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ON AN ESTIMATE FOR OF AN ELLIPTIC PROBLEM THE SOLUTION IN A DOMAIN WITH AN INFINITE BOUNDARY

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Summary. An a priori estimate for the solution of an elliptic boundary value problem in a domain with an infinite boundary is proved. In this case, a certain regularity of the structure of the domain in the vicinity of infinity is assumed. Similar estimates are known for solutions of boundary value problems in bounded and unbounded domains, but with compact boundaries, as well as in a half-space (the boundary of which is a hyperplane).

1 INTRODUCTION AND MAIN RESULTS

The article considers boundary value problems of the form

$$\begin{cases} A(x,\partial)u = f, & x \in \Omega, \\ B_j(x,\partial)u\big|_{\partial\Omega} = 0, & j = 0,\ 1,\ \dots,\ m-1, \end{cases}$$

where A is an elliptic operator of order $2m$, Ω is a (possibly unbounded) domain in the space $\mathbb{R}^n$, whose boundary can be either a compact or non-compact smooth $(n-1)$-dimensional manifold.

Under the condition that the operator is elliptic uniformly in $x \in \overline{\Omega}$, the covering conditions are uniform in $x \in \partial\Omega$, and the boundary operators are normal, and also under the requirements that the structure of the domain in a neighborhood of infinity is regular, we prove (see the theorem) the a priori estimate

$$\| u \|_{H^{2m}(\Omega)} \leq C\{\| f \|_{L_2(\Omega)} + \| u \|_{L_2(\Omega)}\}$$

for any solution $u \in H^{2m}(\Omega)$ of the boundary value problem.

Similar estimates in the scale of Sobolev spaces H^s are known for problems in the entire space, in a half-space, and in (bounded and unbounded) domains with compact boundaries (see, for example, [1], [2]). The author touched upon this topic in [3]. In this article (unlike [3]), there are no additional requirements of uniform ellipticity and uniform fulfillment of the covering conditions for the (actually auxiliary) operator $A + iq^{2m}$ with parameter $q > 0$, and the formulations and proofs are presented more carefully and with a sufficient degree of detail.

The work is theoretical in nature and is aimed at studying the properties of solutions to elliptic boundary value problems in their most general form, namely, in the case of an arbitrary domain and any order of an elliptic operator. Of course, in its substantive and innovative part, we are talking here about unbounded domains with an infinite boundary and,

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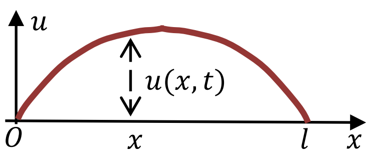
at the same time, about operators of order higher than the second, although all the results remain valid both for domains with compact boundaries and for second-order operators.

A priori estimation of the solution is one of the cornerstones in the study of elliptic boundary value problems. Therefore, it is chosen here as the main object. In this article — and this was its goal — conditions are formulated on the domain and on the operators of the problem, under which the proof of the estimate of the solution in the scale of Sobolev spaces follows a single scenario, regardless of whether the domain is bounded or unbounded, and regardless of the order of the operator.

Many physical problems that describe unbounded media lead to the need to study differential equations in domains with non-compact boundaries, for example, the study of electromagnetic field oscillations outside a conductor, the description of wave diffraction that occurs when rounding an obstacle or due to medium inhomogeneities, and the study of energy dissipation. In addition, finding solutions to a problem in an infinite domain often makes it possible to establish important, fundamental patterns and connections within the physical process, the mathematical model of which is this problem.

Let us recall the beginnings of any course on the equations of mathematical physics (see, for example, [7]).

The oscillation equation $u''_{tt} = a^2 u''_{xx} + f(x,t)$ describes, for example, small transverse vibrations of a string (Fig. 1), longitudinal vibrations of rods, strings, springs, electrical vibrations in wires. During a short period of time, the influence of boundaries (string attachment points) is still insignificant, and instead of a complete boundary value problem with boundary and initial conditions, we can consider a kind of “limiting” problem – the Cauchy problem in an unbounded domain $-\infty < x < +\infty$, $t > 0$ with initial conditions $u(x,0) = \varphi(x)$ and $u'_t(x,0) = \psi(x)$ specifying the initial deviation and initial velocities, respectively.



The function $u(x,t)$ describes string profile at moment time t.

Figure 1. Oscillations of a string

The study of methods for finding solutions to boundary value problems for the equation of oscillations begins with the study of the Cauchy problem (the equation of oscillations of an unbounded string) and leads to the remarkable d'Alembert formula. This formula makes it possible to visually trace the process of oscillations in dynamics and give its physical interpretation, called the method of superposition of two waves (both waves propagate at a speed a, one to the right, in the direction of the abscissa axis, and the other to the left).

The heat equation $u'_t = a^2 u''_{xx} + f(x,t)$ describes, for example, the process of heat propagation in a homogeneous thin rod (Fig. 2), the diffusion process (gas or dissolved substance) in a tube. If the influence of the temperature regime on the boundary of the rod can be neglected (due to the smallness of the time interval and/or the large length of the rod), then it is assumed that the rod has an infinite length and, thus, we arrive at the Cauchy problem in the infinite domain $-\infty < x < +\infty$, $t > 0$ with one initial condition $u(x, 0) = \varphi(x)$ describing the initial temperature distribution in the rod.

The function $u(x, t)$ is the temperature at the point (in section) x at time t.

Figure 2. Distribution of heat in the rod

Finding a solution to the Cauchy problem for the heat equation (as a rule, using the method of separation of variables: $u(x, t) = X(x)T(t)$) leads to an integral representation of the desired solution in the form of the so-called Poisson integral. Moreover, in the resulting formula, one of the factors of the integrand turned out to be so important in the search and study of the properties of solutions to the equation that it received a special name – the fundamental solution (or the function of an instantaneous point source). The concept of a fundamental solution is actively used in the study of partial differential equations of various types.

The study of steady-state (or stationary) processes of various physical nature (for example, oscillations, heat conduction, diffusion, and others) leads, as a rule, to equations of the elliptic type and, accordingly, to elliptic boundary value problems. Examples of such equations are the Poisson equations $\Delta u(x) = f(x)$ and the Helmholtz equations $(\Delta + k^2)u(x) = f(x)$, where Δ is the Laplace operator, $k > 0$.

If we consider elliptic problems in unbounded domains, then it is difficult in the general case to formulate conditions that ensure the existence and uniqueness of the solution. For example, the external Dirichlet boundary value problem for the Laplace equation $\Delta u(x) = 0$ on the plane has a unique solution in the class of bounded functions, and for the unique solvability of the same problem in the three-dimensional case, one has to impose a stronger requirement – the uniform tendency of the solution to zero at infinity. In the case of the Helmholtz operator, the conditions at infinity that ensure the unique solvability of the problem depend essentially on the configuration of the boundary of the domain. Thus, in the whole space or in an unbounded domain with a compact boundary, as well as in the exterior of an infinite cylinder, such conditions are the Sommerfeld radiation conditions in local form (see [7], [8]); for domains "close to a half-space", the unique solvability of boundary value problems in the class of functions with radiation conditions in integral form was proved (see [9], [10]); for a layer between two parallel planes and for the interior of an infinite cylinder, the radiation conditions have a form different from the Sommerfeld conditions and are called partial radiation conditions (see [11], [12]).

Finding conditions at infinity that must be satisfied by a unique and, moreover, interesting from the point of view of physics solution is not an easy task and requires a certain virtuosity. There are two more approaches to identifying a unique solution to stationary boundary value problems – the limiting absorption principle and the limiting amplitude principle (see [7] and also [2] and references there). However, the substantiation of these principles in a particular situation is a non-trivial mathematical problem. The scheme of the principle of limiting absorption in the case of the Helmholtz equation is as follows: consider the auxiliary equation

$(\Delta + k^2 + i\varepsilon)u(x) = f(x)$, where $\varepsilon > 0$, whose characteristic polynomial does not vanish; then they find the limit at $\varepsilon \to 0$ of the only, for example, in the space L_2, solution u_ε of this equation, which, according to the plan, should lead to the solution u of the original equation.

Let us return to the boundary value problem, which is the subject of this paper. If we additionally assume that it is self-adjoint, then it easily follows from the proven a priori estimate that the operator A with domain $D_A = \left\{u \in H^{2m}(\Omega)\colon\ B_j u\big|_{\partial\Omega} = 0,\ \ 0 \le j \le m-1\right\}$ is symmetrical and closed. It turns out (this fact is not proved in this paper) that under the assumptions made, the boundary value problem with the operator $A + i\varepsilon$ instead of A for any right side $f \in L_2(\Omega)$ has, moreover, a unique solution u_ε in $H^{2m}(\Omega)$, which opens up the possibility to study the question of applying the principle of limiting absorption.

Separately, it should be said about second-order operators. In this case, one can significantly weaken the restrictions imposed in the general case both on the domain Ω and on the operators A and B_j. For example (see [13]), (generated by the Dirichlet boundary value problem) the elliptic operator $A_2 = \sum_{k,j=1}^{n} \partial_{x_k}\left(a_{k,j}(x)\partial_{x_j}\right) + b(x)$ in $L_2(\Omega)$, where Ω – a (possibly unbounded) domain with an infinitely smooth boundary, is (under some easy additional assumptions) self-adjoint on some explicitly presented set D_{A_2}, and $D_{A_2} \subset H^1(\Omega)$. Thus, for any function $f \in L_2(\Omega)$ and any $\varepsilon > 0$, the equation $(A_2 + i\varepsilon)u = f$ has a unique solution u_ε belonging to D_{A_2}, and the estimate $\|u_\varepsilon\|_{H^1(\Omega)} \le \frac{C}{\varepsilon}\|f\|_{L_2(\Omega)}$ is valid.

In the modern mathematical literature, the volume of work on differential equations is steadily decreasing. Unfortunately, in the last 5–7 years there have been practically no works on the study of general elliptic boundary value problems. Basically, everything is limited to second-order equations, and extremely rarely in unlimited areas with infinite "curvilinear" boundaries. For example, the article [14] is devoted to the unique solvability of the Neumann–Tricomi problem for the equation $u''_{xx} + yu''_{yy} + \alpha u = 0$ with parameter α in the infinite domain $\{(x;y) : x > 0,\ -x^2/2 < y < +\infty\}$. In [15], [16], an upper bound and a condition for the existence of a solution of a second-order quasilinear elliptic equation in an arbitrary unbounded domain satisfying certain requirements of a geometric nature (the so-called segment property) were obtained. In [17], the solvability was proved and estimates were obtained for the solution (in special weighted spaces) of the first boundary value problem for the second-order singular elliptic equation $\Delta u + ku'_y/y = f$, $k > 0$, in a plane infinite angle.

2 SOME NOTATION

This article considers elliptic problems in some domains of the Euclidean space $\mathbb{R}^n$, the points of which are traditionally denoted by $x = (x_1,\ \dots,\ x_n)$. The standard multi-index notation is used to describe differential operators:

$$\partial \equiv \partial_x = \left(\frac{\partial}{\partial x_1},\ \dots,\ \frac{\partial}{\partial x_n}\right), \qquad \partial^\alpha \equiv \partial_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

where α is a multi-index, i.e. $\alpha = (\alpha_1,\ \dots,\ \alpha_n)$, $\alpha_j \ge 0$, α_j are integers, $|\alpha| \equiv \sum_{j=1}^{n} \alpha_j$.

The notation of function spaces is generally accepted. Let G be an arbitrary domain in $\mathbb{R}^n$. $C_0^\infty(G)$ and $C_0^\infty(\overline{G})$ are sets of infinitely differentiable functions with compact support belonging to G and $\overline{G}$, respectively. For s integers that are non-negative, $H^s(G)$ denotes the space of S.L. Sobolev, consisting of all elements $D'(G)$ whose derivatives up to order s

inclusive belong to $L_2(G)$. (Here, traditionally, $D'(G)$ is the space of generalized functions, and $L_2(G)$ is the set of functions whose square modulus is integrable on the domain G.) The norm in $H^s(G)$ is given by the equality

$$\|u\|^2_{H^s(G)} = \sum_{|\alpha|\le s} \|\partial^\alpha u\|^2_{L_2(G)}.$$

3 REQUIREMENTS FOR A BOUNDARY VALUE PROBLEM

Let Ω be a (possibly unbounded) domain in $\mathbb{R}^n$, $n \ge 2$, whose boundary $\partial\Omega \equiv \Gamma$ is a smooth $(n-1)$-dimensional manifold. Consider in Ω the boundary value problem

$$\begin{cases} Au = f, \quad x \in \Omega, \\ B_j u\big|_\Gamma = 0, \quad 0 \le j \le m-1, \end{cases} \tag{1}$$

where $A \equiv A(x,\partial) = \sum_{|\alpha|\le 2m} a_\alpha(x)\partial^\alpha$, $B_j \equiv B_j(x,\partial) = \sum_{|\beta|\le m_j} b_{j\beta}(x)\partial^\beta$.

Here A is an elliptic operator of order $2m$, B_j is a boundary operator of order $m_j \le 2m - 1$, α and β are n-dimensional multi-indices. It is assumed that the coefficients a_α, $b_{j\beta}$ are functions infinitely differentiable in $\overline{\Omega}$, $f \in L_2(\Omega)$, and the system of operators $\{B_j\}_{j=0}^{m-1}$ is a normal system on Γ, that is, the orders of the operators B_j are pairwise different and the boundary Γ is not a characteristic surface for any of the boundary operators (the above terminology is defined, for example, in the monograph [1]).

In addition, the following requirements will be imposed on the domain Ω and on the operators A, B_j: (I) boundedness, together with the derivatives of the coefficients of the operators, (II) regularity of the structure of the domain at infinity, (III) uniform ellipticity of the equation, uniform (at points along the boundary of the domain) fulfillment of the conditions for covering and normality of boundary operators. Requirements (I)–(III) will be formulated more precisely below.

(I) Assumptions about the coefficients of the operators A and B_j, $0 \le j \le m-1$.

For $x \in \overline{\Omega}$ the following estimates hold:

$$|\partial^\beta a_\alpha| \le C \ \text{ when } \ |\alpha| \le 2\text{m} \ \text{ and } \ \beta_\nu \le \alpha_\nu \ \text{ for all } \ \nu,\ \nu = 1, \dots, n,$$

$$|\partial^\alpha b_{j\beta}| \le C \ \text{ for } \ |\alpha| \le 2m - m_j, \quad |\beta| \le m_j, \quad 0 \le j \le m-1,$$

where the constant $C > 0$ does not depend on α, β, j, x (here $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ are multi-indices).

(II) Requirements for domain Ω.

If the boundary of the domain Ω is compact, then no conditions are imposed on Ω (besides the above smoothness $\partial\Omega \equiv \Gamma$). Otherwise, some regularity of the structure of the domain in the vicinity of infinity is required. Namely, the existence of such a covering $\{U_k\}$, $k = 1, 2, 3, \dots$, of the space $\mathbb{R}^n$ by bounded domains is assumed that:

(II.1) each domain U_k intersects at most M other domains U_j (here M is a fixed number independent of k);

(II.2) for each k, the distance from the boundary of the set $\bigcup_{j\neq k} U_j$ to the boundary U_k is positive and greater than some positive number independent of k;

(II.3) in each domain U_k intersecting with Γ, a smooth non-degenerate coordinate transformation $y = y(x)$ is defined, as a result of which $\Gamma \cap U_k$ becomes an open set in the hyperplane $y_n = 0$, and the adjacent part of the domain Ω becomes a hemisphere $\{y \in \mathbb{R}^n : y_n > 0,\ |y| < 1\}$, moreover, for $x \in U_k$, the following inequalities hold:

$$|\partial_x^\alpha y(x)| \le C, \qquad |\partial_y^\alpha x(y)| \le C, \qquad |\alpha| \le 2m,$$

where the constant $C > 0$ does not depend on k; if U_k does not intersect with Γ, then we set $y = x$ (identical coordinate transformation).

(III) Requirements for operators A and B_j, $0 \le j \le m-1$.

These requirements are that the equation be elliptic uniformly in $x \in \overline{\Omega}$, that the covering conditions (or, equivalently, the Shapiro–Lopatinskii conditions) be satisfied uniformly with respect to $x \in \Gamma$, and that the boundary operators have the normality property uniformly in $x \in \Gamma$.

This means the following. For $x \in U_k \cap \overline{\Omega}$ in the new coordinates y, we denote the symbols of the operators A and B_j by $A(y,\eta)$ and $B_j(y,\eta)$, where $\eta = (\xi, z)$, $\xi \in \mathbb{R}^{n-1}$, $z \in \mathbb{R}^1$ (the one-dimensional variable z corresponds to differentiation along the direction of the normal to Γ). Further, assigning the index zero at the bottom of the symbol of the operator means taking the highest homogeneous part with respect to the variable η.

(III.1) It is assumed that there exists a positive number c_1 such that for all k and any $x \in \overline{\Omega}$

$$|A_0(y,\eta)| \ge c_1 > 0 \quad \text{when} \quad |\eta| = 1.$$

(III.2) Let x^0 be a point of the surface $U_k \cap \Gamma$ and $y^0 \equiv y(x^0)$. It is required that the equation

$$A_0(y^0, \xi, z) = 0, \qquad \xi \in \mathbb{R}^{n-1} \backslash 0, \tag{2}$$

has exactly m complex roots $z_l^+ \equiv z_l^+(y^0, \xi)$, $1 \le l \le m$, in the upper half-plane ($\operatorname{Im} z_l^+ > 0$). Let B_j' be the remainder of the division of $B_{j0}(y^0, \xi, z)$ (considered as a polynomial in z) by $\prod_{l=1}^m (z - z_l^+)$. Then $B_j' = \sum_{k=0}^{m-1} b_{jk}' z^k$, where b_{jk}' depend on y^0 and ξ (the degree in z of the polynomial B_j' obtained in the remainder must be less than the degree of the divisor, which is equal to m). It is assumed that there is a positive constant c_2 such that

$$\left|\det\{b_{jk}'\}_{j,k=0}^{m-1}\right| \ge c_2 > 0$$

for $|\xi| = 1$ and any $x^0 \in \Gamma$.

(III.3) It is assumed that there is a positive constant c_3 such that

$$\left|B_{j0}(y^0, \xi, z)\right| \ge c_3 > 0$$

for $\xi = (0, \dots, 0)$, $z = 1$, any j and all $x^0 \in \Gamma$.

Remark. The requirement of condition (III.2) on the presence of m complex z-roots with a positive imaginary part of equation (2) is ensured by condition (III.1) for $n \ge 3$ always, and also for $n = 2$ in the case when the coefficients A_0 are real numbers.

4 A PRIORI ESTIMATE

Theorem (a priori estimate). Let conditions (I)–(III) be satisfied. Then for the function $u \in H^{2m}(\Omega)$, which is a solution to problem (1), we have the estimate

$$\| u \|_{H^{2m}(\Omega)} \leq C\{\| f \|_{L_2(\Omega)} + \| u \|_{L_2(\Omega)}\}, \tag{3}$$

where the constant C does not depend on u and the function $f \in L_2(\Omega)$, but depends only on the “constraint constants” C, c_1, c_2, c_3 and the number M from conditions (I)–(III).

Proof. It follows from properties (II) of the covering $\{U_k\}$ of the space $\mathbb{R}^n$ that there are functions $\varphi_k \in C_0^\infty(U_k)$ and $\psi_k \in C_0^\infty(U_k)$ satisfying the conditions

$$\text{a)} \quad \sum_{k=1}^{\infty} \varphi_k(x) = 1, \qquad x \in \mathbb{R}^n,$$

$$\text{b)} \quad \psi_k(x) = 1, \qquad x \in \operatorname{supp} \varphi_k, \tag{4}$$

$$\text{c)} \quad |\partial^\alpha \varphi_k(x)| \leq C, \qquad |\partial^\alpha \psi_k(x)| \leq C \tag{5}$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}^n$, $|\alpha| \leq 2m$ (the constant C does not depend on k). The proof of the existence of functions φ_k and ψ_k with the indicated properties is given below in the lemma.

Denote by w, λ_k, μ_k, respectively, the functions u, φ_k, ψ_k in the new coordinates $y = y(x)$ defined in requirements (II) for the domain Ω; for the operators A and B_j in the new variables, we retain the old notation. The function $\lambda_k w$, defined initially only on the image of the map U_k and equal to zero in the neighborhood of the image of the boundary $\partial U_k \cap \overline{\Omega}$ of the map, will be extended by zero to the entire half-space $\mathbb{R}^n_+ \equiv \{y \in \mathbb{R}^n\colon\ y_n > 0\}$, retaining continued function the same notation.

We apply to the function $\lambda_k w$ the standard a priori estimate in the half-space $\mathbb{R}^n_+$ (see, for example, [1], [4], [5])

$$\| \lambda_k w \|_{H^{2m}(\mathbb{R}^n_+)} \leq$$

$$\leq C\left\{\| A(\lambda_k w) \|_{L_2(\mathbb{R}^n_+)} + \sum_{j=0}^{m-1} \| B_j(\lambda_k w) \|_{H^{2m-m_j-1/2}(\mathbb{R}^{n-1})} + \| \lambda_k w \|_{L_2(\mathbb{R}^n_+)}\right\}, \tag{6}$$

where the constant C, due to requirements (I)–(III) on the boundary value problem, will not depend on k. Further, bearing in mind (4), we have

$$A(\lambda_k w) = \lambda_k A w + [A, \lambda_k] w = \lambda_k A w + [A, \lambda_k](\mu_k w), \tag{7}$$

where $[A, \lambda_k]$ is the commutator of the operator A and the operator of multiplication by the function λ_k. Since the order of the differential operator $[A, \lambda_k]$ does not exceed $2m - 1$, it follows from the interpolation inequality, estimates (5) and conditions (I)–(III) on the boundary value problem that for any $\delta > 0$ there exists such a positive number $C(\delta)$ that

$$\| [A, \lambda_k](\mu_k w) \|_{L_2(\mathbb{R}^n_+)} \leq C \| \mu_k w \|_{H^{2m-1}(\mathbb{R}^n_+)} \leq$$

$$\leq C\{\delta \| \mu_k w \|_{H^{2m}(\mathbb{R}^n_+)} + C(\delta) \| \mu_k w \|_{L_2(\mathbb{R}^n_+)}\} \leq$$

$$\leq C\{\delta \| \psi_k u \|_{H^{2m}(U_k \cap \Omega)} + C(\delta) \| \psi_k u \|_{L_2(U_k \cap \Omega)}\} \leq$$

$$\leq C\{\delta \| u \|_{H^{2m}(U_k \cap \Omega)} + C(\delta) \| u \|_{L_2(U_k \cap \Omega)}\}, \tag{8}$$

where the constants C do not depend on k.

The second term from the right side of formula (6) is estimated similarly. Since $B_j u = 0$ on Γ, this term will be estimated through the last expression in inequalities (8).

What has been said together with (6), (7), (8) leads to the estimate

$$\| \varphi_k u \|_{H^{2m}(U_k \cap \Omega)} \leq C\{\| Au \|_{L_2(U_k \cap \Omega)} + \delta \| u \|_{H^{2m}(U_k \cap \Omega)} + C(\delta) \| u \|_{L_2(U_k \cap \Omega)}\},$$

where the constant C does not depend on k.

Since each domain U_k is intersected by at most M other domains U_j (see requirements (II) for the domain Ω), then

$$\| u \|_{H^{2m}(\Omega)} = \| \sum_k \varphi_k u \|_{H^{2m}(\Omega)} \leq \sum_k \| \varphi_k u \|_{H^{2m}(U_k \cap \Omega)} \leq$$

$$\leq C(M+1)\{\| Au \|_{L_2(\Omega)} + \delta \| u \|_{H^{2m}(\Omega)} + C(\delta) \| u \|_{L_2(\Omega)}\}.$$

Choosing δ so that $C(M+1)\delta < 1$, we obtain the required estimate (3). The theorem (a priori estimate) is proved.

Lemma. Let the above requirement (II.2) be satisfied to cover the space $\mathbb{R}^n$ by bounded domains $\{U_k\}$, $k = 1, 2, 3, \ldots$. Then there are functions $\varphi_k \in C_0^\infty(U_k)$ and $\psi_k \in C_0^\infty(U_k)$ satisfying conditions a)–c) formulated in the proof of the theorem (a priori estimate).

Proof. Let, in accordance with (II.2), for each k, the distance from the boundary of the set U_k to the boundary of the set $\bigcup_{j \neq k} U_j$ be greater than some positive number d independent of k.

By U_k^δ, where $\delta > 0$, we denote the δ-narrowing of the domain U_k, that is, the set of points in the domain U_k, the distance from which to the boundary of this domain is greater than δ. From now on, δ will only take values from 0 to d (more specifically: $\frac{d}{5}$, $\frac{2d}{5}$, $\frac{3d}{5}$ and $\frac{4d}{5}$), so $\bigcup_k U_k^\delta = \mathbb{R}^n$.

Through $\chi_{U_k^\delta} \equiv \chi_{U_k^\delta}(x)$ will be denoted the characteristic function of the set U_k^δ, which is equal to 1 (one) for $x \in U_k^\delta$ and 0 (zero) for $x \notin U_k^\delta$.

The main constructive element in creating the desired functions φ_k and ψ_k is the so-called function-cap (in the terminology of [6])

$$\omega_\tau \equiv \omega_\tau(x) = \begin{cases} C_\tau \tau^{-n} e^{-\frac{\tau^2}{\tau^2 - |x|^2}}, & |x| \leq \tau, \\ 0, & |x| > \tau, \end{cases}$$

defined for any number (parameter) $\tau > 0$ and argument $x \in \mathbb{R}^n$. The constant C_τ is chosen so that

$$\int_{\mathbb{R}^n} \omega_\tau(x)\, dx = 1 \quad \Leftrightarrow \quad C_\tau \cdot \int_{|x| \leq 1} e^{-\frac{1}{1-|x|^2}}\, dx = 1.$$

Then, obviously,

$$\omega_\tau \in C_0^\infty(\mathbb{R}^n), \quad \operatorname{supp} \omega_\tau = \{x \in \mathbb{R}^n: \ |x| \leq \tau\}, \quad \omega_\tau(x) > 0 \ \text{ for } \ |x| < \tau.$$

A visual representation of the nature of the functions $y = \omega_\tau(x)$ is given by their graphs shown in Fig. 3.

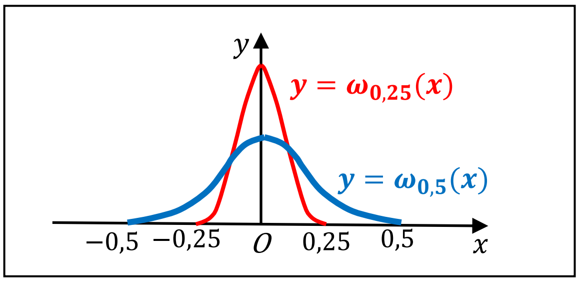


Figure 3. Graphs of functions $y = \omega_\tau(x)$

The construction of the functions φ_k and ψ_k is based on convolutions of the function-cap ω_τ and the characteristic functions $\chi_{U_k^\delta}$ of the sets U_k^δ.

Let for each $k = 1,\ 2,\ 3,\ \dots$

$$\varphi_k^0(x) \equiv \chi_{U_k^{4d/5}} * \omega_{d/5} = \int \chi_{U_k^{4d/5}}(y) \cdot \omega_{d/5}\,(x-y)\ dy.$$

Due to the properties of the function $\omega_\tau(x)$ and the definition of the characteristic function of the set, we have:

$$\varphi_k^0(x) = \int_{U_k^{4d/5}} \omega_{d/5}(x-y)\,dy \in C_0^\infty(\mathbb{R}^n);$$

$$0 \le \varphi_k^0(x) \le \int \omega_{d/5}\,(x-y)\ dy = \int \omega_{d/5}\,(z)\ dz = 1;$$

$$\varphi_k^0(x) = \int_{|x-y|<d/5} \chi_{U_k^{4d/5}}(y) \cdot \omega_{d/5}\,(x-y)\ dy =$$

$$= \begin{cases} \int_{|x-y|<d/5} \omega_{d/5}(x-y)\ dy = \int \omega_{d/5}\,(z)\ dz = 1, & x \in U_k \backslash \bigcup_{j\neq k} U_j\,, \\ 0, & x \notin U_k^{3d/5}, \end{cases}$$

whence, in particular, it follows that $\varphi_k^0(x) \in C_0^\infty(U_k)$.

Let's put now

$$\varphi_k(x) = \frac{\varphi_k^0(x)}{\sum_{j=1}^\infty \varphi_j^0(x)}, \qquad k = 1,\ 2,\ 3,\ \dots$$

Then obviously

$$\varphi_k(x) \in C_0^\infty(U_k), \qquad \sum_{k=1}^\infty \varphi_k(x) = 1 \ \text{ for } \ x \in \mathbb{R}^n, \qquad \operatorname{supp} \varphi_k \subset \overline{U_k^{3d/5}},$$

that is, the set of functions $\{\varphi_k(x)\}$ is a partition of unity into $\mathbb{R}^n$, subject to the cover $\{U_k\}$.

As functions $\psi_k(x)$, $k = 1,\ 2,\ 3,\ \dots$, we take

$$\psi_k(x) \equiv \chi_{U_k^{2d/5}} * \omega_{d/5} = \int \chi_{U_k^{2d/5}}(y) \cdot \omega_{d/5}\,(x-y)\ dy.$$

In the same way as above for the functions $\varphi_k^o(x)$, we obtain that

$$\psi_k(x) \in C_0^\infty(\mathbb{R}^n), \qquad 0 \le \psi_k(x) \le 1 \ \text{ for } \ x \in \mathbb{R}^n,$$

$$\psi_k(x) = \int_{|x-y|<d/5} \chi_{U_k^{2d/5}}(y) \cdot \ \omega_{d/5}\,(x-y)\ dy =$$

$$= \begin{cases} \int_{|x-y|<d/5} \omega_{d/5}(x-y)\, dy = \int \omega_{d/5}(z)\, dz = 1, & x \in U_k^{3d/5}, \\ 0, & x \notin U_k^{d/5}. \end{cases}$$

Thus,

$$\psi_k(x) \in C_0^\infty(U_k) \quad \text{and} \quad \psi_k(x) = 1 \ \text{ for } \ x \in \operatorname{supp} \varphi_k, \quad k = 1,\ 2,\ 3,\ \ldots$$

The uniform boundedness of the derivatives of the functions φ_k and ψ_k up to order $2m$ inclusive for all natural k and $x \in \mathbb{R}^n$ follows from their explicit integral representation. The lemma is proven.

5 CONCLUSION

For a bounded domain and the exterior of a bounded domain – in a word, for (bounded and unbounded) domains with compact boundaries – a priori estimates for solutions of elliptic boundary value problems are known.

For example, for the solution $u = u(x)$ of the elliptic boundary value problem

$$A(x, \partial_x)u = f, \quad x \in \Omega; \qquad B_j(x, \partial_x)u\big|_{\partial\Omega} = \varphi_j, \quad 0 \le j \le m-1,$$

in a bounded domain Ω of the space $\mathbb{R}^n$ the following estimate holds:

$$\|u\|_{H^{s+2m}(\Omega)} \le C\left\{ \|f\|_{H^s(\Omega)} + \sum_{j=0}^{m-1} \left\|\varphi_j\right\|_{H^{s+2m-m_j-1/2}(\partial\Omega)} + \|u\|_{H^0(\Omega)} \right\},$$

where the constant C does not depend on u, $s > \max\limits_{0\le j\le m-1}\{-2m + m_j + 1/2\}$, and also a number of requirements to the boundary value problem are satisfied (for more details, see, for example, [1], [2]).

When proving an a priori estimate for solving an elliptic problem in the exterior of a bounded domain, it is additionally required that the variable coefficients of the elliptic differential operator $A(x, \partial_x)$ stabilize fairly quickly at infinity, and the solution itself is estimated in the norm of the Sobolev–Slobodetsky space $H_\gamma^s(\Omega)$ with weight, which is determined by equality

$$\|u\|_{H_\gamma^s(\Omega)} \equiv \left\|(1 + |x|^2)^{-\gamma/2}\, u\right\|_{H^s(\Omega)},$$

where γ is some real number (in estimates, as a rule, $\gamma > 1$).

In this paper, we consider an elliptic boundary value problem (1) in an *infinite domain with a non-compact boundary*. Among such problems, only problems in the half-space $\mathbb{R}_+^n$ have been well studied (including obtaining a priori estimates for solutions); this is an infinite domain with an infinite but "rectilinear" boundary (the latter is an $(n-1)$-dimensional hyperplane of the form $\{x_n = 0\}$).

In this article, requirements (I)–(III) to the boundary value problem are formulated, under which a priori estimate (3) is proved for the solution, which is similar in form to a priori estimates for solutions to problems in domains with compact boundaries. In this case, the proof of estimate (3) is based on techniques and methods well developed in the theory of elliptic boundary value problems.

Among conditions (I)–(III), requirements (I) and (III) on the operators of the problem are traditional: the operator A is properly elliptic, and the boundary operators $\{B_j\}_{j=0}^{m-1}$ are the normal system and cover the operator A (in this case it is also said that problem (1) satisfies the coercivity conditions or, equivalently, the Shapiro–Lopatinskii conditions are satisfied). These requirements are related to the fact that the boundary operators $\{B_j\}_{j=0}^{m-1}$ cannot be set arbitrarily: there are examples of boundary operators chosen in such a way that the corresponding homogeneous boundary value problem has an infinite-dimensional kernel. In the present paper, it is required that conditions (I) and (III) be satisfied uniformly over the points of the infinite domain Ω and its infinite boundary $\partial\Omega$.

Constraints (II) on the domain Ω determine the regularity of its structure at infinity, which makes it possible to obtain an a priori estimate of the solution on the entire domain, "gluing" it from local estimates.

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