$$

$$-n\nabla_\alpha U^\alpha + n c^{-2} U^\sigma \left(U_\alpha \partial_\sigma U^\alpha \right) \Big] = -n\nabla_\alpha U^\alpha , \tag{28}$$

or

$$n \,\mathrm{D}_u\, n^{-1} = \nabla_\alpha U^\alpha . \tag{28*}$$

Relativistic equation of motion. The equation of motion is derived from the conservation law of the 4-tensor of energy-momentum by convolving it with the projective operator $\Delta^{\alpha\sigma}$ (3):

$$\Delta^\alpha_\sigma \partial_\nu T^{\sigma\nu}(x) = 0 . \tag{29}$$

Substituting expressions (27) for the energy-momentum tensor $T^{\sigma\nu}$ into this equation, we obtain the relativistic equation of motion in another form:

$$c^{-2} hn \,\mathrm{D}_u\, U^\alpha = \nabla^\alpha p - \Delta^\alpha_\nu \nabla_\sigma \Pi^{\nu\sigma} + (hn)^{-1} \Pi^{\alpha\nu} \nabla_\nu p -$$

$$-c^{-2} \left(\Delta^\alpha_\nu \,\mathrm{D}_u\, \mathsf{J}^\nu_q + \mathsf{J}^\alpha_q \nabla_\nu U^\nu + \mathsf{J}^\nu_q \nabla_\nu U^\alpha \right), \tag{30}$$

in which, as before, $h(x)$ is enthalpy per one particle. From this equation one can see that acceleration of relativistic medium is caused by pressure gradients and, besides, by a number of terms of purely relativistic origin. If we neglect the fluxes $\Pi^{\alpha\nu}$ and J^α_q, associated with the dissipative transport phenomena, the equation of motion is reduced to the zero-order equation

$$\mathrm{D}_u\, U^\alpha = c^2 (nh)^{-1} \nabla^\alpha p , \tag{31}$$

which corresponds to the Euler equation for the ideal gas in classical hydromechanics. The expression (31) linking acceleration and pressure gradient plays an important role in deriving relativistic defining relations in those when only relations linear in fluxes associated with transport phenomena are considered.

Relativistic energy equation. The balance equation for specific energy $\varepsilon(x) = en$ of the system is derived using the energy-momentum conservation law (26) with consideration of relations (2), (4), (10), (11), and also definitions (23) for the operator $\mathrm{D}_u := U^\nu \partial_\nu$ and (27) for the 4-tensor $T^{\alpha\nu}$ energy-momentum; as a result, we will have:

$$\mathrm{D}_u\, \varepsilon = -hn\partial_\alpha U^\alpha + \Pi^{\alpha\nu} \partial_\nu U_\alpha - \partial_\alpha \mathsf{J}^\alpha_q + c^{-2} \mathsf{J}^\alpha_q \,\mathrm{D}_u\, U_\alpha . \tag{32}$$

Let us also give an equation for the rate of change of energy $e(x)$ per particle, to derive which we subtract from equation (32) the continuity equation (28) multiplied by $e(x)$; as a result we obtain

$$\mathrm{D} e \equiv n \,\mathrm{D}_u\, e = -p\nabla_\alpha U^\alpha + \Pi^{\alpha\nu} \nabla_\nu U_\alpha - \nabla_\alpha \mathsf{J}^\alpha_q + 2c^{-2} \mathsf{J}^\alpha_q \,\mathrm{D}_u\, U_\alpha . \tag{33}$$

If we neglect the dissipative fluxes $\Pi^{\alpha\beta}$ and J^α_q, then the two equations for energy (32) and (33) will be written in the form of relativistic Euler equations (zero-order equations for energy):

$$\mathrm{D}_u\,\varepsilon = -hn\partial_\alpha U^\alpha\,, \tag{34}$$

$$\mathrm{D}e = -p\partial_\alpha U^\alpha\;. \tag{35}$$

The first law of thermodynamics. The first law of thermodynamics is usually an equation that relates quantities $\mathrm{D}_u\,e + p\,\mathrm{D}_u(1/n)$ to other local quantities. From the continuity equation (28) it follows that $n\,\mathrm{D}_u\,n^{-1} = \nabla_\alpha U_\alpha$.

Combining this expression with (33), we arrive at the equation

$$n\left(\mathrm{D}_u\,e + p\,\mathrm{D}_u\,n^{-1}\right) = \Pi^{\alpha\nu}\nabla_\nu U_\alpha - \nabla_\alpha \mathsf{J}_q^\alpha + 2c^{-2}\mathsf{J}_q^\alpha\,\mathrm{D}_u\,U_\alpha\,, \tag{36}$$

which can be called the first law of relativistic thermodynamics. From equation (36) we see that change energy $e(x)$ occurs due to two terms describing the work, namely the second term in the left part depending on the local hydrostatic pressure $p(x)$, and the first term in the right part depending on the viscous pressure tensor $\Pi^{\alpha\beta}$, and in addition, due to two thermal terms: divergence of the heat flow J_q^α and a purely relativistic term containing, due to relation (31), the pressure gradient

$$\mathrm{D}e + p\,\mathrm{D}n^{-1} = \Pi^{\alpha\nu}\nabla_\nu U_\alpha - \nabla_\alpha \mathsf{J}_q^\alpha + 2\mathsf{J}_q^\alpha(hn)^{-1}\nabla^\alpha p\;. \tag{37}$$

In the absence of values associated with dissipative fluxes by transport processes, this law takes a simple form

$$\mathrm{D}_u\,e + p\,\mathrm{D}_u(1/n) = 0\,, \tag{38}$$

corresponding to the first law of thermodynamics for systems adiabatically isolated from the environment.

The given equations of relativistic hydrodynamics are open because the transfer fluxes $\Pi^{\alpha\beta}, \mathsf{J}_q^\alpha$ entering them still remain undefined. Let us first show how these fluxes can be related linearly with gradients of macro-peak variables by methods of relativistic irreversible thermodynamics.

4 ENTROPY LAW AND ENTROPY BALANCE IN RELATIVISTIC IRREVERSIBLE THERMODYNAMICS.

The content of the concept «second law of thermodynamics» can mean one of the following two statements, or encompass both of them:

(i) ***Gibbs ratio.*** This law states how changes in entropy in space and time are related to changes in the thermodynamic variables that determine the state of the system in equilibrium. A generalization of the Gibbs relation to the relativistic domain was made in [10], in which it was shown that the traditional form of this relation remains valid in the first approximation to the transfer fluxes for a relativistic system in the equilibrium state.

(ii) ***The law of entropy balance***. This equation expresses the fact that the local entropy of a relativistic system can vary both because of the entropy flux J_s^α and because of the entropy production per unit volume per unit time (entropy source intensity) , which, being a non-negative quantity, is expressed through independent fluxes and associated thermodynamic forces, directly related to the measured physical quantities

Formal entropy balance equation. In Section 2, formula (13) determined the entropy density $S := sn$ through flux $S^\alpha(x)$:

$$S := sn := S^\alpha U_\alpha / c^2,$$

where s is entropy per particle. Now we obtain a formal expression for the entropy balance using the identity

$$n\,\mathrm{D}_u\, s \equiv \partial_\alpha (sN^\alpha)\,, \tag{39}$$

which is a consequence of the law of conservation (28) of the number of particles $n(x)$ and definition (23) of the convection time derivative operator D_u . Adding and subtracting the same term, we write equality (39) as

$$n\,\mathrm{D}_u\, s = -\partial_\alpha (S^\alpha - sN^\alpha) + \partial_\alpha S^\alpha\,, \tag{40}$$

which can be interpreted as a balance equation for the entropy per particle. Indeed, it can be rewritten as:

$$n\,\mathrm{D}_u\, s = -\partial_\alpha \mathsf{J}_s^\alpha + \sigma\,, \tag{41}$$

where

$$\mathsf{J}_s^\alpha(x) := S^\alpha(x) - s(x)N^\alpha(x), \quad 0 \le \sigma := \partial_\alpha S^\alpha \tag{42}$$

− respectively entropy flux (by definition) and entropy source intensity − a quantity which is the postulated law of increasing entropy.

Relativistic Gibbs ratio. It is known from CIT that the entropy density is a well-defined function of state parameters necessary for a complete description of a macroscopic equilibrium system (see, for example, [22]. For the considered relativistic liquid such parameters are energy per one particle $e(x)$ and specific volume $n(x)^{-1}$ per one particle. This property for relativistic systems (as well as for classical ones) is expressed by the fact that for systems in equilibrium the so-called Euler relation takes place (see, for example, [10, 16])

$$Ts = e + pn^{-1} - \mu\,, \tag{43}$$

where T is the temperature of the system at equilibrium; p −local hydrostatic pressure, μ −the Gibbs function (thermodynamic potential) per particle.

If we now take the covariant derivative ∂_ν of (43), we obtain the expression

$$T\partial_\nu s = \partial_\nu e + p\partial_\nu n^{-1} + n^{-1}\partial_\nu p - s\partial_\nu T - \partial_\nu \mu\,, \quad (\nu = 0,1,2,3). \tag{44}$$

Since the last three terms in this expression are zero ($(n^{-1}\partial_\nu p - s\partial_\nu T - \partial_\nu \mu = 0$ − a relativistic version of the Gibbs-Duhem relation), the relativistic Gibbs relation valid for local equilibrium conditions follows from (44):

$$T\partial_\nu s = \partial_\nu e + p\partial_\nu n^{-1}\ , \quad (\nu = 0,1,2,3). \tag{45}$$

This relation relates the change (with respect to time t and spatial coordinates $\boldsymbol{x}$) of entropy $s(x)$ per particle to the changes of energy $e(\boldsymbol{x},t)$, density $n(\boldsymbol{x},t)$. Other physical quantities are absolute temperature $T(\boldsymbol{x},t)$ and local hydrostatic pressure $p(\boldsymbol{x},t)$, determined by derivatives:

$$T^{-1} = (\partial s/\partial e)_{1/n}, \quad T^{-1}p = (\partial s/\partial n^{-1})_e\,.$$

The Gibbs relation (45), using the convection time derivative operator $\mathrm{D} := n\,\mathrm{D}_u$, can also be written in the form [10, 16]

$$T\,\mathrm{D}s = \mathrm{D}e + p\,\mathrm{D}n^{-1}\ . \tag{46}$$

Entropy balance based on Gibbs ratio and conservation laws. To find the explicit form of the entropy balance equation (41), combine ratio (46) and equation (37); the result is:

$$Tn\,\mathrm{D}_u\, s = \Pi^{\alpha\nu}\partial_\nu U_\alpha - \nabla_\alpha \mathsf{J}_q^\alpha + \mathsf{J}_q^\alpha (hn)^{-1}\nabla^\alpha p\,. \quad (47$$

This equation, written in the form of the balance equation (41), takes the form:

$$n\,\mathrm{D}_u\, s = -\nabla_\alpha\left(\frac{\mathsf{J}_q^\alpha}{T}\right) + \frac{1}{T}\left\{\Pi^{\alpha\nu}\nabla_\nu U_\alpha - \mathsf{J}_q^\alpha\left(\frac{\nabla_\alpha T}{T} - \frac{\nabla_\alpha p}{hn}\right)\right\}. \tag{48}$$

Comparing (41) with equation (48), the expressions for entropy flux and entropy production:

$$\mathsf{J}_s^\alpha := \frac{1}{T}\mathsf{J}_q^\alpha\,, \quad \sigma := \frac{1}{T}\left\{\Pi^{\alpha\nu}\nabla_\nu U_\alpha - \mathsf{J}_q^\alpha\left(\frac{\nabla_\alpha T}{T} - \frac{\nabla_\alpha p}{hn}\right)\right\} \geq 0\,. \tag{49}$$

5 LINEAR DEFINING RELATIONS IN RELATIVISTIC IRREVERSIBLE THERMODYNAMICS

Entropy production. For the convenience of further operations, let us decompose the viscous pressure tensor $\Pi^{\alpha\nu} := \Delta_\sigma^\alpha T^{\sigma\tau}\Delta_\tau^\nu + p\Delta^{\alpha\nu}$ as follows:

$$\Pi^{\alpha\nu} := -\Pi\Delta^{\alpha\nu} + \overset{\circ}{\Pi}{}^{\alpha\nu}\ . \tag{50}$$

Here $\Delta^{\alpha\beta}(x) := g^{\alpha\beta} - c^2 U^\alpha(x)U^\beta(x)$ is the tensor-projector, which has a number of properties used below (5); $\overset{\circ}{\Pi}{}^{\alpha\nu}$ is the viscous pressure tensor with a zero trace;

$$\Pi := -\tfrac{1}{3}\Pi^{\alpha\nu}\Delta_{\alpha\nu} \tag{51}$$

− the viscous pressure, defined as taken with the sign minus one third of the trace of the viscous pressure tensor, $\Pi := -\tfrac{1}{3}\Pi_\alpha^\alpha$. The latter definition follows from the relation

$$\Pi^{\alpha\nu}\Delta_{\nu\alpha} = \Pi^{\alpha\nu}g_{\nu\alpha} = \Pi_\alpha^\alpha\,, \tag{52}$$

where the first equality is a consequence of : (i) the definition of the tensor-projector (5); (ii) the definition (12) for the pressure tensor $P^{\alpha\beta}$ and the definition

$$\Pi^{\alpha\beta} := P^{\alpha\beta} + p\Delta^{\alpha\beta}; \tag{53}$$

for the viscous pressure tensor; (iii) the orthogonality relations, $\Delta^{\alpha\beta} U_\beta = 0$.

Using the auxiliary formulas listed above, as well as formulas (5) and (50), it is easy to show that the equality

$$\overset{\circ}{\Pi}{}^{\alpha\alpha} = \overset{\circ}{\Pi}{}^{\alpha\nu} g_{\nu\alpha} = \overset{\circ}{\Pi}{}^{\alpha\nu} \nabla_{\nu\alpha} = 0, \tag{54}$$

i.e., that the tensor $\overset{\circ}{\Pi}{}^{\alpha\nu}$ really has a zero trace.

If we now substitute the expansion (50) into the relation (49) for entropy production through viscous processes, we obtain

$$T\sigma_1 = -\Pi\nabla^\nu U_\nu + \overset{\circ}{\Pi}{}^{\alpha\nu} \nabla_\nu U_\alpha . \tag{55}$$

From (55) we see that the total contribution of viscous phenomena to the production of relativistic entropy turns out to be divided into two parts. Of these, the first contribution is due to the presence of viscous pressure (the second is viscosity) As for the second term in the expansion (50), from the fact that the viscosity tensor $\overset{\circ}{\Pi}{}^{\alpha\nu}$ is symmetric and spatially similar, we can conclude that in the covariant derivative $\partial_\nu U_\alpha$ in this product only its symmetric, spatially similar and traceless part is essential: this quantity will be further denoted by the curve bar with a zero sign over it. Thus, instead of (55) we can write

$$T\sigma_1 = -\Pi\nabla^\nu U_\nu + \overset{\circ}{\Pi}_{\nu\alpha} \overset{\circ}{\overbrace{\nabla^\alpha U^\nu}}, \tag{56}$$

where

$$\overset{\circ}{\overbrace{\nabla^\alpha U^\nu}} = \left[\tfrac{1}{2}\left(\Delta^\alpha_\sigma \Delta^\nu_\tau + \Delta^\nu_\sigma \Delta^\alpha_\tau\right) - \tfrac{1}{3}\Delta^{\alpha\nu}\Delta_{\sigma\tau}\right] \nabla^\sigma U^\tau . \tag{57}$$

Finally, substituting (56) into (49), we obtain the final expression for total entropy production

$$T\sigma = -\Pi\nabla^\nu U_\nu + \overset{\circ}{\Pi}{}^{\nu\alpha} \overset{\circ}{\overbrace{\nabla_\nu U_\alpha}} - \mathsf{J}_{q\alpha}\left(\nabla^\alpha \ln T - \frac{k_B T}{h}\nabla^\alpha p\right). \tag{58}$$

Let us now rewrite expression (58), which is the sum of products of irreversible flows and associated thermodynamic forces of different tensor orders, in the following form:

$$T\sigma = \Pi X_U + \mathsf{J}^\alpha_q X_{q\alpha} + \overset{\circ}{\Pi}{}^{\alpha\nu} \overset{\circ}{X}_{\nu\alpha}, \tag{59}$$

where

$$X_U := -\nabla^\alpha U_\alpha \tag{60}$$

− thermodynamic force (4-divergence of hydrodynamic velocity) coupled to the flow due to viscous pressure Π;

$$\overset{\circ}{X}_{\alpha\nu} := \overset{\circ}{\overline{\nabla_\nu U_\alpha}} \tag{61}$$

− thermodynamic force (shear tensor) conjugate to the tensor $\overset{\circ}{\Pi}{}^{\alpha\nu}$;

$$X_{q\alpha} := -\left(\frac{\nabla^\alpha T}{T} - \frac{\nabla^\alpha p}{hn}\right) = -\nabla^\alpha \ln T + \frac{k_B T}{h}\nabla^\alpha \ln p\,, \quad (\alpha = 0,1,2,3) \tag{62}$$

− the thermodynamic force associated with the heat flow J_q^α, including the temperature gradient and the so-called Eckardt term proportional to the pressure gradient (a purely relativistic effect due to the dependence of enthalpy $h = mc^2 + \frac{5}{2}k_B T + ...$ on rest energy mc^2).

Linear defining relations. We use expression (59) to obtain phenomenological defining relations linearly linking independent flows and thermodynamic forces. In principle, each flux component in this case can be a function of the components of all thermodynamic forces. However, flows and thermodynamic forces in (59), as it is easy to see, have different tensor properties: they are 4-scalars, 4-vectors and 4-tensors. This means that the transformational properties of the above objects, determined under ordinary spatial rotations by their behavior with respect to infinitesimal Lorentz transformations, are different [17]. As a result, it may turn out that, due to the symmetry properties of the considered medium, individual components of some flux will not depend on all components of thermodynamic forces (the relativistic Curie symmetry principle). In particular, for an isotropic medium, flows and thermodynamic forces of different tensor dimension do not depend on each other [33].

In accordance with the general concept of construction of phenomenological defining relations in irreversible thermodynamics, relativistic defining relations for isotropic medium, related to the contribution of viscosity to the entropy production, take the form:

$$\Pi = \eta_U X_U = -\eta_U \nabla^\alpha U_\alpha\,, \tag{63}$$

$$\overset{\circ}{\Pi}{}^{\alpha\nu} = 2\eta\,\overset{\circ}{X}_{\alpha\nu} = 2\eta\,\overset{\circ}{\overline{\nabla^\alpha U^\nu}} \tag{64}$$

− tensor law for viscous flow. Here $\eta_U(T, n^{-1})$, $\eta(T, n^{-1})$ are, respectively, the scalar coefficient of bulk viscosity and the scalar coefficient of shear viscosity.

The linear relation for the heat flux J_q^α, takes the form:

$$\mathsf{J}_q^\alpha = l_{qq} X_q^\alpha = -l_{qq}\left(\nabla^\alpha \ln T - \frac{k_B T}{h}\nabla^\alpha \ln p\right) =$$

$$= -\lambda\left(\nabla^{\alpha} T - \frac{T}{hn}\nabla^{\alpha} p\right), \quad (\alpha = 0,1,2,3), \tag{65}$$

where $\lambda(T, n^{-1}) = l_{qq} / T$ is the thermal conductivity coefficient.

The results of relativistic kinetic theory [13, 24] can be used as kinetic transfer coefficients. In particular, in [35] analytical and numerical results were obtained for the first three approximations to the transfer coefficients of a gas consisting of massive particles with a constant differential section; the full temperature behavior of these coefficients for both a weakly relativistic and an ultrarelativistic ideal gas was also studied.

Relativistic hydrodynamics in a class of first-order theories. Using the phenomenological linear laws (63)-(65) and the equations of state

$$p = nk_B T, \tag{66}$$

$$e = mc^2 + \tfrac{3}{2} k_B T + \ldots \tag{67}$$

it is possible to transform the balance equations for particle number density $n(x)$, hydrodynamic velocity $U^{\alpha}(x)$ and energy $e(x)$ per particle into a system of partial differential equations for the variables n, U^{α}, and T.

However, if we simply substitute the linear laws (64) and (65) in the balance equations, we will get a cumbersome result that apparently has no practical value. Therefore, it is usually assumed that gradients of field quantities can be considered small, which allows linearizing the balance equations by these gradients, i.e., neglecting the terms containing products of fluxes and gradients. In this case, the transfer coefficients can be assumed constant.

The continuity equation (28), which does not contain any irreversible quantities, retains its form

$$\mathrm{D}_u\, n = -n\nabla_{\nu} U^{\nu}. \tag{68}$$

The relativistic equation of motion (30) after substitution of expression (64) and linearization by gradients takes the form:

$$c^{-2} hn\, \mathrm{D}_u\, U^{\alpha} = \nabla^{\alpha} p - \eta_U \nabla_{\sigma} U^{\sigma} - 2\eta \nabla_{\nu} \overset{\circ}{\overbrace{\nabla^{\alpha} U^{\nu}}} - c^{-2}\, \mathrm{D}_u\, \mathrm{J}_q^{\alpha}. \tag{69}$$

The equation for energy (33), which by virtue of equations of state (66) and (67) and the linear relationship (65) for heat flux after linearization by gradients, takes the form :

$$nc_v\, \mathrm{D}_u\, T = -p\nabla_{\alpha} U^{\alpha} + \lambda\left(\nabla^2 T - \frac{T}{hn}\nabla^2 p\right). \tag{70}$$

Here $\nabla^2 := \nabla^{\alpha}\nabla_{\alpha}$; $c_v := \partial e / \partial T$ is the heat capacity per one particle. Using the ratio of heat capacities per one particle $\gamma = 1 + k_B / c_v = \frac{5}{3} - \frac{5}{3} k_B T / mc^2 + \ldots$, the energy equation (70) can be rewritten as follows:

$$\frac{D_u T}{T} = (1-\gamma)\left[\nabla_\alpha U^\alpha + \frac{\lambda}{p}\left(\nabla^2 T - \frac{T}{hn}\nabla^2 p\right)\right]. \tag{71}$$

Express now the last term of the equation of motion (71) through the structural parameters n, U^α, and T. Using equations (66), (68), and (71), it is easy to obtain the relation $D_u\, p = \gamma p \nabla_\alpha U^\alpha$, with which the desired representation for the time derivative of the heat flux J_q^α has the form:

$$c^{-2} D_u \mathsf{J}_q^\alpha = \xi \nabla^\alpha \nabla_\nu U^\nu, \text{ where } \quad \xi := \frac{\lambda T}{c^2 h}\left[(1-\gamma)h + \gamma k_B T\right]. \tag{72}$$

Thus, equations (68), (69), (71) and (72) form a self-consistent system of partial differential equations that completely describe the time evolution of a relativistic fluid provided that the corresponding initial and boundary conditions are specified. These equations are a relativistic generalization of the Navier-Stokes equations of classical fluid dynamics for ideal media. They differ from these equations in the presence of terms that are proportional to transfer coefficients and that describe dissipative effects in the relativistic system.

In conclusion of this section we note the following: The system of dissipative equations of relativistic hydrodynamics obtained here has two drawbacks. Firstly, this first-order hydrodynamics predicts the existence of infinite velocity of thermal and viscous disturbances, which, generally speaking, is unacceptable from the point of view of the relativistic theory. Second, the transfer equations obtained using phenomenological linear laws are characterized by a general instability: in fact, in the presence of small perturbations, their solutions diverge exponentially from the equilibrium state [18].

These drawbacks can be avoided by using the methodology of extended irreversible thermodynamics in the design of relativistic fluid dynamics. Note that ensuring the finiteness of the thermal and viscous propagation rate was, precisely, one of the motivations for the emergence and development of the RST theory (see [25-30]).

6 SECOND-ORDER CLOSURE EVOLUTIONARY MODELS DERIVED FROM RELATIVISTIC EXTENDED IRREVERSIBLE THERMODYNAMICS

Let us now proceed to the main goal of this paper − the derivation of the defining relations describing relaxation of relativistic dissipative flows within the EIT formalism [26, 27, 36]. Let us note at once that the EIT (which is valid outside the local equilibrium approximation) does not generally provide any general criterion for finding them, except only for the restrictions that the second law of thermodynamics imposes on these equations. For this reason, the equations of evolution cannot take an arbitrary form, since they must satisfy the second law $\sigma \geq 0$ (hence, the linear relationship between the flows and the associated thermodynamic forces).

The natural way to obtain differential equations for relaxation dissipative flows describing nonequilibrium but stable states of the relativistic medium is to modify the linear laws obtained in Section 4 in the framework of the CIT.

Generalized Gibbs ratio. Just as in CIT, entropy and the Gibbs identity play a central role in relativistic extended irreversible thermodynamics. In the context of the EIT formalism we will postulate that there exists a generalized no equilibrium entropy

$$s = s(e, n^{-1}, \Pi, J_q^\alpha, \overset{\circ}{\Pi}{}^{\alpha\nu}), \tag{73}$$

depending not only on the classical variables - energy $e(x)$ and specific volume $n(x)^{-1}$, but also on and fluxes $\Pi(x)$, $J_q^\alpha(x)$ and $\overset{\circ}{\Pi}{}^{\alpha\nu}(x)$, appearing in the equations of the balance of the number of particles, momentum and energy. Then the differential form of the generalized entropy takes the form of:

$$\mathrm{D}_u s = \overset{\circ}{\Pi}{}^{\alpha\nu}\left(\frac{\partial s}{\partial e}\right)_{n^{-1},\Pi,J_q^\alpha,\overset{\circ}{\Pi}{}^{\alpha\nu}} \mathrm{D}_u e + \left(\frac{\partial s}{\partial n^{-1}}\right)_{e,\Pi,J_q^\alpha,\overset{\circ}{\Pi}{}^{\alpha\nu}} \mathrm{D}_u n^{-1} + \left(\frac{\partial s}{\partial J_q^\alpha}\right)_{e,n^{-1},\Pi,\overset{\circ}{\Pi}{}^{\alpha\nu}} \mathrm{D}_u J_q^\alpha +$$

$$+\left(\frac{\partial s}{\partial \Pi}\right)_{e,n^{-1},J_q^\alpha,\overset{\circ}{\Pi}{}^{\alpha\nu}} \mathrm{D}_u \Pi + \left(\frac{\partial s}{\partial \overset{\circ}{\Pi}{}^{\alpha\nu}}\right)_{e,n^{-1},\Pi,J_q^\alpha} \mathrm{D}_u \overset{\circ}{\Pi}{}^{\alpha\nu}. \tag{74}$$

By analogy with the classical theory of irreversible processes, we define the nonequilibrium absolute temperature [ii] $\theta(x)$ and the no equilibrium thermodynamic pressure $\pi(x)$ by the equations:

$$\theta^{-1}(e, n^{-1}, \Pi, J_q^\alpha, \overset{\circ}{\Pi}{}^{\alpha\nu}) := \left(\partial s / \partial e\right)_{n^{-1},\Pi,J_q^\alpha,\overset{\circ}{\Pi}{}^{\alpha\nu}}, \tag{75}$$

$$\theta^{-1}\pi(e, n^{-1}, \Pi, J_q^\alpha, \overset{\circ}{\Pi}{}^{\alpha\nu}) := \left(\partial s / \partial n^{-1}\right)_{e,\Pi,J_q^\alpha,\overset{\circ}{\Pi}{}^{\alpha\nu}}. \tag{76}$$

The remaining partial derivatives in (74) will be assumed to be linear in fluxes. Let us introduce notations:

$$\left(\partial s / \partial \Pi\right)_{e,n^{-1},J_q^\alpha,\overset{\circ}{\Pi}{}^{\alpha\nu}} := -T^{-1} n^{-1} \alpha_{00} \Pi, \tag{77}$$

$$\left(\partial s / \partial J_q^\alpha\right)_{e,n^{-1},\Pi,\overset{\circ}{\Pi}{}^{\alpha\nu}} := -T^{-1} n^{-1} \alpha_{10} J_q^\alpha, \tag{78}$$

$$(\partial s / \partial \overset{\circ}{\Pi}{}^{\alpha\nu})_{e,n^{-1},\Pi,J_q^\alpha} := -T^{-1} n^{-1} \alpha_{21} \overset{\circ}{\Pi}{}^{\alpha\nu}. \tag{79}$$

Here the coefficients α_{00}, α_{10} and α_{21} are unknown scalar functions of the parameters $e(x)$ and $n(x)^{-1}$. Substituting expressions (77)-(79) into (74), leads to a generalized Gibbs relation for a nonequilibrium relativistic system

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[ii] Note that temperature θ is not a measure of the average energy of the translational degrees of freedom of particles in a relativistic fluid medium.

$$n\,\mathrm{D}_u s = \theta^{-1} n\,\mathrm{D}_u e + \theta^{-1}\pi\, n\,\mathrm{D}_u n^{-1} -$$

$$-T^{-1}\alpha_{00}\Pi(\mathrm{D}_u\Pi) - T^{-1}\alpha_{10}\mathrm{J}_q^{\alpha}(\mathrm{D}_u\mathrm{J}_q^{\alpha}) - T^{-1}\alpha_{21}\overset{\circ}{\Pi}{}^{\alpha\nu}(\mathrm{D}_u\overset{\circ}{\Pi}{}^{\alpha\nu})\,. \qquad (80)$$

The evolution of entropy is determined by the law of balance (41), and now it is necessary to find the corresponding expressions for the entropy flux $\mathrm{J}_s^{\alpha}(x)$ and entropy production $\sigma(x)$ in this nonequilibrium case. To do this, we substitute in (80) the expressions for $n\,\mathrm{D}_u e$ and $n\,\mathrm{D}_u n^{-1}$ from the energy and mass balance laws (33). (28). Immediate calculations allow us to obtain the equality:

$$n\,\mathrm{D}_u s = \frac{1}{T}\left(-\nabla^{\alpha}\mathrm{J}_q^{\alpha} + 2\mathrm{J}_q^{\alpha}\frac{\nabla^{\alpha}p}{hn} - \Pi\nabla^{\nu}U_{\nu} + \overset{\circ}{\Pi}_{\alpha\nu}\,\overset{\circ}{\overbrace{\nabla^{\alpha}U^{\nu}}}\right) -$$

$$-\alpha_{00}\Pi(\mathrm{D}_u\,\Pi) - \alpha_{10}\mathrm{J}_q^{\alpha}(\mathrm{D}_u\,\mathrm{J}_q^{\alpha}) - \alpha_{21}\overset{\circ}{\Pi}{}^{\alpha\nu}(\mathrm{D}_u\,\overset{\circ}{\Pi}_{\alpha\nu})\,. \qquad (81)$$

In obtaining this result, we used the pressure tensor expansion $P^{\alpha\nu} = -(\pi + \Pi)\Delta^{\alpha\nu} + \overset{\circ}{\Pi}{}^{\alpha\nu}$, which is a well-known expansion in classical relativistic hydrodynamics, in which $p(x)$ the nonequilibrium pressure $\pi(x)$ is replaced with. The further replacement of $\pi(x)$ and $\theta(x)$ by $p(x)$ and $T(x)$ in relation (81) is explained by the fact that the second-order contributions in the flows are negligible.

Generalized entropy flux and entropy production. Before proceeding to finding a generalized expression for entropy production, defined by equality (41)

$$\sigma = n\,\mathrm{D}_u\, s + \partial_{\alpha}\mathrm{J}_S^{\alpha} \ge 0\,, \qquad (82)$$

we need to determine the corresponding expression for the entropy flux J_S^{α}. For isotropic relativistic systems, the most general such representation in terms of the basis independent variables e, n^{-1}, Π, J_q^{α} and $\overset{\circ}{\Pi}{}^{\alpha\nu}$ and taking into account terms at least of the second order, is the following equality:

$$\mathrm{J}_S^{\alpha} := \theta^{-1}\mathrm{J}_q^{\alpha} + \beta''\Pi\mathrm{J}_q^{\alpha} + \beta'''\overset{\circ}{\Pi}{}^{\alpha\nu}\,\mathrm{J}_{q\nu}\,, \qquad (83)$$

where the coefficients β'' and β''' represent in general the functions $e(x)$ and $n^{-1}(x)$. Here, in order to show the connection with the thermal conductivity problem, the inverse nonequilibrium temperature $\theta(x)^{-1}$ is chosen as the coefficient in the heat fluxes $\mathrm{J}_q^{\alpha}(x)$.

The expression for entropy production can be easily obtained from (82) by replacing $n\,\mathrm{D}_u\, s$ and J_s^{α} with the corresponding expressions (81) and (83). As a result, we obtain

$$0 \le \sigma = \frac{1}{T}\left(-\nabla_\alpha \mathsf{J}_q^\alpha + \mathsf{J}_q^\alpha \frac{2\nabla^\alpha p}{hn} - \Pi \nabla^\alpha U_\alpha + \overset{\circ}{\Pi}{}^{\alpha\nu}\, \overset{\circ}{\overline{\nabla_\alpha U_\nu}} \right) - \alpha_{00}\Pi(\mathrm{D}_u \Pi) -$$

$$-\alpha_{10}\mathsf{J}_q^\alpha(\mathrm{D}_u\, \mathsf{J}_{q\alpha}) - \alpha_{21}\overset{\circ}{\Pi}{}^{\alpha\nu}(\mathrm{D}_u\, \overset{\circ}{\Pi}_{\alpha\nu}) + \partial_\alpha \left\{ \theta^{-1}\mathsf{J}_q^\alpha + \beta''\Pi \mathsf{J}_q^\alpha + \beta'''\, \overset{\circ}{\Pi}{}^{\alpha\nu}\, \mathsf{J}_{q\nu} \right\}. \quad (84)$$

This expression has a bilinear form

$$\sigma = \Pi X_U + \mathsf{J}_q^\alpha X_{q\alpha} + \overset{\circ}{\Pi}{}^{\alpha\nu} \overset{\circ}{X}_{\alpha\nu}, \quad (85)$$

where the following thermodynamic forces, coupled with dissipative flows Π, J_q^α and $\overset{\circ}{\Pi}{}^{\alpha\nu}$:

$$X_U := -\frac{1}{T}\nabla^\alpha U_\alpha - \alpha_{00}(\mathrm{D}_u \Pi) + \beta''\nabla_\alpha \mathsf{J}_q^\alpha, \quad (86)$$

$$X_{q\alpha} = \frac{1}{T}\left(-\nabla^\alpha \ln T + \frac{\nabla^\alpha p}{hn} \right) - \alpha_{10}(\mathrm{D}_u\, \mathsf{J}_q^\alpha) + \beta''\nabla^\alpha \Pi + \beta'''\nabla_\nu\, \overset{\circ}{\Pi}{}^{\alpha\nu}, \quad (87)$$

$$\overset{\circ}{X}_{\alpha\nu} = \frac{1}{T}\overset{\circ}{\overline{\nabla^\alpha U^\nu}} - \alpha_{21}(\mathrm{D}_u\, \overset{\circ}{\Pi}{}^{\alpha\nu}) + \beta'''\overset{\circ}{\overline{\nabla^\alpha \mathsf{J}_q^\nu}}. \quad (88)$$

In writing the relations (86)-(88) the identity transformations were used:

$$\partial_\alpha(\theta^{-1}\mathsf{J}_q^\alpha) = \nabla_\alpha(\theta^{-1}\mathsf{J}_q^\alpha) + c^{-2}U_\alpha\, \mathrm{D}_u(\theta^{-1}\mathsf{J}_q^\alpha) =$$

$$= \nabla_\alpha(\theta^{-1}\mathsf{J}_q^\alpha) - c^{-2}\theta^{-1}\mathsf{J}_q^\alpha\, \mathrm{D}_u U^\alpha \cong T^{-1}\nabla_\alpha \mathsf{J}_q^\alpha - \mathsf{J}_q^\alpha \frac{\nabla_\alpha T}{T^2} - \frac{1}{T}\mathsf{J}_q^\alpha \frac{\nabla^\alpha p}{hn}, \quad (89)$$

$$\partial_\alpha(\beta''\Pi \mathsf{J}_q^\alpha) = \beta''\Pi\nabla_\alpha \mathsf{J}_q^\alpha + \beta''\mathsf{J}_q^\alpha \nabla_\alpha \Pi - \beta'' c^{-2}\Pi \mathsf{J}_q^\alpha \left(\mathrm{D}_u\, U^\alpha\right), \quad (90)$$

$$\partial_\alpha(\beta'''\, \overset{\circ}{\Pi}{}^{\alpha\nu}\mathsf{J}_{q\nu}) = \beta'''\mathsf{J}_{q\nu}\nabla_\alpha\, \overset{\circ}{\Pi}{}^{\alpha\nu} + \beta'''\, \overset{\circ}{\Pi}{}^{\alpha\nu}\, \nabla_\alpha \mathsf{J}_{q\nu} - \beta''' c^{-2}\mathsf{J}_{q\nu}\, \overset{\circ}{\Pi}{}^{\alpha\nu}\left(\mathrm{D}_u\, U^\alpha\right), \quad (91)$$

in obtaining which are used:

(i) formula (22) for $\partial^\alpha = c^{-2}U^\alpha\, \mathrm{D}_u + \nabla^\alpha$;

(ii) instead of the time derivative $\mathrm{D}_u\, U^\alpha$, the zero-order equation of motion (31), valid for an ideal fluid, $c^{-2}\, \mathrm{D}_u\, U^\alpha = (hn)^{-1}\nabla^\alpha p$;

(iii) orthogonality conditions (10), $\mathsf{J}_q^\alpha U_\alpha = 0$, $\Pi^{\alpha\nu}U_\nu = 0$;

(iv) the neglect of the last terms in relations (90) and (91), since they lead to a higher approximation order (higher than the second one).

Relaxation equations. According to the general concept of construction of defining relations in irreversible thermodynamics, the relativistic evolution equations for flows J_q^α, Π and $\overset{\circ}{\Pi}{}^{\alpha\nu}$ in an isotropic relativistic medium can be chosen proportional to , respectively, $X_{q\alpha}$, X_U and $\overset{\circ}{X}_{\alpha\nu}$ with positive proportionality coefficients and so as to ensure that the value of entropy production σ is positive. As a consequence, the equations of evolution of fluxes have the form:

a) ***The phenomenological linear equation for viscous pressure***. This equation has the form:

$$\Pi = T\eta_U X_U = -\eta_U\left[\nabla_\nu U^\nu + \frac{1}{Tnk_B}\left(\hat{a}_{00}\,\mathrm{D}_u\Pi - \hat{\beta}''\nabla_\alpha J_q^\alpha\right)\right]. \tag{92}$$

Here η_U is the bulk viscosity coefficient, for which the relativistic kinetic theory for an ideal gas gives the following expression in the first approximation of the Ritz variational method [37-40].

$$\eta_U = \frac{k_B T}{c\,\sigma(T)}\frac{[(5-3\gamma)\hat{h}-3\gamma]^2}{A^{22}}, \tag{93}$$

where $\sigma(T)$ - effective differential cross-section; $\hat{h} = h/k_B T$ - reduced enthalpy; $z := mc^2/k_B T$; $\gamma := c_p/c_v = 1 + k_B/c_v$ - ratio of heat capacities. In the same approximation for the coefficients $\hat{a}_{00} = a_{00}Tp$ and $\hat{\beta}'' = \beta''Tp$, in the case of ideal gas $p = k_B nT$, we have:

i) in the nonrelativistic ideal gas limit, when $z \to \infty$:

$$\hat{\alpha}_{00} = 6z^2/5, \quad \hat{\beta}'' = 4z/5, \tag{94}$$

ii) in the ultrarelativistic ideal gas limit, when $z \to 0$:

$$\hat{\alpha}_{00} = 216/z, \quad \hat{\beta}'' = 6/z^2. \tag{95}$$

b) ***Linear evolution equation for heat flow.*** This equation has the form:

$$J_q^\alpha = T^2\lambda X_{q\alpha} = T\lambda\left[-\frac{\nabla^\alpha T}{T} + \frac{\nabla^\alpha p}{hn} - \frac{1}{nk_B T}\left(\hat{\alpha}_{10}(\mathrm{D}_u J_q^\alpha) - \hat{\beta}''\nabla^\alpha\Pi - \hat{\beta}'''\nabla_\nu\overset{\circ}{\Pi}{}^{\alpha\nu}\right)\right], \tag{96}$$

where

$$\lambda = \frac{3ck_B}{\sigma(T)}\left(\gamma/(\gamma-1)\right)^2\frac{1}{B^{11}} \tag{97}$$

− is the heat transfer coefficient obtained in the first approximation of the Ritz variational method (Anderson, 1975); for the coefficients of $\hat{\beta}'' = Tp\beta''$, $\hat{\beta}''' = Tp\beta'''$ and $\hat{\alpha}_{10} = pT\alpha_{10}$ we have:

i) in the nonrelativistic ideal gas limit, when $z \to \infty$:

$$\hat{\beta}'' = 4z/5, \quad \hat{\alpha}_{10} = 2z/5c^2, \quad \hat{\beta}''' = 2/5; \tag{98}$$

ii) in the ultra-relativistic ideal gas limit, when $z \to 0$:

$$\hat{\beta}'' = 6/z^2, \quad \hat{\alpha}_{10} = 5/4c^2, \quad \hat{\beta}''' = 1/4. \tag{99}$$

c) ***Equation for pressure in the presence of shear viscosity***. Relativistic extended irreversible thermodynamics leads to the following result:

$$\overset{\circ}{\Pi}{}^{\alpha\nu} = 2\eta T \overset{\circ}{X}_{\alpha\nu} = 2\eta\left[\overset{\circ}{\overbrace{\nabla^{\alpha}U^{\nu}}} - \frac{1}{nk_BT}\left(\hat{\alpha}_{21}(\mathrm{D}_u \overset{\circ}{\Pi}{}^{\alpha\nu}) - \hat{\beta}'''\nabla \overset{\circ}{\overbrace{{}^{\alpha}\mathsf{J}_q^{\nu}}}\right)\right], \tag{100}$$

where

$$\eta = \frac{k_BT}{c\sigma(T)}\frac{10\hat{h}}{C^{00}} \tag{101}$$

− shear viscosity coefficient in the first approximation of the Ritz variational method [37]; the coefficients $\hat{\alpha}_{21} = nk_BT^2\alpha_{21}$ and $\hat{\beta}''' = nk_BT^2\beta'''$ tend to values:

i) in the nonrelativistic limit to

$$\hat{\alpha}_{21} = 1/2, \quad \hat{\beta}''' = 2/5; \tag{102}$$

ii) in the ultrarelativistic limit to the values of

$$\hat{\alpha}_{21} = 3/4, \quad \hat{\beta}''' = 1/2. \tag{103}$$

Here the matrix elements A^{22}, B^{11} and C^{00} are bracket expressions, which are calculated taking into account the type of interaction of the particles.

If the linear laws (92), (96) and (100) are used to exclude thermodynamic forces from (85), the entropy gain can be represented as

$$T\sigma = \frac{1}{\eta_U}\Pi^2 + \frac{1}{T\lambda}\mathsf{J}_q^{\alpha}\mathsf{J}_q^{\alpha} + \frac{1}{2\eta}\overset{\circ}{\Pi}{}^{\alpha\nu}\,\overset{\circ}{\Pi}{}^{\alpha\nu} \geq 0. \tag{104}$$

Thus, the generalized linear laws are compatible with non-negative entropy growth. The found expression for the entropy increment agrees with the traditional thermodynamics of irreversible processes.

Second-order relativistic hydrodynamics. The three equations (92), (96), and (100) for viscous pressure, heat flow, and pressure in the presence of shear viscosity obtained within the framework of extended irreversible thermodynamics are generalizations of linear laws (63)-(65) derived by CST methods. Here they are derived by choosing the Eckart hydrodynamic velocity**.** These relations partially coincide with similar formulas derived by de Groot et al. [10], Stewart [2] and Israel and Stewart [3] in their thermodynamic theory. The most striking property of generalized linear laws is the appearance of time derivatives of fluxes. Thus, they take into account the relaxation effect with a typical time scale of the order of the mean free time.

The system of hydrodynamic equations (28), (30) and (33) linearized by gradients of structural parameters and linear constitutive equations for dissipative flows, rewritten as relaxation equations, has the form:

$$\mathrm{D}_u\, n = -n\nabla_\alpha U^\alpha \,, \tag{105}$$

$$c^{-2} hn\, \mathrm{D}_u\, U^\alpha = \nabla^\alpha p + \nabla_\nu \left(\mathit{\Pi}\Delta^{\alpha\nu} - \overset{\circ}{\mathit{\Pi}}{}^{\alpha\nu} \right) - c^{-2}\, \mathrm{D}_u\, \mathsf{J}_q^\alpha \,, \tag{106}$$

$$n\, \mathrm{D}_u\, e = -\nabla_\alpha \mathsf{J}_q^\alpha - (p + \mathit{\Pi})\nabla_\alpha U^\alpha + \overset{\circ}{\mathit{\Pi}}{}^{\alpha\nu}\, \overbrace{\overset{\circ}{\nabla_\nu U_\alpha}} \,, \tag{107}$$

$$\tau_0\, \mathrm{D}_u \mathit{\Pi} = -\mathit{\Pi} - \eta_U \nabla_\nu U^\nu + \frac{\eta_U}{Tnk_B} \hat{\beta}'' \nabla_\alpha \mathsf{J}_q^\alpha \,, \tag{108}$$

$$\tau_2 (\mathrm{D}_u\, \overset{\circ}{\mathit{\Pi}}{}^{\alpha\nu}) = -\overset{\circ}{\mathit{\Pi}}{}^{\alpha\nu} + 2\eta\, \overbrace{\overset{\circ}{\nabla^\alpha U^\nu}} + 2\eta\, p^{-1} \hat{\beta}''' \overbrace{\overset{\circ}{\nabla^\alpha \mathsf{J}_q^\nu}}, \tag{109}$$

$$\tau_1 (\mathrm{D}_u \mathsf{J}_q^\alpha) = -\mathsf{J}_q^\alpha + T\lambda \left[-\frac{\nabla^\alpha T}{T} + \frac{\nabla^\alpha p}{hn} + \frac{1}{nk_B T} \left(\hat{\beta}'' \nabla^\alpha \mathit{\Pi} + \hat{\beta}''' \nabla_\nu \overset{\circ}{\mathit{\Pi}}{}^{\alpha\nu} \right) \right]. \tag{110}$$

These five conservation laws (105)-(107) , as well as the nine relaxation equations (108)-(110) represent a system of 14 equations with 14 unknown variables $n, e, U_\alpha, \mathit{\Pi}, \mathsf{J}_q^\alpha$ and $\overset{\circ}{\mathit{\Pi}}{}^{\alpha\nu}$, in which the values $\tau_0 := \eta_U \hat{a}_{00} / nk_B T$, $\tau_1 := \lambda \hat{\alpha}_{10} / nk_B$ and $\tau_2 := 2\eta \hat{\alpha}_{21} / nk_B T$ being the relaxation times of the viscous pressure, heat flow, and pressure respectively in the presence of shear viscosity, account for the relaxation effect with the typical time scale of the mean free time between collisions. From relations (107)-(109) it is clear that deviations from the linear law take place when the relaxation times of the system overlap the macroscopic time scale. For nonstationary flows, these equations are of the hyperbolic type, while similar hydrodynamic equations obtained by the CST method are parabolic. Similarly, stationary equations (104)-(109) have elliptic (at low velocities) and hyperbolic (at high velocities) types depending on the flow velocity, while stationary equations (68)-(71) are always of elliptic type. The results of a study of the properties of wave phenomena in these relativistic hyperbolic heat-conducting and viscous fluids are presented in [41].

7. CONCLUSION

The paper presents a modern formulation of relativistic extended irreversible thermodynamics for dissipative fluid medium taking into account the second-order terms for dissipative flows. Relativistic thermodynamics is presented here as field theory. The field equations are based on the postulated conservation laws of such fundamental macroscopic quantities as 4-vector particle flux N^α, 4-vector energy-momentum $T^{\alpha\beta}$ and 4-vector entropy flux S^α. The linear defining relations for dissipative flows of heat conduction and viscosity are derived from an explicit entropy balance equation based on the postulated relativistic Gibbs relation for both equilibrium and nonequilibrium states.

It is shown that the theory based on the relativistic Gibbs relation for equilibrium systems contains a fundamental drawback, which leads to parabolic differential equations and, consequently, to infinite propagation velocities for heat flow and viscosity, which contradicts the principle of causality.

Thus, this approach is not effective for many phenomena in high energy astrophysics related to steep gradients or rapid changes of structural parameters. It is valid only when using terms up to the first order for deviations from equilibrium; however, in order to find effective phenomenological defining relations based on an explicit expression for entropy production, the second order is already required. For this reason, the goal of this paper is to attempt to eliminate these drawbacks by systematically conserving second-order terms on fluxes in the 4-vector of entropy flux and obtaining on its basis the defining relaxation equations. This has led to the necessity of using the methods of the so-called extended irreversible thermodynamics which, going beyond the local equilibrium hypothesis, uses dissipative flows of corresponding physical quantities as additional independent structural parameters.

The relativistic thermo-hydrodynamics constructed in the class of theories of the second order has not only purely conceptual meaning: this theory has its applications in such important fields of knowledge as nuclear physics, astrophysics and cosmology. In particular, in viscous cosmological models the volumetric viscosity acts as a cause of dissipation, which has a significant impact on the processes in the Universe.

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REFERENCE

[1] H. Grad, *Principles of the Kinetic Theory of Gases*, Springer, Berlin. 1958.

[2] J. M. Stewart. *Relativistic thermodynamics and kinetic theory/ with applications to Cosmology"* (*in: E. Schatzman (Ed.), Cargese lectures in physics 6, pp. 175-226*). Gordon and Breach, New York, 1973.

[3] W. Israel, J. M. Stewart, "Transient relativistic thermodynamics and kinetic theory", *Ann. Physics*, **118**, 341-372 (1979).

[4] C.M. Marle, "Relativistic extension of the Chapman-Enskog method", In: Gravitational waves and radiations; International Conference, Paris, France, 1973, *Transactions. (A75-26747 11-90) Paris, Centre National de la Recherche Scientifique*, 313-330 (1974). In French.

[5] M. Kranyš, "Phase and signal velocities of waves in dissipative media. Special relativistic theory", *Archive for Rational Mechanics and Analysis*, **48**(4**)**. 274-301 (1972).

[6] M. Kranyš, "Kinetic derivation of nonstationary general relativistic thermodynamics", *Nuovo Cimento B*, **8**(2), 417-441 (1972).

[7] C. Eckart, "The thermodynamics of irreversible processes III. Relativistic theory of the simple fluid", *Phys. Rev*., **58**, 919-928 (1940).

[8] G.A. Kluitenberg, S.R. de Groot, P. Mazur, "Relativistic thermodynamics of irreversible processes I. Heat conduction, diffusion, viscous flow and chemical reactions; Formal part", *Physica*, **19**, 689-794 (1953).

[9] L.D. Landau, E.M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford. 1959.

[10] S.R. de Groot, C.G. van Weert, W.T. Hermens, W.A. van Leeuwen, "On relativistic kinetic gas theory I. The second law for a gas mixture outside equilibrium", *Physica*, **40**. 257-276 (1968).

[11] S.R. de Groot, C.G. van Weert, W.T. Hermens, W.A. van Leeuwen, "On relativistic gas theory II. Reciprocal relations between transport phenomena", *Physica*, **40**, 581-593 (1969).

[12] S.R. de Groot, C.G. van Weert, W.T. Hermens, W.A. van Leeuwen, "On relativistic kinetic gas theory III. The non-relativistic limit and its range of validity", *Physica*, **42**, 309-319. (1969).

[13] S.R. de Groot, W.A. van Leeuwen, P.H. Meltzeril, "Transport Coefficients of a Neutrino Gas", *Nuovo Cimento A*, **25** (2), 229-251 (1975).

[14] T. Alts, I. Müller, "Relativistic thermodynamics of simple heat conducting fluids", *Archive for Rational Mechanics and Analysis*, **48 (**4), 245-273 (1972).
[15] W. Israel, "Relativistic kinetic theory of a simple gas", *J. Math. Phys.*, **4**, 1163-1181 (1963).
[16] W. Israel, "Nonstationary irreversible thermodynamics: a causal relativistic theory", *Ann. Physic*., **100**, 310-331 (1976).
[17] S. Weinberg, *Gravitation and cosmology. Principles and applications of the theory of relativity*, John Wiley & Sons, Inc., New York, 1972.
[18] W.A. Hiscock, L. Lindblom. "Generic instabilities in first-order dissipative relativistic fluid theories", *Physical Review D*, **31 (**4), 725-733 (1985).
[19] C. Cattaneo, "Sulla conduzione del calore", *Atti Seminario Mat. Fis. University Modena,* **3**, 83-101 (1948).
[20] L. Herrera, N. Falcon, "Secular stability behaviour of nuclear burning before relaxation", A*strophysics and Space Science*, **229**(1), 105-115 (1995).
[21] L. Herrera, N. Falcon, "Convection theory before relaxation". *Astrophysics and Space Science*, **234**(1), 139-152 (1995).
[22] I. Müller, "On the entropy inequality", *Archive for Rational Mechanics and Analysis,* **26**(2), 118-141 (1967).
[23] P. Havas, R.J. Swenson, "Relativistic thermodynamics of fluids. I.", *Annals of Physics*, **118**. 259-306 (1979).
[24] I-S. Liu, T. Muller, B. Ruggeri, "Relativistic Thermodynamics of Gases", *Annals of Physics*, **169**, 191-219 (1986).
[25] I. Müller, T. Ruggeri, *Rational Extended Thermodynamic*s, Springer. Berlin. Heidelberg. New York, 1998.
[26] D. Jou, J. Casas-Vazquez, G, Lebon, *Extended Irreversible Thermodynamics,* Springer. Berlin Heidelberg New York. 2001
[27] D. Jou, J. Casas-Vazquez, G, Lebon, M. Grmela, "A phenomenological scaling approach for heat transport in nano-systems", *Appl. Math. Lett.*, **18**, 963-967 (2005).
[28] G. Lebon, J. Casas-Vazquez, D. Jou, "Questions and answers about a thermodynamic theory of the third type", *Contemp. Phys.*, **33**, 41-51 (1992).
[29] G. Lebon, M. Torrissi, A. Valenti. A nonlocal thermodynamic analysis of second sound propagation in crystalline dielectrics, *J. Phys.*, **7**, 1461-1474 (1995).
[30] G. Lebon, D. Jou, J. Casas-Vazquez, U*nderstanding Non-equilibrium Thermodynamics: Foundations, Applications, Frontiers,* Springer-Verlag, Berlin, Heidelberg, 2008.
[31] A.V. Kolesnichenko, "On the thermodynamic derivation of differential equations for turbulent flow transfer in a compressible heat-conducting fluid", *Solar System Research*, **44**(4), 334-347 (2010).
[32] J. Casas-Vazquez, D. Jou. "Temperature in non-equilibrium states: a review of open problems and current proposals", *Rep. Prog. Phys.*, **66**, 1937-2023 (2003).
[33] S.R. de Groot, P. Mazur. *Non-Equilibrium Thermodynamics*, North-Holland, Amsterdam, 1962.
[34] S. Chandrasekhar, "The post-newtonian equations of hydrodynamics in general relativity", *Astroph. J.*, **142,** 1488-1512 (1965).
[35] A. J. Kox, S. R. de Groot, W.A. van Leeuwen, "On relativistic kinetic gas theory XVI. The temperature dependence of the transport coefficients for a simple gas of hard spheres", *Physica A:Statistical Mechanics and its Application,* **84A**, 155-164 (1976).
[36] J. Keizer, "On the relationship between fluctuating irreversible thermodynamics and'extended' irreversible thermodynamics", *J. Stat. Phys.*, **31**, 485-497 (1983).
[37] J. L. Anderson, "Variational principles for relativistic transport coefficients", *Physica A:Statistical Mechanics and its Application*, **79,** 569-582 (1975).
[38] J. L. Anderson, A. C. Jr. Payne, "The relativistic Burnett equations and sound Propagation". *Physica A:Statistical Mechanics and its Application,* **85,** 261-286 (1976),

[39] J. L. Anderson, “Variational principles for calculation of transport coefficients of relativistic multi-component systems”, *Physica A:Statistical Mechanics and its Application,* **85,** 287-309 (1976).
[40] J. L. Anderson, A. J. Kox, “On the correct forms of the transport coefficients for a relativistic gas of hard spheres”, *Physica A:Statistical Mechanics and its Application,* **89,** 408-410 (1977).
[41] Ch.G. van Weert, “Generalized hydrodynamics from relativistic kinetic theory physica”. *Physica A: Statistical Mechanics and its Applications,* **111**(3), 537-552(1982).

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