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ON THE AVERAGE ORDER OF THE GCD-SUM FUNCTION OVER THE SET OF SQUARE INTEGERS

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Summary. The gcd-sum function is one of the most important functions that has been studied by many researchers in recent years (Broughan, Bordellès, etc.). The gcd-sum function appears in a specific lattice point problem, where it can be used to estimate the number of integer coordinate points under the square root curve. In this paper, we give an average order of the gcd-sum function over the set of squares.

1 INTRODUCTION

We use the following notation: $\mathbb{N} = \{1,2,\ldots\}$, $\times$ is the usual Dirichlet convolution product, I_d the completely multiplicative function defined by $I_d(n) = n$, $\mathbf{1}$ is the unite function defined by $\mathbf{1}(n) = 1$ for all n, μ denoting the Möbius function, φ is Euler's totient function, ζ denoting the Euler-Riemann zeta function, $\tau(n)$ denoting the number of divisors of n, $\Omega(n)$ denoting the number of prime power divisors of n.

The Pillai's ([1]) arithmetical function is defined by

$$P(n) := \sum_{i=1}^{n} (i,n),$$

where (a,b) denotes the greatest common divisor (gcd) of $a, b \in \mathbb{N}$.

The function P appears in a specific lattice point problem (see, e.g., [2- 3]), where it can be used to estimate the number of integer coordinate points under the square-root curve.

In recent years, we find for example in 2001, Broughan [2, Theorem 4.7] he studied this function and showed that P is multiplicative, and satisfies the convolution identity

$$P = \varphi \times I_d,$$

and in the same paper, he showed that

$$\sum_{n\leq x} P(n) = \frac{x^2 log x}{2\zeta(2)} + \frac{\zeta(2)^2}{2\zeta(3)} x^2 + O\left(x^{3/2} log x\right), \text{ for any real number } x > 1.$$

Also, in 2007 Bordellès [4] showed that

$$P = \mu \times (I_d . \tau)$$

and using this representation the author proved the following asymptotic formula: for every $\varepsilon > 0$,

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$$\sum_{n\le x} P(n) = \frac{x^2}{2\zeta(2)}\left(logx + 2\gamma - \frac{1}{2} + \frac{\zeta'(2)}{\zeta(2)}\right) + O\left(x^{1+\theta+\varepsilon}\right), \quad x > 1,$$

where γ is Euler's constant and θ is the number appearing in Dirichlet's divisor problem, that is

$$\sum_{n\le x} \tau(n) = xlogx + (2\gamma - 1)x + O\left(x^{1+\theta}\right), \ \frac{1}{4} \le \theta \le \frac{131}{416}, \ x > 1. \quad (1)$$

Note also, that for any arithmetical function f,

$$\begin{aligned} P_f(n) &= \sum_{i=1}^{n} f\big((i,n)\big) \\ &= \sum_{d|n} f(d) \sum_{k\le\frac{n}{d},\ \left(k,\frac{n}{d}\right)=1} 1 \\ &= \sum_{d|n} f(d)\varphi\left(\frac{n}{d}\right) \\ &= (f \times \varphi)(n), \end{aligned} \quad (2)$$

which is a result of E. Cesàro in 1885 (see, e.g., [5], [6, p.127]). Bordellès [7], introduced a more general situation of this function and he was able to give unified asymptotic formulae for $\sum_{n\le x} P_f(n)$ in cases of real-valued multiplicative functions that lie in certain special classes (four classes) and then he gave some applications.

Let S be an arbitrary nonempty subset of $\mathbb{N}$, then we define

$$G_S(n) = \sum_{\substack{1\le i\le n \\ (i,n)\in S}} (i,n), \quad n \in \mathbb{N}. \quad (3)$$

Note that if $S = \mathbb{N}$, we have $G_S(n) = P(n)$.

The aim of this paper is to estimate $\sum_{n\le x} G_S(n)$, if S is the set of squares. In this case, we will simply denote $G_S(n)$ by $G(n)$, when S is a set of squares. Indeed, we prove the following result:

Theorem 1.1 If S is the set of squares and for any real numbers $x > 1$, we have

$$\sum_{n\le x} G(n) = \left(\frac{1}{2} logx + \gamma - \frac{1}{4}\right) x^2 + O\left(x^{3/2} logx\right),$$

where γ is Euler's constant.

2 MAIN RESULTS

This section is devoted to the presentation of some lemmas used in the proof of the previous theorem.

Define the function μ_S named the Möbius function of S (see, e.g., [8]) by

$$\mu_S \times \mathbf{1} = \varrho_S, \tag{4}$$

where ϱ_S is the characteristic function of S. That is, $\varrho_S(n) = 1$ or 0 according as n is or is not an element of S and

$$\sum_{d|n} \mu_S(d) = \varrho_S(n).$$

Moreover, let $\varphi_S(n)$ denote the number of integers $r(mod\, n)$ such that $(r, n) \in S$. It can be shown (see, e.g., [8, Corollary 4.1]) that

$$\varphi_S(n) = \mu_S \times I_d \tag{5}$$

r equivalently

$$\varphi_S(n) = \sum_{d|n} d\mu_S\left(\frac{n}{d}\right).$$

L.Tòth [9, p.29], he noted that if S is the set of squares, then $\mu_S = \lambda$ is the Liouville function defined by $\lambda(n) = (-1)^{\Omega(n)}$ and $\varphi_S = \beta$ given by

$$\beta(n) = \sum_{d|n} d\lambda\left(\frac{n}{d}\right).$$

Lemma 2.1 For any nonempty subset S of $\mathbb{N}$, we have

$$G = (I_d . \mu_S) \times \big((I_d . \tau) \times \mu\big).$$

Proof. Since the function G represents a particular sum over a set S, then

$$G(n) = \sum_{i=1}^{n} \varrho_S\big((i,n)\big) . I_d\big((i,n)\big) = \sum_{i=1}^{n} (\varrho_S . I_d)\big((i,n)\big).$$

From result (2) of Cesàro, we can see that

$$G(n) = (\varrho_S . I_d \times \varphi)(n),$$

then, from (4) it comes that

$$G(n) = (I_d . (\mu_S \times \mathbf{1}) \times \varphi)(n). \tag{6}$$

Notice now from the result (5) that when $S = \mathbb{N}$, we obtain the known result

$$\varphi(n) = (\mu \times I_d)(n).$$

We apply this last identity in (6) and the fact that $\mathbf{1} \times \mathbf{1} = \tau$, we get

$$\begin{aligned} G &= I_d . (\mu_S \times \mathbf{1}) \times (\mu \times I_d) \\ &= (I_d . \mu_S) \times (I_d . \mathbf{1}) \times (I_d . \mathbf{1} \times \mu) \\ &= (I_d . \mu_S) \times \big(I_d . (\mathbf{1} \times \mathbf{1})\big) \times \mu \\ &= (I_d . \mu_S) \times \big((I_d . \tau) \times \mu\big). \end{aligned}$$

Hence the proof of the lemma 2.1.

Lemma 2.2 For any real numbers $x \geq 1$ and $\varepsilon > 0$, we have

$$\sum_{n\leq x}\big((I_d.\tau)\times \mu\big)(n) = \frac{1}{2\zeta(2)}\left(logx + 2\gamma - \frac{\zeta'(2)}{\zeta(2)} - \frac{1}{2}\right)x^2 + O\big(x^{1+\theta+\varepsilon}\big), \qquad (7)$$

where θ is the number appearing in Dirichlet's divisor problem.

Proof. We put $f(n) = \big((I_d.\tau)\times \mu\big)(n)$. Then, we get

$$\sum_{n\leq x} f(n) = \sum_{d\leq x}\mu(d)\sum_{k\leq \frac{x}{d}} k\tau(k).$$

We start by estimating the inner sum. Then, using formula (1) and Abel's summation, it is easy to prove

$$\sum_{k\leq \frac{x}{d}} k\tau(k) = \frac{x^2}{2d^2}log\left(\frac{x}{d}\right) + \frac{x^2}{d^2}\left(\gamma - \frac{1}{4}\right) + O\left(\left(\frac{x}{d}\right)^{1+\theta+\varepsilon}\right).$$

So, he comes

$$\sum_{n\leq x} f(n) = \sum_{d\leq x}\mu(d)\left(\frac{x^2}{2d^2}log\left(\frac{x}{d}\right) + \frac{x^2}{d^2}\left(\gamma - \frac{1}{4}\right) + O\left(\left(\frac{x}{d}\right)^{1+\theta+\varepsilon}\right)\right)$$

$$= x^2\left(\frac{logx}{2} + \gamma - \frac{1}{4}\right)\sum_{d\leq x}\frac{\mu(d)}{d^2} - x^2\sum_{d\leq x}\frac{\mu(d)logd}{2d^2} + O\left(x^{1+\theta+\varepsilon}\left|\sum_{d\leq x}\frac{\mu(d)}{d^{1+\theta+\varepsilon}}\right|\right).$$

Recall that if $s \in \mathbb{C}$ such that $Re(s) > 1$, we have

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

which its differentiation gives

$$\sum_{n=1}^{\infty}\frac{\mu(n)logn}{n^s} = \frac{\zeta'(s)}{\zeta^2(s)}.$$

As we also have for all $x > 1$ and all $k > 1$

$$\sum_{d>x}\frac{|\mu(d)|}{d^k} \leq x^{1-k}\delta_{c_0}(x),$$

where $c_0 \in (0,1)$ and $\delta_{c_0}(x) := e^{-c_0(logx)^{3/5}(loglogx)^{-1/5}}$ (see,e.g., [10, P.236]), which implies that

$$\sum_{d\leq x}\frac{\mu(d)}{d^{1+\theta+\varepsilon}} = O(1), \quad \sum_{d\leq x}\frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} + O\left(\frac{1}{x}\right)$$

and

$$\sum_{d\le x}\frac{\mu(d)logd}{d^2}=\frac{\zeta'(2)}{2\zeta^2(2)}+O\left(\frac{logx}{x}\right).$$

Therefore,

$$\sum_{n\le x}f(n)=x^2\left(\frac{logx}{2}+\gamma-\frac{1}{4}\right)\left(\frac{1}{\zeta(2)}+O\left(\frac{1}{x}\right)\right)-x^2\left(\frac{\zeta'(2)}{2\zeta^2(2)}+O\left(\frac{logx}{x}\right)\right)+O\left(x^{1+\theta+\varepsilon}\right)$$
$$=\frac{1}{2\zeta(2)}\left(logx+2\gamma-\frac{\zeta'(2)}{\zeta(2)}-\frac{1}{2}\right)x^2+O\left(x^{1+\theta+\varepsilon}\right).$$

Lemma 2.3 Let x a real number where $x\ge 1$, $\theta\in\left[\frac{1}{4},\frac{131}{416}\right]$ and S is the set of squares, we have

$$\sum_{n\le x}\frac{\mu_S(n)}{n}=\zeta(2)+O\left(\frac{1}{x^{1/2}}\right), \tag{8}$$

$$\sum_{n\le x}\frac{\mu_S(n)logn}{n}=-2\zeta'(2)+O\left(\frac{logx}{x^{1/2}}\right), \tag{9}$$

$$\sum_{n\le x}\frac{\mu_S(n)}{n^{\theta+\varepsilon}}=O\left(x^{\frac{1}{2}-\theta-\varepsilon}\right)\quad\text{or}\sum_{n\le x}\frac{\mu_S(n)}{n^{\theta+\varepsilon}}=O(1),\quad\text{for }\varepsilon>0 \tag{10}$$

Proof. Note that for any integer $n\in\mathbb{N}$, we have $\mu_S(n)$ is non-zero if $n=m^2$ where m are positive integer. This last one leads us to the result of L.Tòth mentioned before i.e. $\mu_S=\lambda$. So,

$$\sum_{n\le x}\frac{\mu_S(n)}{n}=\sum_{m=1}^{\infty}\frac{\lambda(m^2)}{m^2}-\sum_{m>x^{1/2}}\frac{\lambda(m^2)}{m^2}=\zeta(2)+O\left(\frac{1}{x^{1/2}}\right).$$

Also,

$$\sum_{n\le x}\frac{\mu_S(n)logn}{n}=\sum_{m=1}^{\infty}\frac{\lambda(m^2)logm^2}{m^2}-\sum_{m>x^{\frac{1}{2}}}\frac{\lambda(m^2)logm^2}{m^2}$$
$$=2\sum_{m=1}^{\infty}\frac{logm}{m^2}-2\sum_{m>x^{\frac{1}{2}}}\frac{logm}{m^2}$$
$$=-2\zeta'(2)+O\left(\frac{logx}{x^{\frac{1}{2}}}\right).$$

Now to prove the estimate (10), we apply the following known formula

$$\sum_{n\le x}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}+\zeta(s)+O(x^{-s})\ \text{ if } s>0,\ \ s\ne 1\ \ (\text{see, e.g., [11, Theoreme 3.2, p. 55]}).$$

Then, for $\varepsilon > 0$ such that $\theta + \varepsilon < \frac{1}{2}$, we get

$$\sum_{n \le x} \frac{\mu_S(n)}{n^{\theta+\varepsilon}} = \sum_{m \le x^{\frac{1}{2}}} \frac{\lambda(m^2)}{m^{2(\theta+\varepsilon)}} = \sum_{m \le x^{\frac{1}{2}}} \frac{1}{m^{2(\theta+\varepsilon)}} = O\left(x^{\frac{1}{2}-\theta-\varepsilon}\right).$$

In the case where $\theta + \varepsilon > \frac{1}{2}$, we find

$$\sum_{n \le x} \frac{\mu_S(n)}{n^{\theta+\varepsilon}} = \sum_{m \le x^{\frac{1}{2}}} \frac{\lambda(m^2)}{m^{2(\theta+\varepsilon)}} = \sum_{m \le x^{\frac{1}{2}}} \frac{1}{m^{2(\theta+\varepsilon)}} = O(1).$$

3 PROOF OF THEOREM 1.1

To begin the proof of the Theorem 1.1, we put

$$f(n) = \big((I_d.\tau) \times \mu\big)(n)$$

and

$$g(n) = (I_d.\mu_S)(n).$$

Therefore, we have

$$\sum_{n \le x} G(n) = \sum_{n \le x} (g \times f)(n) = \sum_{n \le x} g(n) \sum_{k \le \frac{x}{n}} f(k).$$

Using the result (7) in this last formula, we get

$$\begin{aligned}\sum_{n \le x} G(n) &= \sum_{n \le x} n\mu_S(n)\left(\frac{1}{2\zeta(2)}\left(log\left(\frac{x}{n}\right) + 2\gamma - \frac{\zeta'(2)}{\zeta(2)} - \frac{1}{2}\right)\left(\frac{x}{n}\right)^2 + O\left(\left(\frac{x}{n}\right)^{1+\theta+\varepsilon}\right)\right) \\ &= \frac{x^2}{\zeta(2)}\left(\frac{1}{2}logx + \gamma - \frac{1}{4} - \frac{\zeta'(2)}{\zeta(2)}\right)\sum_{n \le x} \frac{\mu_S(n)}{n} - \frac{x^2}{2\zeta(2)}\sum_{n \le x} \frac{\mu_S(n)logn}{n} \\ &\quad + O\left(x^{1+\theta+\varepsilon}\left|\sum_{n \le x} \frac{\mu_S(n)}{n^{\theta+\varepsilon}}\right|\right). \qquad (11)\end{aligned}$$

Now by inserting (8) , (9) and (10) into (11), we obtain

$$\sum_{n \le x} G(n) = \left(\frac{1}{2}logx + \gamma - \frac{1}{4}\right)x^2 + O\big(x^{3/2}logx\big).$$

This completes the proof.

4 CONCLUSIONS

Let S denote the set of all square integers. In this paper, we obtain asymptotic formula on the sum $\sum_{n \le x} \sum_{i=1}^{\infty} gcd(i,n)$ if $gcd(i,n) \in S$ where n and x respectively represent a positive integer and a real number strictly greater than 1. On the other hand, the goal of the next paper is to generalize this property when the gcd is applied for several integers.

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