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CUBIC INTUITIONISTIC STRUCTURES OF A SEMIGROUP IN KU-ALGEBRA

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Summary. An intuitionistic fuzzy set was exhibited by Atanassov in 1986 as a generalization of the fuzzy set. So, we introduce cubic intuitionistic structures on a KU-semigroup as a generalization of the fuzzy set of a KU-semigroup. A cubic intuitionistic k-ideal and some related properties are introduced. Also, a few characterizations of a cubic intuitionistic k-ideal are discussed and new cubic intuitionistic fuzzy sets in a KU-semigroup are defined.

1 INTRODUCTION

A fuzzy set is introduced by Zadehin 1956 [1]. Many papers studied this concept in different branches of mathematic, for example, vector space, group theory, topological space, ring theory and module theory. Atanassov [2] introduced the generalization of a fuzzy set is called an intuitionistic fuzzy set. In [3], Jun and Kim studied intuitionistic fuzzy ideals on BCK-algebras. Kim [4] studied intuitionistic (T; S)-normed fuzzy subalgebras of BCK-algebras. Many authors introduced intuitionistic fuzzy sets in different ways and applied them in many structures; see [5, 6, 7, 8 and 9]. In this paper, a cubic intuitionistic k-ideal and some important properties of cubic intuitionistic k-ideals are discussed. By apply a homomorphism; we can prove some results about a cubic intuitionistic fuzzy$k$-ideal.

2 BASIC CONCEPTS

Some basic concepts that are necessary for the main part of the paper we will include in this section.

Definition 2.1 [10,11].

A KU-algebra$(\Gamma,*,0)$ is a set Γ with a binaryoperation$*$ and constant 0 and itissatisfiesthefollowing,forall$\alpha,\beta,\delta \in \Gamma$

(ku_1) $(\alpha * \beta) * [(\beta * \delta) * (\alpha * \delta)] = 0$,

$(ku_2)$$\alpha * 0 = 0$,

(ku_3) $0 * \alpha = \alpha$,

(ku_4) $\alpha * \beta = 0$ and $\beta * \alpha = 0$ implies $\alpha = \beta$,

(ku_5) $\alpha * \alpha = 0$.

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A relation $\leq$ on Γ is defined as follows $\alpha \leq \beta \Leftrightarrow \beta * \alpha = 0$. Then $(\Gamma, *, 0)$ is satisfies the following, for all $\alpha, \beta, \delta \in \Gamma$,

$(ku_{1'})$ $(\beta * \delta) * (\alpha * \delta) \leq (\alpha * \beta)$,

$(ku_{2'})$ $0 \leq \alpha$,

$(ku_{3'})$ $\alpha \leq \beta$ *and* $\beta \leq \alpha$ implies $\alpha = \beta$

$(ku_{4'})$ $\beta * \alpha \leq \alpha$.

Theorem 2.2 [12].

For a KU-algebra $(\Gamma, *, 0)$, the following axioms can be realized, for all $\alpha, \beta, \delta \in \Gamma$,

(1) $\alpha \leq \beta$ imply $\beta * \delta \leq \alpha * \delta$,

(2) $\alpha * (\beta * \delta) = \beta * (\alpha * \delta)$,

(3) $\big((\beta * \alpha) * \alpha\big) \leq \beta$.

Example 2.3 [10].

Let $\Gamma = \{0, a, b, c\}$ be a set and $*$ a binary operation,

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	b
b	0	b	0	a
c	0	0	0	0

Then $(\Gamma, *, 0)$ is a KU-algebra.

Definition 2.4 [13].

A nonempty set Γ with $*, \circ$ *and* 0 is called a KU-semigroup if

(I) A nonempty set Γ with $*, 0$ is a KU-algebra,

(II) A nonempty set Γ with $\circ, 0$ is a semigroup,

(III) $\alpha \circ (\beta * \delta) = (\alpha \circ \beta) * (\alpha \circ \delta)$ and $(\alpha * \beta) \circ \delta = (\alpha \circ \delta) * (\beta \circ \delta)$, $\forall \alpha, \beta, \delta \in \Gamma$.

Definition 2.5 [13].

A non-empty subset S of a KU-semigroup Γ is called a *sub*KU-*semigroup* if $\alpha * \beta, \alpha \circ \beta \in$ S, for all $\alpha, \beta \in$ S.

Example 2.6 [13]**.**

Let $\Gamma = \{0, a, b, c\}$ be a set. Define $*$- operation and $\circ$- operation by the following tables

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	c
b	0	a	0	c
c	0	0	0	0

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then$(\Gamma,*,\circ,0)$ is a KU-semigroup and $\{0,b\}$ is a *sub*KU-*semigroup*.

Definition 2.7 [13].

For a KU-semigroup$(\Gamma,*,\circ,0)$, let $\varphi \neq S\subseteq\Gamma$. Then S is called *S-ideal*, if

(i) $0 \in S$,

(ii) $\alpha * \beta \in S$ and $\alpha \in S \Rightarrow \beta \in S$,

(iii) For all $\alpha \in \Gamma, a \in S$, we have $\alpha \circ a \in S$ and $a \circ \alpha \in S$.

Definition 2.8 [13].

For a KU-semigroup$(\Gamma,*,\circ,0)$, if $\varphi \neq S\subseteq\Gamma$. Then S is called a *k-ideal* ,if

i) $0 \in S$,

ii) For all $\alpha,\beta,\delta \in \Gamma$, $(\alpha * (\beta * \delta)) \in S$, $\beta \in S$imply $\alpha * \delta \in S$,

iii) For all $\alpha \in \Gamma$,$a \in$ S, we have $\alpha \circ a \in$ S and $a \circ \alpha \in$ S.

Definition 2.9 [13].

A mapping $f\colon \Gamma \to \Gamma'$ is called a KU-semigroup homomorphism if $f(\alpha * \beta) = f(\alpha) *' f(\beta)$ and $f(\alpha \circ \beta) = f(\alpha) \circ' f(\beta) \forall \alpha, \beta \in \Gamma$, where Γ and $\Gamma'$are two KU-semigroups.

Definition 2.10 [14].

In a KU-semigroup $(\Gamma,*,\circ,0)$. A cubic set Θ in Γ is the set
$\Theta = \{\langle\alpha, \tilde{\mu}_\Theta(\alpha), \lambda_\Theta(\alpha)\rangle : \alpha \in \Gamma\}$, where $\tilde{\mu}_\Theta\colon \Gamma \to D[0,1]$ such that $\tilde{\mu}_\Theta(\alpha) = [\tilde{\mu}_\Theta{}^L(\alpha), \tilde{\mu}_\Theta{}^U(\alpha)]$
Is an interval valued fuzzy set in $\Gamma$and $\lambda_\Theta(\alpha)$ is a fuzzy set in Γ.

Definition 2.11 [14]**.**

In a KU-semigroup$(\Gamma,*,\circ,0)$. A cubic set $\Theta = \langle\tilde{\mu}_\Theta, \lambda_\Theta\rangle$ in Γ is called a cubic sub KU-semigroup if: for all $\alpha, \beta \in \Gamma$

(1) $\tilde{\mu}_\Theta(\alpha * \beta) \geq rmin\,\{\tilde{\mu}_\Theta(\alpha)\,, \tilde{\mu}_\Theta(\beta)\}$,$\lambda_\Theta(\alpha * \beta) \leq max\,\{\lambda_\Theta(\alpha)\,, \lambda_\Theta(\beta)\}$

(2) $\tilde{\mu}_\Theta(\alpha \circ \beta) \geq rmin\,\{\tilde{\mu}_\Theta(\alpha)\,,\ \tilde{\mu}_\Theta(\beta)\}$, $\lambda_\Theta(\alpha \circ \beta) \leq max\,\{\lambda_\Theta(\alpha)\,,\ \lambda_\Theta(\beta)\}$.

Example 2.12 [14].

Let $\Gamma = \{0,1,2,3\}$ be a set. We define two operations by the following tables

*	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then $(\Gamma,*,\circ,0)$ is a KU-semigroup. Define $\tilde{\mu}_{\Theta}(\alpha)$ and $\lambda_{\Theta}(\alpha)$ by

$$\tilde{\mu}_{\Theta}(\alpha)=\begin{cases}[0.2,0.8] & if \quad \alpha=\{0,1,2\}\\ [0.1,0.3] & if \quad \alpha=3\end{cases},\ \lambda_{\Theta}(\alpha)=\begin{cases}0.2 & if \quad \alpha=\{0,1,2\}\\ 0.4 & if \quad \alpha=3\end{cases},$$

Then $\Theta=\langle\tilde{\mu}_{\Theta},\lambda_{\Theta}\rangle$ is a cubicsub KU-semigroup.

Definition 2.13 [14].

A cubic set $\Theta=\langle\tilde{\mu}_{\Theta},\lambda_{\Theta}\rangle$ is called a cubic ideal of a KU-semigroup$(\Gamma,*,\circ,0)$ if for all $\alpha,\beta\in\Gamma$

(II_1) $\tilde{\mu}_{\Theta}(0)\geq\tilde{\mu}_{\Theta}(\alpha)$ and $\lambda_{\Theta}(0)\leq\lambda_{\Theta}(\alpha)$,

(II_2) $\tilde{\mu}_{\Theta}(\beta)\geq rmin\{\tilde{\mu}_{\Theta}(\alpha*\beta),\tilde{\mu}_{\Theta}(\alpha)\}$, $\lambda_{\Theta}(\beta)\leq max\{\lambda_{\Theta}(\alpha*\beta),\lambda_{\Theta}(\alpha)\}$,

(II_3) $\tilde{\mu}_{\Theta}(\alpha\circ\beta)\geq rmin\{\tilde{\mu}_{\Theta}(\alpha),\tilde{\mu}_{\Theta}(\beta)\}$, $\lambda_{\Theta}(\alpha\circ\beta)\leq max\{\lambda_{\Theta}(\alpha),\lambda_{\Theta}(\beta)\}$.

Definition 2.14 [14].

A cubic set $\Theta=\langle\tilde{\mu}_{\Theta},\lambda_{\Theta}\rangle$ is called a cubic k-ideal of a KU-semigroup$(\Gamma,*,\circ,0)$if for all $\alpha,\beta,\delta\in\Gamma$

$(\boldsymbol{Ck_1})$ $\tilde{\mu}_{\Theta}(0)\geq\tilde{\mu}_{\Theta}(\alpha)$ and $\lambda_{\Theta}(0)\leq\lambda_{\Theta}(\alpha)$,

$(\boldsymbol{Ck_2})$ $\tilde{\mu}_{\Theta}(\alpha*\delta)\geq rmin\{\tilde{\mu}_{\Theta}(\alpha*(\beta*\delta)),\tilde{\mu}_{\Theta}(\beta)\}$,

$\lambda_{\Theta}(\alpha*\delta)\leq max\{\lambda_{\Theta}(\alpha*(\beta*\delta)),\lambda_{\Theta}(\beta)\}$,

$(\boldsymbol{Ck_3})$ $\tilde{\mu}_{\Theta}(\alpha\circ\beta)\geq rmin\{\tilde{\mu}_{\Theta}(\alpha),\tilde{\mu}_{\Theta}(\beta)\}$,$\lambda_{\Theta}(\alpha\circ\beta)\leq max\{\lambda_{\Theta}(\alpha),\lambda_{\Theta}(\beta)\}$.

3 CUBIC INTUITIONISTIC K-IDEAL OF A SEMIGROUP IN KU-ALGEBRA

Consider, two elements $D_1=[a^L,a^U]$ and $D_2=[b^L,b^U]$ in D [0, 1] are defined by $r\min(D_1,D_2)=[\min(a^L,b^L),\min(a^U,b^U)]$ and $r\max(D_1,D_2)=\left[\max(a^L,b^L),\max(a^U,b^U)\right]$.

The operations $\geq,\leq$ and = are refined by

1- $D_1\leq D_2$ iff $a^L\leq b^L, a^U\leq b^U$;

2- $D_1\geq D_2$ iff $a^L\geq b^L, a^U\geq b^U$;

3- $D_1=D_2$ iff $a^L=b^L, a^U=b^U$.

In this paper, we use the interval-valued intuitionistic fuzzy set

$$A = \{(\alpha, M_A(\alpha), N_A(\alpha): \alpha \in \Gamma\}.$$

In which $M_A(\alpha)$ and $N_A(\alpha)$ are closed subintervals of [0; 1] for all $\alpha \in \Gamma$. Also, we use the notations $M_A^L(\alpha)$ and $M_A^U(\alpha)$ to mean the left end point and the right endpoint of the interval $M_A(\alpha)$, respectively, and so we have $M_A(\alpha) = [M_A^L(\alpha), M_A^U(\alpha)]$. For the sake of simplicity, we shall use the symbol $A = (M_A(\alpha), N_A(\alpha))$for the interval-valued intuitionisticfuzzy set $A = \{(\alpha, M_A(\alpha), N_A(\alpha): \alpha \in \Gamma\}$. A cubic intuitionistic set in Γ we mean a structure

$\Theta = \{\langle \alpha, A(\alpha), \lambda(\alpha)\rangle: \alpha \in \Gamma\}$ in which A is an interval-valued intuitionistic fuzzy set in Γ and λ is an intuitionistic fuzzy set in Γ. A cubic intuitionistic set $\Theta = \{\langle \alpha, A(\alpha), \lambda(\alpha)\rangle: \alpha \in \Gamma\}$ is simply denoted by $\Theta = \langle A, \lambda\rangle$.

Definition 3.1.

An (CIFS) $\Theta = \langle A, \lambda\rangle$ of Γ is called a CIF-sub KU-semigroup of Γ if ,$\forall \alpha, \beta \in \Gamma$

(i) $M_{\mathrm{A}}(\alpha * \beta) \geq rmin\{M_{\mathrm{A}}(\alpha), M_{\mathrm{A}}(\beta)\}$, $N_{\mathrm{A}}(\alpha * \beta) \leq rmax\{N_{\mathrm{A}}(\alpha), N_{\mathrm{A}}(\beta)\}$,

And $\omega_\lambda(\alpha * \beta) \geq min\{\omega_\lambda(\alpha), \omega_\lambda(\beta)\}$, $\rho_\lambda(\alpha * \beta) \leq max\{\rho_\lambda(\alpha), \rho_\lambda(\beta)\}$,

(ii) $M_{\mathrm{A}}(\alpha \circ \beta) \geq rmin\{M_{\mathrm{A}}(\alpha), M_{\mathrm{A}}(\beta)\}$, $N_{\mathrm{A}}(\alpha \circ \beta) \leq rmax\{N_{\mathrm{A}}(\alpha), N_{\mathrm{A}}(\beta)\}$ and

$\omega_\lambda(\alpha \circ \beta) \geq min\{\omega_\lambda(\alpha), \omega_\lambda(\beta)\}$, $\rho_\lambda(\alpha \circ \beta) \leq max\{\rho_\lambda(\alpha), \rho_\lambda(\beta)\}$.

Lemma 3.2.

If $\Theta = \langle A, \lambda\rangle$ is a CIF-sub KU-semigroup of Γ, then $M_{\mathrm{A}}(0) \geq M_{\mathrm{A}}(\alpha)$, $N_{\mathrm{A}}(0) \leq N_{\mathrm{A}}(\alpha)$ and $\omega_\lambda(0) \geq \omega_\lambda(\alpha)$, $\rho_\lambda(0) \leq \rho_\lambda(\alpha)$, for all $\alpha \in \Gamma$.

Proof.

The prove is straightforward, by applying Definition3.1.

Example 3.3.

Let $\Gamma = \{0, a, b, c\}$ be a set. Define $*$- operation and $\circ$- operation by the following tables

*	0	a	b	c
0	0	a	b	c
a	0	0	0	c
b	0	a	0	c
c	0	0	0	0

∘	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then$(\Gamma, *, \circ, 0)$is a KU-semigroup. We can define a (CIFS) $\Theta = \langle A, \lambda\rangle$ as follows $\Theta = \{(0,1,0,0.8,0.2), (a,0.8,0.2,0.6,0.4), (b,0.6,0.4,0.7,0.3), (c,0.6,0.4,0.8,0.2)\}$.

Then$\Theta = \langle A, \lambda\rangle$ is a CIF-sub KU-semigroup.

Definition 3.4.

A (CIFS) $\Theta = \langle A, \lambda\rangle$ of Γ is called a (CIF)-ideal, if: for all $\alpha, \beta, \delta \in \Gamma$

(cif$_1$)$M_{\mathrm{A}}(0) \geq M_{\mathrm{A}}(\alpha)$, $N_{\mathrm{A}}(0) \leq N_{\mathrm{A}}(\alpha)$ and $\omega_\lambda(0) \geq \omega_\lambda(\alpha)$, $\rho_\lambda(0) \leq \rho_\lambda(\alpha)$, (cif$_2$)$M_A(\beta) \geq rmin\{M_A(\alpha * \beta), M_A(\alpha)\}$,$N_A(y) \leq rmax\{N_A(\alpha * \beta), N_A(\alpha)\}$, and $\omega_\lambda(\beta) \geq min\{\omega_\lambda(\alpha * \beta), \omega_\lambda(\alpha)\}$,$\rho_\lambda(\beta) \leq max\{\rho_\lambda(\alpha * \beta), \rho_\lambda(\alpha)\}$,

(cif$_3$) $M_A(\alpha \circ \beta) \geq rmin\{M_A(\alpha), M_A(\beta)\}$, $N_A((\alpha \circ \beta)) \leq rmax\{N_A(\alpha), N_A(\beta)\}$and $\omega_\lambda((\alpha \circ \beta)) \geq min\{\omega_\lambda(\alpha), \omega_\lambda(\beta)\}$, $\rho_\lambda((\alpha \circ \beta) \leq max\{\rho_\lambda(\alpha), \rho_\lambda(\beta)\}$.

Definition 3.5.

A(CIFS) $\Theta = \langle A, \lambda \rangle$ of Γ is called a (CIF) k- ideal, if: for all$\alpha, \beta, \delta \in \Gamma$

(CIF$_1$)$M_A(0) \geq M_A(\alpha)$, $N_A(0) \leq N_A(\alpha)$ and $\omega_\lambda(0) \geq \omega_\lambda(\alpha)$, $\rho_\lambda(0) \leq \rho_\lambda(\alpha)$,

(CIF$_2$)$M_A(x * \tau) \geq rmin\{M_A(x * (y * \tau)), M_A(y)\}$,$N_A(x * \tau) \leq rmax\{N_A(x * (y * \tau),N_A y\}$, and $\omega_\lambda(x*\tau) \geq min\{\omega_\lambda x*(y*\tau),\omega_\lambda y\}$,

$\rho_\lambda(x * \tau) \leq max\{\rho_\lambda(x * (y * \tau)), \rho_\lambda(y)\}$,

(CIF$_3$)$M_A(x \circ y) \geq rmin\{M_A(x), M_A(y)\}$, $N_A(x \circ y) \leq rmax\{N_A(x), N_A(y)\}$ and $\omega_\lambda(x \circ y) \geq min\{\omega_\lambda(x), \omega_\lambda(y)\}$, $\rho_\lambda(x \circ y) \leq max\{\rho_\lambda(x), \rho_\lambda(y)\}$.

Example 3.6.

Let $\Gamma = \{0, a, b, c, d\}$ be a set. Define $*$- operation and $\circ$- operation by the following

*	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	a	0	c	d
c	0	a	0	0	d
d	0	0	0	0	0

∘	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	b	c
d	0	a	b	c	d

Then $(\Gamma, *, \circ, 0)$ is a KU-semigroup. We can define a (CIFS) $\Theta = \langle A, \lambda \rangle$ as follows: $\Theta = \{(0,1,0,0.5,0.3), (a,0.7,0.2,0.6,0.5), (b,0.6,0.4,0.8,0.1), (c,0.9,0.1,0.6,0.4), (d,0.5,0.2,0.4,0.2)\}$ Then$\Theta = \langle A, \lambda \rangle$ is a (CIF) k- ideal of Γ.

Lemma 3.7.

Let $\Theta = \langle A, \lambda \rangle$ be a (CIF) k-.ideal of a KU-semigroup Γ and $\chi \leq \gamma$, then $M_A(x) \geq M_A(y)$, $N_A(x) \leq N_A(y)$ and $\omega_\lambda(x) \geq \omega_\lambda(y)$, $\rho_\lambda(x) \leq \rho_\lambda(y)$, for all $x, y \in \Gamma$.

Proof.

Let $\chi \leq \gamma$, then by the definition of "$\leq$", we get $\gamma * \chi = 0$.

Now, since $\Theta = \langle A, \lambda \rangle$ is a (CIF)k- ideal, then

$$M_A(x) = M_A(0 * x) \geq rmin\{M_A(0 * (y * x)), M_A(y)\} = rmin\{M_A(0 * 0), M_A(y)\}$$
$$= M_A(y),$$

and

$N_{\mathrm{A}}(x) = N_{\mathrm{A}}(0 * x) \leq rmax\{N_{\mathrm{A}}(0 * (y * x)), N_{\mathrm{A}}(y)\} = rmax\{N_{\mathrm{A}}(0 * 0), N_{\mathrm{A}}(y)\} = N_{\mathrm{A}}(y)$.

Also**,**

$\omega_{\lambda}(x) = \omega_{\lambda}(0 * x) \geq min\{\omega_{\lambda}(0 * (y * x)), \omega_{\lambda}(y)\} = min\{\omega_{\lambda}(0 * 0), \omega_{\lambda}(y)\} = \omega_{\lambda}(y)$

and

$\rho_{\lambda}(x) = \rho_{\lambda}(0 * x) \leq max\{\rho_{\lambda}(0 * (y * x)), \rho_{\lambda}(y)\} = max\{\rho_{\lambda}(0 * 0), \rho_{\lambda}(y)\} = \rho_{\lambda}(y)$.

Theorem 3.8.

Let $(\Gamma,*,\circ,0)$ be a KU-semigroup. A(CIFS) $\Theta = \langle A, \lambda\rangle$of Γ is an (CIF)-ideal if and only if $\Theta = \langle A, \lambda\rangle$ is a (CIF) k-ideal.

Proof.

$N_{\mathrm{A}}(x * \tau) \leq rmax\{N_{\mathrm{A}}(x * (y * \tau)), N_{\mathrm{A}}(y)\}$ and $\omega_{\lambda}(x * \tau) \geq min\{$($\Rightarrow$) Let $\Theta = \langle \mathrm{A}, \lambda\rangle$ be a (CIF)-ideal of Γ ,then by the condition (cif_2) we get

$M_{\mathrm{A}}(x * \tau) \geq rmin\{M_{\mathrm{A}}(y * (x * \tau)), M_{\mathrm{A}}(y)\}$, $N_{\mathrm{A}}(x * \tau) \leq rmax\{N_{\mathrm{A}}(y * (x * \tau)), N_{\mathrm{A}}(y)\}$ and $\omega_{\lambda}(x * \tau) \geq min\{\omega_{\lambda}(y * (x * \tau)), \omega_{\lambda}(y)\}$,$\rho_{\lambda}(x * \tau) \leq max\{\rho_{\lambda}(y * (x * \tau)), \rho_{\lambda}(y)\}$, by Theorem (2.2)(2), we get $M_{\mathrm{A}}(x * \tau) \geq rmin\{M_{\mathrm{A}}(x * (y * \tau)), M_{\mathrm{A}}(y)\}$,

$\omega_{\lambda}(x * (y * \tau)), \omega_{\lambda}(y)\}$,$\rho_{\lambda}(x * \tau) \leq max\{\rho_{\lambda}(x * (y * \tau)), \rho_{\lambda}(y)\}$. And since $\Theta = \langle \mathrm{A}, \lambda\rangle$ is a (CIF)-ideal of KU-semigroup Γ, then the condition(cif_3) is hold. It follows that $\Theta = \langle \mathrm{A}, \lambda\rangle$ is a (CIF) k-ideal of Γ.

($\Leftarrow$)Let $\Theta = \langle \mathrm{A}, \lambda\rangle$ be a (CIF) k-ideal of Γ . If we put $\chi = 0$ in the condition(CIF_2), we get $M_{\mathrm{A}}(\tau) \geq rmin\{M_{\mathrm{A}}(y * \tau)), M_{\mathrm{A}}(y)\}$, $N_{\mathrm{A}}(\tau) \leq rmax\{N_{\mathrm{A}}(y * \tau)), N_{\mathrm{A}}(y)\}$ and

$\omega_{\lambda}(\tau) \geq min\{\omega_{\lambda}(y * \tau)), \omega_{\lambda}(y)\}$,$\rho_{\lambda}(\tau) \leq max\{\rho_{\lambda}(y * \tau)), \rho_{\lambda}(y)\}$, and since $\Theta = \langle \mathrm{A}, \lambda\rangle$ is a (CIF) k-ideal of KU-semigroup Γ, then the condition(CIF_3) is hold. It follows that $\Theta = \langle \mathrm{A}, \lambda\rangle$ is a (CIF)-ideal of Γ.

Definition 3.9.

Let $\Theta = \langle A, \lambda\rangle$ be a (CIFS) of a KU-semigroup Γ. For $[t_1, t_2] \in D[0,1]$ and $\theta \in [0,1]$, the set

$K(M_{\mathrm{A}}, \omega_{\lambda}, [t_1, t_2]) = \{x \in \Gamma: M_{\mathrm{A}}(x) \geq [t_1, t_2], \omega_{\lambda}(x) \geq [t_1, t_2]\}$ is called an upper $[t_1, t_2]$ -level of Θ and $L(N_{\mathrm{A}}, \rho_{\lambda}, \theta) = \{x \in \Gamma: N_{\mathrm{A}}(x) \leq \theta, \rho_{\lambda}(x) \leq \theta\}$ is called an lowerθ-level of Θ**.**

Theorem 3.10.

A (CIFS) $\Theta = \langle A, \lambda\rangle$ of Γ is a (CIF) k-ideal if and only if the upper $[t_1, t_2]$-level and the lowerθ-level of Θ are k-ideals of Γ.

Proof.

For $[t_1, t_2] \in D[0,1]$ and $\theta \in [0,1]$, since $K(M_{\mathrm{A}}, \omega_{\lambda}, [t_1, t_2]) \neq \varphi$, for any element $x \in K(M_{\mathrm{A}}, \omega_{\lambda}, [t_1, t_2])$, then

$M_{\mathrm{A}}(x) \geq [t_1, t_2], \omega_{\lambda}(x) \geq [t_1, t_2]$. It follows that $x = 0$, hence $0 \in K(M_{\mathrm{A}}, \omega_{\lambda}, [t_1, t_2])$.

Let $\left(x*(y*\tau)\right)\in K(M_{\mathrm{A}},\omega_{\lambda},[t_1,t_2]) and\ y\in K(M_{\mathrm{A}},\omega_{\lambda},[t_1,t_2])$,then $M_{\mathrm{A}}(x*(y*\tau))\geq[t_1,t_2]$, $M_{\mathrm{A}}(y)\geq[t_1,t_2]$ and $\omega_{\lambda}(x*(y*\tau))\geq[t_1,t_2]$, $\omega_{\lambda}(y)\geq[t_1,t_2]$.

We have $M_{\mathrm{A}}(x*\tau)\geq rmin\{M_{\mathrm{A}}\left(x*(y*\tau)\right),M_{\mathrm{A}}(y)\}\geq[t_1,t_2]$ and

$\omega_{\lambda}(x*\tau)\geq min\{\omega_{\lambda}\left(x*(y*\tau)\right),\omega_{\lambda}(y)\}\geq[t_1,t_2]$, then$(x*\tau)\in K(M_{\mathrm{A}},\omega_{\lambda},[t_1,t_2])$.

It follows that $M_{\mathrm{A}}(x\circ y)\geq rmin\{M_{\mathrm{A}}(x),M_{\mathrm{A}}(y)\}\geq[t_1,t_2]$ and

$\omega_{\lambda}(x\circ y)\geq min\{\omega_{\lambda}(x),\omega_{\lambda}(y)\}\geq[t_1,t_2]$, then we get $x\circ y\in K(M_{\mathrm{A}},\omega_{\lambda},[t_1,t_2])$ and by the same way $y\circ x\in K(M_{\mathrm{A}},\omega_{\lambda},[t_1,t_2])$. So $K(M_{\mathrm{A}},\omega_{\lambda},[t_1,t_2])$ is *a* *k*- ideal of Γ.

By the similar way, we can prove that $L(N_{\mathrm{A}},\rho_{\lambda},\theta)=\{x\in\Gamma: N_{\mathrm{A}}(x)\leq\theta,\rho_{\lambda}(x)\leq\theta\}$ is also a *k*- ideal of Γ.

Conversely, assume that $K(M_{\mathrm{A}},\omega_{\lambda},[t_1,t_2])$ and $L(N_{\mathrm{A}},\rho_{\lambda},\theta)$ are *k*-ideals of Γ. For any $[t_1,t_2]\in D[0,1]$ and $\theta\in[0,1]$, suppose that $x,y,\tau\in\Gamma$ such that $M_{\mathrm{A}}(0)\geq M_{\mathrm{A}}(x)$ and $\omega_{\lambda}(0)\geq\omega_{\lambda}(x)$. Now, let

$[\varepsilon_0,\varepsilon_1]=\frac{1}{2}[M_{\mathrm{A}}(0)+M_{\mathrm{A}}(x)]$ and $[\varepsilon_0,\varepsilon_1]=\frac{1}{2}[\omega_{\lambda}(0)+\omega_{\lambda}(x)]$. It follows that $[\varepsilon_0,\varepsilon_1]<M_{\mathrm{A}}(x)$, $[0,0]\leq M_{\mathrm{A}}(0)<[\varepsilon_0,\varepsilon_1]<[1,1]$ and $[\varepsilon_0,\varepsilon_1]<\omega_{\lambda}(x)$, $[0,0]\leq\omega_{\lambda}(0)<[\varepsilon_0,\varepsilon_1]<[1,1]$ then $x\in K(M_{\mathrm{A}},\omega_{\lambda},[\varepsilon_0,\varepsilon_1])$ and since $K(M_{\mathrm{A}},\omega_{\lambda},[\varepsilon_0,\varepsilon_1])$ is a *k*- ideal of Γ, then $[0,0]\in K(M_{\mathrm{A}},\omega_{\lambda},[\varepsilon_0,\varepsilon_1])\Rightarrow M_{\mathrm{A}}(0)\geq[\varepsilon_0,\varepsilon_1]$, and

$\omega_{\lambda}(0)\geq[\varepsilon_0,\varepsilon_1]$, this is a contradiction. Therefore, we have

$M_{\mathrm{A}}(0)\geq M_{\mathrm{A}}(x)$ and $\omega_{\lambda}(0)\geq\omega_{\lambda}(x)$, for all $x\in\Gamma$. Similarly, by taking

$[\delta_0,\delta_1]=\frac{1}{2}[N_{\mathrm{A}}(0)+N_{\mathrm{A}}(x)]$ and $[\delta_0,\delta_1]=\frac{1}{2}[\rho_{\mathrm{A}}(0)+\rho_{\mathrm{A}}(x)]$, we can show that

$N_{\mathrm{A}}(0)\leq N_{\mathrm{A}}(x)$ and $\rho_{\lambda}(0)\leq\rho_{\lambda}(x)$, for any $x\in\Gamma$.

If assume $x,y,\tau\in\Gamma$ such that $M_{\mathrm{A}}(x*\tau)<rmin\{M_{\mathrm{A}}(x*(y*\tau)),M_{\mathrm{A}}(y)\}$.

Put $[\varepsilon_0,\varepsilon_1]=\frac{1}{2}\left[M_{\mathrm{A}}(x*\tau)+rmin\{M_{\mathrm{A}}\left(x*(y*\tau)\right),M_{\mathrm{A}}(y)\}\right]$.

And $[\varepsilon_0,\varepsilon_1]=\frac{1}{2}\left[\omega_{\mathrm{A}}(x*\tau)+min\{\omega_{\mathrm{A}}\left(x*(y*\tau)\right),\omega_{\mathrm{A}}(y)\}\right]$.

$\Rightarrow[\varepsilon_0,\varepsilon_1]>M_{\mathrm{A}}(x*\tau)$, $[\varepsilon_0,\varepsilon_1]<rmin\{M_{\mathrm{A}}\left(x*(y*\tau)\right),M_{\mathrm{A}}(y)\}$and

$[\varepsilon_0,\varepsilon_1]>\omega_{\mathrm{A}}(x*\tau)$, $[\varepsilon_0,\varepsilon_1]<min\{\omega_{\mathrm{A}}\left(x*(y*\tau)\right),\omega_{\mathrm{A}}(y)\}$.

$\Rightarrow[\varepsilon_0,\varepsilon_1]>M_{\mathrm{A}}(x*\tau),[\varepsilon_0,\varepsilon_1]<M_{\mathrm{A}}\left(x*(y*\tau)\right)$and $[\varepsilon_0,\varepsilon_1]<M_{\mathrm{A}}(y)$, also

$[\varepsilon_0,\varepsilon_1]>\omega_{\mathrm{A}}(x*\tau),[\varepsilon_0,\varepsilon_1]<\omega_{\mathrm{A}}\left(x*(y*\tau)\right)$ and $[\varepsilon_0,\varepsilon_1]<\omega_{\mathrm{A}}(y)$.

So, $(x*\tau)\notin K(M_{\mathrm{A}},\omega_{\lambda},[\varepsilon_0,\varepsilon_1])$,$(x*(y*\tau)\in K(M_{\mathrm{A}},\omega_{\lambda},[\varepsilon_0,\varepsilon_1])$ and $y\in K(M_{\mathrm{A}},\omega_{\lambda},[\varepsilon_0,\varepsilon_1])$, this is a contradiction .

Hence $M_{\mathrm{A}}(x*\tau)\geq rmin\{M_{\mathrm{A}}(x*(y*\tau)),M_{\mathrm{A}}(y)\}$ and $\omega_{\lambda}(x*\tau)\geq min\{\omega_{\lambda}(x*(y*\tau)),\omega_{\lambda}(y)\}$, for all $x,y,\tau\in\Gamma$.

Similarly, we can show that

$N_{\mathrm{A}}(x*\tau)\leq rmax\{N_{\mathrm{A}}(x*(y*\tau)),N_{\mathrm{A}}(y)\}$, and
$\rho_{\lambda}(x*\tau)\leq max\{\rho_{\lambda}(x*(y*\tau)),\rho_{\lambda}(y)\}$, $\forall x,y,\tau\in\Gamma$.

Also, we can prove that $M_{\mathrm{A}}(x\circ y)\geq rmin\{M_{\mathrm{A}}(x),M_{\mathrm{A}}(y)\}$, $N_{\mathrm{A}}(x\circ y)\leq rmax\{N_{\mathrm{A}}(x),N_{\mathrm{A}}(y)\}$and $\omega_{\lambda}(x\circ y)\geq min\{\omega_{\lambda}(x),\omega_{\lambda}(y)\}$,
$\rho_{\lambda}(x\circ y)\leq max\{\rho_{\lambda}(x),\rho_{\lambda}(y)\}$. Therefore, $\Theta=\langle A,\lambda\rangle$ is a(CIF) k -ideal of Γ.

4 HOMOMORPHISM OF A (CIFS) OF KU-SEMIGROUPS

Definition 4.1.

Let a mapping $h:\Gamma\to\Gamma'$ be a homomorphism of two KU-semigroups Γ and Γ'. For any (CIFS) $\Theta=\langle A,\lambda\rangle$ of Γ, we can define a new (CIFS)$A^{h}=(x,M_{A}^{h},N_{A}^{h},\omega_{A}^{h},\rho_{A}^{h})$ as follows:
$M_{A}^{h}(x)=M_{\mathrm{A}}(h(x))$, $N_{A}^{h}(x)=N_{\mathrm{A}}(h(x))$,$\omega_{A}^{h}(x)=\omega_{\mathrm{A}}(h(x))$,$\rho_{A}^{h}(x)=\rho_{\mathrm{A}}(h(x))$,
where $x\in\Gamma$.

Theorem 4.2.

Let Γ and Γ' be two KU-semigroups and h be a homomorphism and ontomapping fromΓ into Γ'. Then

(1) If $A=(x',M_{\mathrm{A}},N_{\mathrm{A}},\omega_{\mathrm{A}},\rho_{\mathrm{A}})$ is a (CIF) k-ideal of Γ', then $A^{h}=(x,M_{A}^{h},N_{A}^{h},\omega_{A}^{h},\rho_{A}^{h})$ is a (CIF) k -ideal of Γ.

(2) If $A^{h}=(x',M_{A}^{h},N_{A}^{h},\omega_{A}^{h},\rho_{A}^{h})$ is a (CIF) k -ideal of Γ', then $A=(x,M_{\mathrm{A}},N_{\mathrm{A}},\omega_{\mathrm{A}},\rho_{\mathrm{A}})$ is a (CIF) k -ideal of Γ.

Proof.

(1) Since$h$ is onto mapping, it follows that there exists $x\in\Gamma$, for any $x'\in\Gamma'$such that
$h(x)=x'$. We have $M_{A}^{h}(0)=M_{\mathrm{A}}\big(h(0)\big)=M_{\mathrm{A}}(0')\geq M_{\mathrm{A}}(x')=M_{\mathrm{A}}\big(h(x)\big)=M_{A}^{h}(x)$,
$N_{A}^{h}(0)=N_{\mathrm{A}}\big(h(0)\big)=N_{\mathrm{A}}(0')\leq N_{\mathrm{A}}(x')=N_{\mathrm{A}}\big(h(x)\big)=N_{A}^{h}(x)$ and

$$\omega_{A}^{h}(0)=\omega_{\mathrm{A}}\big(h(0)\big)=\omega_{\mathrm{A}}(0')\geq\omega_{\mathrm{A}}(x')=\omega_{\mathrm{A}}\big(h(x)\big)=\omega_{A}^{h}(x),$$
$$\rho_{A}^{h}(0)=\rho_{\mathrm{A}}\big(h(0)\big)=\rho_{\mathrm{A}}(0')\leq\rho_{\mathrm{A}}(x')=\rho_{\mathrm{A}}\big(h(x)\big)=\rho_{A}^{h}(x).$$

Now, let $x,y\in\Gamma$ and $\tau'\in\Gamma'$, then there exists $\tau\in\Gamma$, such that $h(\tau)=\tau'$. Then

$$M_{A}^{h}(x*y)=M_{\mathrm{A}}\big(h(x*y)\big)=M_{\mathrm{A}}\big(h(x)*h(y)\big)\geq rmin\{M_{\mathrm{A}}(h(x)*(\tau'*h(y))),M_{\mathrm{A}}(\tau')\}$$
$$=rmin\big\{M_{\mathrm{A}}(h(x)*(h(\tau)*h(y))),M_{\mathrm{A}}\big(h(\tau)\big)\big\}=rmin\Big\{M_{A}^{h}\big(x*(\tau*y)\big),M_{A}^{h}(\tau)\},$$ and
$$\omega_{A}^{h}(x*y)=\omega_{\mathrm{A}}\big(h(x*y)\big)=\omega_{\mathrm{A}}\big(h(x)*h(y)\big)\geq min\{\omega_{\mathrm{A}}(h(x)*(\tau'*h(y))),\omega_{\mathrm{A}}(\tau')\}$$
$$=min\big\{\omega_{\mathrm{A}}(h(x)*(h(\tau)*h(y))),\omega_{\mathrm{A}}\big(h(\tau)\big)\big\}=min\Big\{\omega_{A}^{h}\big(x*(\tau*y)\big),\omega_{A}^{h}(\tau)\}.$$

Also, we have

$$N_{A}^{h}(x*y)=N_{\mathrm{A}}\big(h(x*y)\big)=N_{\mathrm{A}}\big(h(x)*h(y)\big)\leq rmax\{N_{\mathrm{A}}(h(x)*(\tau'*h(y))),N_{\mathrm{A}}(\tau')\}$$

$= rmax\{N_{\mathrm{A}}(h(x) * (h(\tau) * h(y))), N_{\mathrm{A}}(h(\tau))\} = rmax\left\{N_A^h(x * (\tau * y)), N_A^h(\tau)\right\}$, and

$\rho_A^h(x * y) = \rho_{\mathrm{A}}(h(x * y)) = \rho_{\mathrm{A}}(h(x) * h(y)) \le max\{\rho_{\mathrm{A}}(h(x) * (\tau' * h(y))), \rho_{\mathrm{A}}(\tau')\}$

$= max\{\rho_{\mathrm{A}}(h(x) * (h(\tau) * h(y))), \rho_{\mathrm{A}}(h(\tau))\} = max\left\{\rho_A^h(x * (\tau * y)), \rho_A^h(\tau)\right\}$.

We must prove the condition (CIF_3)

$M_A^h(x \circ y) = M_{\mathrm{A}}(h(x \circ y)) = M_{\mathrm{A}}(h(x) \circ h(y)) \ge rmin\{M_{\mathrm{A}}(h(x)), M_{\mathrm{A}}(h(y))\} = rmin\{M_A^h(x), M_A^h(y)\}$, and

$\omega_A^h(x \circ y) = \omega_{\mathrm{A}}(h(x \circ y)) = \omega_{\mathrm{A}}(h(x) \circ h(y)) \ge min\{\omega_{\mathrm{A}}(h(x)), \omega_{\mathrm{A}}(h(y))\} = min\{\alpha_A^h(x), \alpha_A^h(y)\}$,

Also,

$N_A^h(x \circ y) = N_{\mathrm{A}}(h(x \circ y)) = N_{\mathrm{A}}(h(x) \circ h(y)) \le rmax\{N_{\mathrm{A}}(h(x)), N_{\mathrm{A}}(h(y))\} = rmax\{N_A^h(x), N_A^h(y)\}$,

$\rho_A^h(x \circ y) = \rho_{\mathrm{A}}(h(x \circ y)) = \rho_{\mathrm{A}}(h(x) \circ h(y)) \le max\{\rho_{\mathrm{A}}(h(x)), \rho_{\mathrm{A}}(h(y))\} = max\{\rho_A^h(x), \rho_A^h(y)\}$,

Hence $A^h = (x, M_A^h, N_A^h, \omega_A^h, \rho_A^h)$ is a (CIF) k -ideal of Γ.

(2) Since $h: \Gamma \to \Gamma'$ is onto mapping, it follows that for all $x', y', \tau' \in \Gamma'$, there exist $x, y, \tau \in \Gamma$ such that $h(x) = x'$, $h(y) = y'$ and $h(\tau) = \tau'$. It follows that

$M_{\mathrm{A}}(h(x) * h(y)) = M_{\mathrm{A}}(h(x * y)) = M_A^h(x * y) \ge rmin\{M_A^h(x * (\tau * y)), M_A^h(\tau)\} = rmin\{M_{\mathrm{A}}(h(x) * (h(\tau) * h(y))), M_{\mathrm{A}}(h(\tau))\}$.

And

$$\begin{aligned}\omega_{\mathrm{A}}(h(x) * h(y)) = \omega_{\mathrm{A}}(h(x * y)) = \omega_A^h(x * y) \ge min\{\omega_A^h(x * (\tau * y)), \omega_A^h(\tau)\} \\ = min\{\omega_{\mathrm{A}}(h(x) * (h(\tau) * h(y))), \omega_{\mathrm{A}}(h(\tau))\}\end{aligned}$$

Also, we obtain

$N_{\mathrm{A}}(h(x) * h(y)) = N_{\mathrm{A}}(h(x * y)) = N_A^h(x * y) \le rmax\{N_A^h(x * (\tau * y)), N_A^h(\tau)\} = rmax\{N_{\mathrm{A}}(h(x) * (h(\tau) * h(y))), N_{\mathrm{A}}(h(\tau))\}$

and

$\rho_{\mathrm{A}}(h(x) * h(y)) = \rho_{\mathrm{A}}(h(x * y)) = \rho_A^h(x * y) \le max\{\rho_A^h(x * (\tau * y)), \rho_A^h(\tau)\} = max\{\rho_{\mathrm{A}}(h(x) * (h(\tau) * h(y))), \rho_{\mathrm{A}}(h(\tau))\}$.

We must prove the condition(CIF_3)

$M_{\mathrm{A}}(h(x) \circ h(y)) = M_{\mathrm{A}}(h(x \circ y)) = M_A^h(x \circ y) \ge rmin\{M_A^h(x), M_A^h(y)\} = rmin\{M_{\mathrm{A}}(h(x)), M_{\mathrm{A}}(h(y))\}$, and

$\omega_{\mathrm{A}}\big(h(x) \circ h(y)\big) = \omega_{\mathrm{A}}\big(h(x \circ y)\big) = \omega_A^h(x \circ y) \geq min\{\omega_A^h(x), \omega_A^h(y)\} = min\{\omega_{\mathrm{A}}\big(h(x)\big), \omega_{\mathrm{A}}\big(h(y)\big)\}$,

Also,

$N_{\mathrm{A}}\big(h(x) \circ h(y)\big) = N_{\mathrm{A}}\big(h(x \circ y)\big) = N_A^h(x \circ y) \leq rmax\{N_A^h(x), N_A^h(y)\} = rmax\{N_{\mathrm{A}}\big(h(x)\big), N_{\mathrm{A}}\big(h(y)\big)\}$

and

$\rho_{\mathrm{A}}\big(h(x) \circ h(y)\big) = \rho_{\mathrm{A}}\big(h(x \circ y)\big) = \rho_A^h(x \circ y) \leq max\{\rho_A^h(x), \rho_A^h(y)\} = max\{\rho_{\mathrm{A}}\big(h(x)\big), \rho_{\mathrm{A}}\big(h(y)\big)\}$.

Hence $A = (x, M_{\mathrm{A}}, N_{\mathrm{A}}, \omega_{\mathrm{A}}, \rho_{\mathrm{A}})$ is a (CIF) k -ideal of Γ.

Definition 4.3.

Let $A_1 = (x, M_{A_1}, N_{A_1}, \omega_{A_1}, \rho_{A_1})$ and $A_2 = (x, M_{A_2}, N_{A_2}, \omega_{A_2}, \rho_{A_2})$ be two cubic intuitionistic fuzzy sets of a KU-semigroupΓ. The Cartesian product $A_1 \times A_2 : \Gamma \times \Gamma \to [0,1]$ is defined by $A_1 \times A_2 = [(x,y), M_{A_1} \times M_{A_2}, N_{A_1} \times N_{A_2}, \omega_{A_1} \times \omega_{A_2}, \rho_{A_1} \times \rho_{A_2}]$ such that

$M_{A_1} \times M_{A_2}(x,y) = rmin\{M_{A_1}(x), M_{A_2}(y)\}$, $N_{A_1} \times N_{A_2}(x,y) = rmax\{N_{A_1}(x), N_{A_2}(y)\}$,

$\omega_{A_1} \times \omega_{A_2}(x,y) = min\{\omega_{A_1}(x), \omega_{A_2}(y)\}$ and $\rho_{A_1} \times \rho_{A_2}(x,y) = max\{\rho_{A_1}(x), \rho_{A_2}(y)\}$.

Theorem 4.4.

Let $A_1 = (x, M_{A_1}, N_{A_1}, \omega_{A_1}, \rho_{A_1})$ and $A_2 = (x, M_{A_2}, N_{A_2}, \omega_{A_2}, \rho_{A_2})$ be two a (CIF) -ideals of a KU-semigroupΓ, then $A_1 \times A_2$ is a(CIF)-ideal of $\Gamma \times \Gamma$.

Proof.

(i) For any$(x,y) \in \Gamma \times \Gamma$, we have

$M_{A_1} \times M_{A_2}(0,0) = rmin\{M_{A_1}(0), M_{A_2}(0)\} \geq rmin\{M_{A_1}(x), M_{A_2}(y)\} = M_{A_1} \times M_{A_2}(x,y)$

and

$N_{A_1} \times N_{A_2}(0,0) = rmax\{N_{A_1}(0), N_{A_2}(0)\} \leq rmax\{N_{A_1}(x), N_{A_2}(y)\} = N_{A_1} \times N_{A_2}(x,y)$

Also,

$\omega_{A_1} \times \omega_{A_2}(0,0) = min\{\omega_{A_1}(0), \omega_{A_2}(0)\} \geq min\{\omega_{A_1}(x), \omega_{A_2}(y)\} = \omega_{A_1} \times \omega_{A_2}(x,y)$

and

$\rho_{A_1} \times \rho_{A_2}(0,0) = max\{\rho_{A_1}(0), \rho_{A_2}(0)\} \leq max\{\rho_{A_1}(x), \rho_{A_2}(y)\} = \rho_{A_1} \times \rho_{A_2}(x,y)$

(i) Let $(a,b), (c,d) \in \Gamma \times \Gamma$, then

$$\begin{aligned} M_{A_1} \times M_{A_2}(c,d) &= rmin\{M_{A_1}(c), M_{A_2}(d)\} \\ &\geq rmin\{rmin\{M_{A_1}(a*c), M_{A_2}(a)\}, rmin\{M_{A_1}(b*d), M_{A_2}(b)\}\} \\ &= rmin\{rmin\{M_{A_1}(a*c), M_{A_1}(b*d)\}, rmin\{M_{A_2}(a), M_{A_2}(b)\}\} \\ &= rmin\{M_{A_1} \times M_{A_2}(a*c, b*d), M_{A_1} \times M_{A_2}(a,b)\} \\ &= rmin\{M_{A_1} \times M_{A_2}(a*c, a), M_{A_1} \times M_{A_2}(b*d, b)\} \end{aligned}$$

and

$$\begin{aligned} N_{A_1} \times N_{A_2}(c,d) &= rmax\{N_{A_1}(c), N_{A_2}(d)\} \\ &\leq rmax\{rmax\{N_{A_1}(a*c), N_{A_2}(a)\}, rmax\{N_{A_1}(b*d), N_{A_2}(b)\}\} \\ &= rmax\{rmax\{N_{A_1}(a*c), N_{A_1}(b*d)\}, rmax\{N_{A_2}(a), N_{A_2}(b)\}\} \\ &= rmax\{N_{A_1} \times N_{A_2}(a*c, b*d), N_{A_1} \times N_{A_2}(a,b)\} \\ &= rmax\{N_{A_1} \times N_{A_2}(a*c, a), N_{A_1} \times N_{A_2}(b*d, b)\} \end{aligned}$$

Also,

$$\begin{aligned} \omega_{A_1} \times \omega_{A_2}(c,d) &= min\{\omega_{A_1}(c), \omega_{A_2}(d)\} \\ &\geq min\{min\{\omega_{A_1}(a*c), \omega_{A_2}(a)\}, min\{\omega_{A_1}(b*d), \omega_{A_2}(b)\}\} min\{min\{\omega_{A_1}(a \\ &* c), \omega_{A_1}(b*d)\}, min\{\omega_{A_2}(a), \omega_{A_2}(b)\}\} \\ &= min\{\omega_{A_1} \times \omega_{A_2}(a*c, b*d), \omega_{A_1} \times \omega_{A_2}(a,b)\} \\ &= min\{\omega_{A_1} \times \omega_{A_2}(a*c, a), \omega_{A_1} \times \omega_{A_2}(b*d, b)\} \end{aligned}$$

and

$$\begin{aligned} \rho_{A_1} \times \rho_{A_2}(c,d) &= max\{\rho_{A_1}(c), \rho_{A_2}(d)\} \\ &\leq max\{max\{\rho_{A_1}(a*c), \rho_{A_2}(a)\}, max\{\rho_{A_1}(b*d), \rho_{A_2}(b)\}\} \\ &= max\{max\{\rho_{A_1}(a*c), \rho_{A_1}(b*d)\}, max\{\rho_{A_2}(a), \rho_{A_2}(b)\}\} \\ &= max\{\rho_{A_1} \times \rho_{A_2}(a*c, b*d), \rho_{A_1} \times \rho_{A_2}(a,b)\} \\ &= max\{\rho_{A_1} \times \rho_{A_2}(a*c, a), \rho_{A_1} \times \rho_{A_2}(b*d, b)\} \end{aligned}$$

(i) Let $(a,b),(c,d) \in \Gamma \times \Gamma$, then we have

$$\begin{aligned} M_{A_1} \times M_{A_2}(a \circ b, c \circ d) &= rmin\{M_{A_1}(a \circ b), M_{A_2}(c \circ d)\} \\ &\geq rmin\{rmin\{M_{A_1}(a), M_{A_2}(b)\}, rmin\{M_{A_1}(c), M_{A_2}(d)\}\} \\ &= rmin\{rmin\{M_{A_1}(a), M_{A_1}(c)\}, rmin\{M_{A_2}(b), M_{A_2}(d)\}\} \\ &= rmin\{M_{A_1} \times M_{A_2}(a,c), M_{A_1} \times M_{A_2}(b,d)\} \end{aligned}$$

and

$$\begin{aligned} N_{A_1} \times N_{A_2}(a \circ b, c \circ d) &= rmax\{N_{A_1}(a \circ b), N_{A_2}(c \circ d)\} \\ &\leq rmax\{rmax\{N_{A_1}(a), N_{A_2}(b)\}, rmax\{N_{A_1}(c), N_{A_2}(d)\}\} \\ &= rmax\{rmax\{N_{A_1}(a), M_{A_1}(c)\}, rmax\{N_{A_2}(b), N_{A_2}(d)\}\} \\ &= rmax\{N_{A_1} \times N_{A_2}(a,c), N_{A_1} \times N_{A_2}(b,d)\} \end{aligned}$$

Also,

$$\begin{aligned} \omega_{A_1} \times \omega_{A_2}(a \circ b, c \circ d) &= min\{\omega_{A_1}(a \circ b), \omega_{A_2}(c \circ d)\} \\ &\geq min\{min\{\omega_{A_1}(a), \omega_{A_2}(b)\}, min\{\omega_{A_1}(c), \omega_{A_2}(d)\}\} \\ &= min\{min\{\omega_{A_1}(a), \omega_{A_1}(c)\}, min\{\omega_{A_2}(b), \omega_{A_2}(d)\}\} \\ &= min\{\omega_{A_1} \times \omega_{A_2}(a,c), \omega_{A_1} \times \omega_{A_2}(b,d)\} \end{aligned}$$

and

$$\begin{aligned}\rho_{A_1} \times \rho_{A_2}(a \circ b, c \circ d) &= max\{\rho_{A_1}(a \circ b), \rho_{A_2}(c \circ d)\} \\ &\leq max\{max\{\rho_{A_1}(a), \rho_{A_2}(b)\}, max\{\rho_{A_1}(c), \rho_{A_2}(d)\}\} \\ &= max\{max\{\rho_{A_1}(a), \rho_{A_1}(c)\}, max\{\rho_{A_2}(b), \rho_{A_2}(d)\}\} \\ &= max\{\rho_{A_1} \times \rho_{A_2}(a, c), \rho_{A_1} \times \rho_{A_2}(b, d)\}\end{aligned}$$

Then $A_1 \times A_2$ is a (CIF)-ideal of $\Gamma \times \Gamma$.

5 CONCLUSION

We have studied the concept of cubic intuitionistic structures on a KU-semigroup as a generalization of the fuzzy set of a KU-semigroup. We discussed a few properties of this concept and we investigate some related results. A cubic intuitionistic k-ideals are studied and the product of two cubic intuitionistic k-ideals to product of KU-semigroup are established. The notion of homomorphism of new cubic intuitionistic fuzzy sets in a KU-semigroup is defined.

The main purpose of our future work is to study of cubic intuitionistic fuzzy ideals for other algebraic structures. Also, we can introduce the notion of cubic intuitionistic fuzzy graph for a KU-semigroup.

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