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GENERALIZATION OF LOHWATER-POMMERENKE'S THEOREM

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Summary. In this paper, as an application of Zalcman's lemma in $\mathbb{C}^n$, we give a sufficient condition for normality of holomorphic functions of several complex variables, which generalizes previous known one-dimensional criterion of A.J. Lohwater and Ch. Pommerenke.

1 INTRODUCTION

A "heuristic principle" attributed to Andrè Bloch says that a family of holomorphic functions which have a property P in common in a domain $\Omega \subset \mathbb{C}$ is [apt to be] a normal family in Ω if P cannot be possessed by non-constant holomorphic functions in the whole plane $\mathbb{C}$. [An example of such a P is "uniform boundedness."] A rigorous formulation and proof of this was given in 1975 by Zalcman [5]. Zalcman's work was inspired by the result of Lohwater and Pommerenke [4]. Their theorem deals with normal functions, not normal families, but the proofs are almost identical.

It is the purpose of this note to give a generalization of the result of Lohwater and Pommerenke [4] for normal functions defined on bounded domains of $\mathbb{C}^n$.

It is known that the notion of normality can be generalized in various ways to higher dimensions. Here, we adopt the definition of Cima and Krantz [1, p. 305].

2 PRELIMINARY RESULTS

Let Ω be a bounded domain in $\mathbb{C}^n$. By $B(a,r)$ we denote the ball in $\mathbb{C}^n$ with center a and radius r. Thus $B(a,r)$ consist of all $z \in \mathbb{C}^n$ such that $|z-a|<r$.

For every function φ of class we define at each point $z \in \Omega$ a Hermitian form

$$L_z(\varphi, v) := \sum_{k,l=1}^{n} \frac{\partial^2 \varphi}{\partial z_k \partial \overline{z}_l}(z) v_k \overline{v}_l$$

and call it the Levi form of the function φ at z.

For a holomorphic function f in Ω, set

$$f^{\#}(z) := \sup_{|\nu|=1} \sqrt{L_z(\log(1+|f|^2), \nu)} \tag{0.1}$$

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This quantity $f^{\#}(z)$ is well defined since the Levi form $L_z(\log(1+|f|^2),v)$ is nonnegative for all $z\in\Omega$.

Let U be a unit disk in $\mathbb{C}$. The infinitesimal Kobayashi metric on Ω is given by $K_\Omega(z,v):=\inf\{\alpha:\alpha>0 \quad \textit{and} \quad \exists g:\Omega\to U \quad \textit{holomorphic}, \quad g(0)=a \quad \textit{and} \quad g'(0)=v/\alpha\}$.

Definition 1 A holomorphic function $f:\Omega\to\mathbb{C}$ is called normal if exists a constant C, $0<C<\infty$, such that

$$L_z(\log(1+|f|^2),v)\le C\cdot K_\Omega^2(z,v) \tag{0.2}$$

for all $(z,v)\in\Omega\times\mathbb{C}^n$.

A family $\mathcal{F}$ of holomorphic functions on a domain $\Omega\subset\mathbb{C}^n$ is normal in Ω if every sequence of functions $\{f_j\}\subseteq\mathcal{F}$ contains either a subsequence which converges to a limit function $f\ne\infty$ uniformly on each compact subset of Ω, or a subsequence which converges uniformly to ∞ on each compact subset.

Theorem 1 (Marty's Criterion, see [2]) *A family $\mathcal{F}$ of functions holomorphic on Ω is normal on $\Omega\subset\mathbb{C}^n$ if and only if for each compact subset $K\subset\Omega$ there exists a constant $M(K)$ such that at each point $z\in K$*

$$f^{\#}(z)\le M(K)$$

for all $f\in\mathcal{F}$.

Theorem 2 (Zalcman's Lemma, see [2]) *Suppose that a family $\mathcal{F}$ of functions holomorphic on $\Omega\subset\mathbb{C}^n$ is not normal at some point $z_0\in\Omega$ if and only if there exist sequences $f_j\in\mathcal{F}$, $z_j\to z_0$, $\rho_j = 1/f^{\#}_j(z_j)\to 0$ $\rho_j=1/f_j^{\sharp}(z_j)\to 0$, such that the sequence*

$$g_j(z)=f_j(z_j+\rho_j z)$$

converges locally uniformly in $\mathbb{C}^n$ to a non-constant entire function g satisfying $g^{\#}(z)\le g^{\#}(0)=1$.

In one-dimensional case there are many criteria known for a meromorphic function to be normal, and the Lohwater and Pommerenke add a very valuable criterion to this list: *a nonconstant function f meromorphic in unit disc* U $\subset\mathbb{C}$ *is normal if and only if there do not exist sequences $\{z_n\}$ and $\{\rho_n\}$ with $z_n\in U$, $\rho_n>0$, $\rho_n\to 0$, such that}*

$$\lim_{n\to\infty} f(z_n+\rho_n t)=g(t)$$

locally uniformly in $\mathbb{C}$, where $g(t)$ is a nonconstant meromorphic function in $\mathbb{C}$.

Lohwater and Pommerenke [4, Theorem 1] originally stated their theorem with no restriction on the speed at which $\rho_n\to 0$. In proving their theorem they asserted, "if f is normal and $f(z_n+\rho_n\zeta)\to g(\zeta)$ locally uniformly, then $\rho_n/(1-|z_n|)\to 0$ ". The statement in quotes is false as one can see from $f(z)=z$, $z_n=1-n^{-3}$, $\rho_n=n^{-2}$, $g(\zeta)\equiv 1$.

3 GENERALIZATION OF LOHWATER-POMMERENKE'S THEOREM

Theorem 3 *A non-constant function* f *holomorphic on* $\Omega\subset\mathbb{C}^n$ *is non-normal if there exist sequences* $z_j\in\Omega$, $\rho_j=1/f^\sharp(z_j)\to 0$, *such that the sequence*

$$g_j(\zeta)=f(z_j+\rho_j\zeta)$$

converges locally uniformly in $\mathbb{C}^n$ *to a non-constant entire function* g *satisfying* $g^\sharp(\zeta)\le g^\sharp(0)=1$.

Proof. Let $\{p_j\}$ be an arbitrary sequence of points in Ω, then $B(p_j,\delta_j)\subset\Omega$, where $\delta_j=dist(p_j,\partial\Omega)$. By the distance-decreasing property of Kobayashi metric

$$K_\Omega(z,v)\le K_{B(p_j,\delta_j)}(z,v)$$

for all $(z,v)\in B(p_j,\delta_j)\times\mathbb{C}^n$.

The Kobayashi metric of $B(p_j,\delta_j)$ is given by

$$K_{B(p_j,\delta_j)}(z,v)=\frac{[(\delta_j^2-|z-p_j|^2)|v|^2+|(z-p_j,v)|^2]^{1/2}}{\delta_j^2-|z-p_j|^2}$$

which clearly satisfy the inequality:

$$K_{B(p_j,\delta_j)}(z,v)\le\frac{\delta_j|v|}{\delta_j^2-|z-p_j|^2}.$$

If f is normal in Ω and then from (0.2) follows

$$f^\sharp(p_j+\delta_j\zeta)\le\frac{\sqrt{C}\delta_j}{\delta_j^2-|\delta_j\zeta|^2}. \tag{0.3}$$

Set $g_j(\zeta):=f(p_j+\delta_j\zeta)$. By the invariance of the Levi form under biholomorphic mappings, we have

$$L_\zeta(\log(1+|g_j|^2),v)=L_{p_j+\delta_j\zeta}(\log(1+|f_j|^2),\delta_j v)$$

and so

$$g_j^\sharp(\zeta)=\delta_j f^\sharp(p_j+\delta_j\zeta).$$

It follows from (0.3) that

$$g_j^\sharp(\zeta)\le\frac{\sqrt{C}}{1-|\zeta|^2}$$

for all j and all $\zeta \in B(0,1)$. By Marty's Criterion ([2, Theorem 2.1) the family $\{g_j(\zeta)\}$ is normal in the unit ball $B(0,1)$.

So if f is not normal function in Ω, then there exists a sequence $\{p_j\}$ in Ω such that $\{g_j(\zeta) := f(p_j + \delta_j \zeta)\}$ is not a normal sequence in a point, say, ζ_0, $\zeta_0 \in B(0,1)$. It follows from Zalcman's lemma [2, Theorem 3.1] that there exist $\zeta_j \to \zeta_0$, $\rho_j = 1/g_j^\sharp(\zeta_j) \to 0$, such that the sequence

$$g_j(\zeta) = f_j(p_j + \delta_j(\zeta_j + \rho_j \zeta))$$

converges locally uniformly in $\mathbb{C}^n$ to a non-constant entire function g satisfying $g^\#(\zeta) \le g^\#(0) = 1$.

A simple calculation shows that $\delta_j \rho_j = 1/f^\sharp(p_j + \delta_j \zeta_j)$ and therefore

$$g_j(\zeta) = f_j(p_j + \delta_j \zeta_j + \zeta / f^\sharp(p_j + \delta_j \zeta_j))$$

converges locally uniformly in $\mathbb{C}^n$ to a non-constant entire function g satisfying $g^\sharp(\zeta) \le g^\sharp(0) = 1$. It follows $z_j = p_j + \delta_j \zeta_j$, $\rho_j = 1/f^\sharp(p_j + \delta_j \zeta_j)$ do the work. This completes the proof of Theorem 3. ■

The next result is closely related to the preceding theorem and is essentially a reformulation of (0.2).

Theorem 4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. If $f:\Omega \to \mathbb{C}$ is a normal holomorphic function, then for every choice of sequences $\{p_j\}$ in Ω and $\{r_j\}$, $r_j > 0$, with $\lim_{j\to\infty} r_j/\delta_j = 0$, where $\delta_j = dist(p_j, \Omega)$, the sequence $\{f(p_j + r_j \zeta)\}$ converges locally uniformly to a constant function in $\mathbb{C}^n$.*

Proof. Set $R_j = \delta_j / r_j$. It is clear that $R_j \to \infty$. Without restriction we can assume that $R_j > j$. Then for all $\zeta \in \mathbb{C}^n$ such that $|\zeta| < j$, we have

$$|p_j + r_j \zeta - p_j| = r_j |\zeta| < \delta_j$$

so that $p_j + r_j \zeta \in B(p_j, \delta_j) \subset \Omega$.

Hence $g_j(\zeta) := f(p_j + r_j \zeta)$ is a holomorphic function on the ball $B(0,j) = \{\zeta \in \mathbb{C}^n : |\zeta| < j\}$.

It is an immediate consequence of the definition that since f is the normal function, then there exists a positive constant C such that

$$L_z(\log(1+|f|^2), v) \le C \cdot K_\Omega^2(z,v)$$

for all $(z,v) \in \Omega \times \mathbb{C}^n$.

Since $B(p_j, \delta_j)$ is contained in Ω the distance-decreasing property yields

$$K_\Omega(z,v) \le K_{B(p_j,\delta_j)}(z,v)$$

for all $(z,v) \in B(p_j,\delta_j) \times \mathbb{C}^n$.

Since

$$K_{B(p_j,\delta_j)}(z,v) = \frac{[(\delta_j^2 - |z-p_j|^2)|v|^2 + |(z-p_j,v)|^2]^{1/2}}{\delta_j^2 - |z-p_j|^2}$$

we have

$$K_{B(p_j,\delta_j)}(z,v) \le \frac{\delta_j |v|}{\delta_j^2 - |z-p_j|^2}.$$

Hence

$$K_\Omega(z,v) \le \frac{\delta_j |v|}{\delta_j^2 - |z-p_j|^2}$$

for all $(z,v) \in B(p_j,\delta_j) \times \mathbb{C}^n$.

Therefore,

$$\sqrt{L_{p_j+r_j\zeta}(\log(1+|f_j|^2),v)} \le \frac{\sqrt{C}\delta_j |v|}{\delta_j^2 - |r_j\zeta|^2}$$

for all $(\zeta,v) \in B(0,j) \times \mathbb{C}^n$.
Taking sup on both sides over $|v|=1$, we have

$$f^\sharp(p_j + r_j\zeta) \le \frac{\sqrt{C}\delta_j}{\delta_j^2 - |r_j\zeta|^2} \tag{0.4}$$

By the invariance of the Levi form under biholomorphic mappings, we have

$$L_\zeta(\log(1+|g_j|^2),v) = L_{p_j+r_j\zeta}(\log(1+|f_j|^2),r_jv)$$

and hence

$$g_j^\sharp(\zeta) = r_j f^\sharp(p_j + r_j\zeta). \tag{0.5}$$

We note from (0.4), (0.5), and $\delta_j / r_j > j$, that

$$g_j^\sharp(\zeta) \le \frac{\sqrt{C} r_j \delta_j}{\delta_j^2 - |r_j\zeta|^2} \le \frac{\sqrt{C}/j}{1-(1/j)^2|\zeta|^2}$$

for all j sufficiently large and all ζ, $|\zeta|< j$.

For every $m \in N$ the sequence $\{g_j\}_{j>m}$ is normal in $|\zeta|< m$ by Marty's Theorem [2, Theorem 2.1]. The well-known Cantor diagonal process yields a subsequence $\{g_k = g_{j_k}\}$

which converges uniformly on every ball $|\zeta|< R$. The limit function g is holomorphic and satisfies $g^{\sharp}(\zeta)=0$ which yields: $dg(\zeta)=0$ for all $\zeta\in\mathbb{C}^n$, i.e. $g(\zeta)\equiv$ constant in $\mathbb{C}^n$.■

Theorem 4 can be restated in the following way.

Corollary. *Let $\Omega\subset\mathbb{C}^n$ be a bounded domain. If $f:\Omega\to\mathbb{C}$ is a holomorphic function, and there exist sequences $\{p_j\}$ in Ω and $\{r_j\}$, $r_j>0$, with $\lim_{j\to\infty} r_j/\delta_j=0$, where $\delta_j=\delta_\Omega(p_j)$, such that $\{f(p_j+r_j\zeta)\}$ converges locally uniformly to a non-constant holomorphic function in $\mathbb{C}^n$, then f is non-normal.*

Remark. In [3], Theorem 4 was proven for the case of the unit ball in $\mathbb{C}^n$.

4 CONCLUSUONS

In this paper, as an application of Marty's Criterion and Zalcman's Lemma in $\mathbb{C}^n$, we obtain a sufficient condition for normality of holomorphic functions of several complex variables, which generalizes previous known one-dimensional theorem of A.J. Lohwater and Ch. Pommerenke [4, Theorem 1].

An honest generalization of Theorem 1 of [4] to more than one complex variable does not hold as has been shown by an example [3].

REFERENCES

[1] J. A. Cima and S. G. Krantz, “The Lindelöf principle and normal functions of several complex variables”, *Duke Math. J,* **50**, 303-328 (1983).

[2] P. V. Dovbush, “Zalcman’s lemma in $\mathbb{C}^n$”, *Complex Variables and Elliptic Equations,* **65**(5), 796-800 (2020).

[3] Kyong T, Hahn, “Higher-dimensional generalizations of some classical theorems on normal meromorphic functions”. *Complex Variables Theory Appl.*, **6** (2-4), 109–121 (1986).

[4] A. J. Lohwater and C. Pommerenke, “On normal meromorphic functions”, *Ann. Acad. Sci. Fenn. Ser. A,* **I** (550), 1-12 (1973).

[5] L. Zalcman, “A heuristic principle in complex function theory”. *Am. Math. Mon*., **82**, 813-818 (1975).

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