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THE GENERALIZED BIVARIATE FIBONACCI AND LUCAS MATRIX POLYNOMIALS

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Summary. The main object of the present paper is to consider the matrix polynomials for the generalized bivariate Fibonacci and Lucas polynomials. Working with matrix properties for these new matrix polynomials, some identities of the generalized bivariate Fibonacci and Lucas polynomials will be researched. Finally, we build the relationships between the generalized bivariate Fibonacci and Lucas matrix polynomials

1 INTRODUCTION

As with any very well-studied subject in mathematics, the Fibonacci, Lucas, Pell, Chebyshev numbers possess many kinds of generalizations. One of the most important generalizations is the Fibonacci polynomial [1-3, 8, 12, 15, 16, 18-21, 26, 29, 31]. Because of the common usage of this polynomial in the applied sciences, its some generalizations have been defined in the literature. In [14] and its references, concerned readers may find a short history and comprehensive information about the Fibonacci polynomial. The Fibonacci numbers are defined as

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1 \tag{1.1}$$

for $n \geq 2$. In [13], the authors gave a new generalization of the Fibonacci and Lucas polynomials which are called generalized bivariate Fibonacci and Lucas polynomials. For $n \geq 2$ and $p(x,y)$, $q(x,y)$ polynomials with real coefficients, the generalized bivariate Fibonacci and Lucas polynomials are described by

$$H_n(x,y) = p(x,y)H_{n-1}(x,y) + q(x,y)H_{n-2}(x,y) \tag{1.2}$$

and

$$K_n(x,y) = p(x,y)K_{n-1}(x,y) + q(x,y)K_{n-2}(x,y), \tag{1.3}$$

where $H_0(x,y) = 0, H_1(x,y) = 1, K_0(x,y) = 2, K_1(x,y) = p(x,y)$ and $p^2(x,y) + 4q(x,y) > 0$. The relation of generalized bivariate Fibonacci and Lucas polynomials is (see[13])

$$K_n(x,y) = H_{n+1}(x,y) + q(x,y)H_{n-1}(x,y). \tag{1.4}$$

For the different $p(x,y)$ and $q(x,y)$, we obtain different polynomial sequences by using recursive relation. These polynomial sequences are given in Table 1 below:

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$p(x,y)$	$q(x,y)$	$H_n(x,y)$	$K_n(x,y)$
x	y	Bivariate Fibonacci, $F_n(x,y)$	Bivariate Lucas, $L_n(x,y)$
x	1	Fibonacci, $F_n(x)$	Lucas, $L_n(x)$
$2x$	1	Pell, $P_n(x)$	Pell-Lucas, $Q_n(x)$
1	$2x$	Jacobsthal, $J_n(x)$	Jacobsthal-Lucas, $j_n(x)$
$2x$	-1	Chebyshev of the second kind, $U_{n-1}(x)$	Chebyshev of the first kind, $2T_n(x)$
$3x$	-2	Fermat, $\mathrm{F}_n(x)$	Fermat-Lucas, $\mathrm{F}_n(x)$

Table 1: Special conditions of the generalized bivariate Fibonacci and Lucas polynomials (see [13])

But then, the matrix sequences have received attention from many authors ([4-7, 9-11, 17, 22-25, 27, 28, 30, 32, 33]). In [6],[7], the authors established the bi-periodic Fibonacci and Lucas matrix sequences and acquired nth general term of these matrix sequences.

Therefore, the key purpose of this study is to examine the relations between the generalized bivariate Fibonacci and Lucas matrix polynomials. Firstly, we determine the generalized bivariate Fibonacci and Lucas matrix polynomials. Then, we obtain the generating functions, Binet formulas and summation formulas for these matrix polynomials. By considering the results in Section 2, we have a big chance to get new properties in Section 3.

2 THE MATRIX POLYNOMIALS OF GENERALIZED BIVARIATE FIBONACCI AND LUCAS POLYNOMIALS

In this part of the study, the relations and properties for the matrix polynomials of the generalized bivariate Fibonacci and Lucas polynomials are investigated.

Then, we will first give the definition of the generalized bivariate Fibonacci and Lucas matrix polynomials.

Definition 2.1 For $n \in \mathrm{N}$, the generalized bivariate Fibonacci $\left(\mathrm{H}_n\right)$ and Lucas matrix polynomials $\left(\mathrm{K}_n\right)$ are satisfy the following recurrence relations

$$\mathrm{H}_{n+2}(x,y) = p(x,y)\mathrm{H}_{n+1}(x,y) + q(x,y)\mathrm{H}_n(x,y), \tag{2.1}$$

and

$$\mathrm{K}_{n+2}(x,y) = p(x,y)\mathrm{K}_{n+1}(x,y) + q(x,y)\mathrm{K}_n(x,y), \tag{2.2}$$

respectively, with initial conditions

$$\mathrm{H}_0(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathrm{H}_1(x,y) = \begin{pmatrix} p(x,y) & 1 \\ q(x,y) & 0 \end{pmatrix},$$

and

$$\mathrm{K}_0(x,y)=\begin{pmatrix} p(x,y) & 2 \\ 2q(x,y) & -p(x,y) \end{pmatrix}, \mathrm{K}_1(x,y)=\begin{pmatrix} p^2(x,y)+2q(x,y) & p(x,y) \\ p(x,y)q(x,y) & 2q(x,y) \end{pmatrix}.$$

We would like mention hereafter that, we write $H_n(x,y)=H_n$, $K_n(x,y)=K_n$ and $p=p(x,y), q=q(x,y)$ for brevity. We obtain the nth general terms of the matrix polynomials in (2.1) and (2.2) by the generalized Fibonacci and Lucas polynomials as in the following.

Theorem 2.2 We have

$$\mathrm{H}_n(x,y)=\begin{pmatrix} H_{n+1} & H_n \\ qH_n & qH_{n-1} \end{pmatrix} \tag{2.3}$$

and

$$\mathrm{K}_n(x,y)=\begin{pmatrix} K_{n+1} & K_n \\ qK_n & qK_{n-1} \end{pmatrix}, \tag{2.4}$$

where $n \geq 0$.

Proof. The proof will be done by induction. For $n=-1$, we obtain the equality $H_{-1}=1/q$ by using equation (1.2) which gives the first step of the finite induction.

$$\mathrm{H}_0=\begin{pmatrix} H_1 & H_0 \\ qH_0 & qH_{-1} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let assume that the equation in (2.3) satisfies all the conditions $n=k \in \mathrm{Z}^+$. Then, by considering (1.2) and (2.1), we need to present that the part also holds for $n=k+1$. So we obtain

$$\begin{aligned} \mathrm{H}_{k+1} &= p\mathrm{H}_k+q\mathrm{H}_{k-1} \\ &= p\begin{pmatrix} H_{k+1} & H_k \\ qH_k & qH_{k-1} \end{pmatrix}+q\begin{pmatrix} H_k & H_{k-1} \\ qH_{k-1} & qH_{k-2} \end{pmatrix} \\ &=\begin{pmatrix} H_{k+2} & H_{k+1} \\ qH_{k+1} & qH_k \end{pmatrix}. \end{aligned}$$

For another part of the proof, we require the following nearly like approximation by using (1.3). Similarly, as in the above part, the end step of the induction can be acquired by $\mathrm{K}_{k+1}=p\mathrm{K}_k+q\mathrm{K}_{k-1}$ as in the following

$$\mathrm{K}_{k+1}=\begin{pmatrix} K_{k+2} & K_{k+1} \\ qK_{k+1} & qK_k \end{pmatrix}.$$

This completes the proof.

The generating functions for the generalized bivariate Fibonacci and Lucas matrix polynomials play a vital role to find out many important identities for these matrix polynomials. In the following theorem, we construct the generating functions for these matrix polynomials.

Theorem 2.3 For the generalized bivariate Fibonacci and Lucas matrix polynomials, we have the generating functions

$$\sum_{i=0}^{\infty} \mathrm{H}_i t^i = \frac{1}{1-pt-qt^2}\begin{pmatrix} 1 & t \\ qt & 1-pt \end{pmatrix}$$

and

$$\sum_{i=0}^{\infty} \mathrm{K}_i t^i = \frac{1}{1-pt-qt^2}\begin{pmatrix} p+2qt & 2-pt \\ 2q-pqt & -p+(p^2+2q)t \end{pmatrix},$$

respectively.

Proof. We will disregard the proof for Fibonacci because it will be similar. Accept that $G(t)$ is the generating function for the $\{\mathrm{K}_n\}_{n\in\mathrm{N}}$. Then we obtain

$$\begin{aligned} G(t) &= \sum_{i=0}^{\infty} \mathrm{K}_i t^i \\ &= \mathrm{K}_0 + \mathrm{K}_1 t + \sum_{i=2}^{\infty} \mathrm{K}_i t^i. \end{aligned}$$

From Definition 2.1, we get

$$\begin{aligned} G(t) &= \mathrm{K}_0 + \mathrm{K}_1 t + pt\sum_{i=2}^{\infty} \mathrm{K}_{i-1} t^{i-1} + qt^2 \sum_{i=0}^{\infty} \mathrm{K}_i t^i \\ &= \mathrm{K}_0 + \mathrm{K}_1 t + pt\left(G(t) - \mathrm{K}_0\right) + qt^2 G(t). \end{aligned}$$

Now, rearrangement of the above equation will show that

$$G(t) = \frac{\mathrm{K}_0 + \mathrm{K}_1 t - pt\mathrm{K}_0}{1-pt-qt^2}$$

which match to the $\sum_{i=0}^{\infty} \mathrm{K}_i t^i$ in theorem.

In [13], the authors obtain the generating functions for the generalized bivariate Fibonacci and Lucas polynomials. However, here, we will get these functions in terms of the generalized bivariate Fibonacci and Lucas matrix polynomials as a result of Theorem 2.3.

Corollary 2.4 There always exist

$$\sum_{i=0}^{\infty} H_i t^i = \frac{t}{1-pt-qt^2}$$

and

$$\sum_{i=0}^{\infty} K_i t^i = \frac{2-pt}{1-pt-qt^2}.$$

Theorem 2.5 We can note the Binet formulas for the generalized bivariate Fibonacci and Lucas matrix polynomials

$$\mathrm{H}_n = A_1\alpha^n + B_1\beta^n \quad \text{and} \quad \mathrm{K}_n = A_2\alpha^n + B_2\beta^n,$$

where

$$A_1 = \frac{\mathrm{H}_1 - \beta\mathrm{H}_0}{\alpha-\beta}, B_1 = \frac{\alpha\mathrm{H}_0 - \mathrm{H}_1}{\alpha-\beta}$$

and

$$A_2 = \frac{\mathrm{K}_1 - \beta\mathrm{K}_0}{\alpha-\beta}, B_2 = \frac{\alpha\mathrm{K}_0 - \mathrm{K}_1}{\alpha-\beta}$$

such that α, β are roots of equations of (2.1) and (2.2), $n \in \mathrm{N}$.

Proof. We omit the generalized bivariate Lucas matrix polynomial case since the proof is quite similar. The proof will be done by Theorem 2.3. Let α,β are roots of equations of (2.1), it is easily seen that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathrm{H}_n t^n &= \frac{1}{1-pt-qt^2}\begin{pmatrix}1 & t\\ qt & 1-pt\end{pmatrix}\\
&= \frac{1}{\alpha-\beta}\begin{pmatrix}\frac{\alpha}{1-\alpha t}-\frac{\beta}{1-\beta t} & \frac{1}{1-\alpha t}-\frac{1}{1-\beta t}\\ \frac{q}{1-\alpha t}-\frac{q}{1-\beta t} & \frac{-\beta}{1-\alpha t}+\frac{\alpha}{1-\beta t}\end{pmatrix}\\
&= \frac{1}{\alpha-\beta}\begin{pmatrix}\sum_{n=0}^{\infty}\left(\alpha^{n+1}-\beta^{n+1}\right)t^n & \sum_{n=0}^{\infty}\left(\alpha^{n}-\beta^{n}\right)t^n\\ q\sum_{n=0}^{\infty}\left(\alpha^{n}-\beta^{n}\right)t^n & q\sum_{n=0}^{\infty}\left(\alpha^{n-1}-\beta^{n-1}\right)t^n\end{pmatrix}\\
&= \sum_{n=0}^{\infty}\alpha^n t^n\begin{pmatrix}\frac{\alpha}{\alpha-\beta} & \frac{1}{\alpha-\beta}\\ \frac{q}{\alpha-\beta} & \frac{q}{\alpha(\alpha-\beta)}\end{pmatrix} + \sum_{n=0}^{\infty}\beta^n t^n\begin{pmatrix}\frac{-\beta}{\alpha-\beta} & \frac{-1}{\alpha-\beta}\\ \frac{-q}{\alpha-\beta} & \frac{-q}{\beta(\alpha-\beta)}\end{pmatrix}\\
&= \sum_{n=0}^{\infty}\left(A_1\alpha^n + B_1\beta^n\right)t^n.
\end{aligned}$$

Thus, by the equality of generating function, we obtain $\mathrm{H}_n = A_1\alpha^n + B_1\beta^n$.

In [13], the writers find the Binet formulas for the generalized bivariate Fibonacci and Lucas polynomials. Now as a different approximation and so as a result of Theorems 2.2 and 2.5, we will show these formulas by matrix polynomials in the following corollary.

Corollary 2.6 The Binet formulas for the generalized bivariate Fibonacci and Lucas polynomials in terms of their matrix polynomials are given by

$$H_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

and

$$K_n = \alpha^n + \beta^n,$$

where $n \geq 0$.

Proof. Firstly, by considering Definition 2.1 and Theorem 2.5, we can write

$$\begin{aligned} \mathrm{H}_n &= A_1\alpha^n + B_1\beta^n \\ &= \frac{\mathrm{H}_1 - \beta\mathrm{H}_0}{\alpha - \beta}\alpha^n + \frac{\alpha\mathrm{H}_0 - \mathrm{H}_1}{\alpha - \beta}\beta^n \\ &= \frac{\alpha^n}{\alpha - \beta}\begin{pmatrix} p-\beta & 1 \\ q & \frac{q}{\alpha} \end{pmatrix} + \frac{\beta^n}{\alpha - \beta}\begin{pmatrix} \alpha - p & -1 \\ -q & \frac{-q}{\beta} \end{pmatrix}. \end{aligned}$$

Here, by Theorem 2.2 and α, β are roots of the equation $\lambda^2 - p\lambda - q = 0$, we clearly have

$$\begin{pmatrix} H_{n+1} & H_n \\ qH_n & qH_{n-1} \end{pmatrix} = \frac{\alpha^n}{\alpha - \beta}\begin{pmatrix} \alpha & 1 \\ q & \frac{q}{\alpha} \end{pmatrix} + \frac{\beta^n}{\alpha - \beta}\begin{pmatrix} -\beta & -1 \\ -q & \frac{-q}{\beta} \end{pmatrix}.$$

Now, if we compare the 1*st* row and 2*nd* column entries with the matrices in the above equation, so we get

$$H_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Secondly, similarly, by using Theorem 2.2, Theorem 2.5 and Definition 2.1, we can write

$$\begin{aligned} \mathrm{K}_n &= A_2\alpha^n + B_2\beta^n \\ &= \frac{\mathrm{K}_1 - \beta\mathrm{K}_0}{\alpha - \beta}\alpha^n + \frac{\alpha\mathrm{K}_0 - \mathrm{K}_1}{\alpha - \beta}\beta^n \\ &= \frac{\alpha^n}{\alpha - \beta}\begin{pmatrix} p^2 + 2q - p\beta & p - 2\beta \\ pq - 2q\beta & 2q + p\beta \end{pmatrix} + \frac{\beta^n}{\alpha - \beta}\begin{pmatrix} p\alpha - p^2 - 2q & 2\alpha - p \\ 2q\alpha - pq & -p\alpha - 2q \end{pmatrix}. \end{aligned}$$

In here, by Theorem 2.2 and α, β are roots of the equation $\lambda^2 - p\lambda - q = 0$, we have

$$\begin{pmatrix} K_{n+1} & K_n \\ qK_n & qK_{n-1} \end{pmatrix} = \alpha^n\begin{pmatrix} \alpha & 1 \\ q & \frac{q}{\alpha} \end{pmatrix} + \beta^n\begin{pmatrix} \beta & 1 \\ q & \frac{q}{\beta} \end{pmatrix}.$$

Finally, if we compare the 1*st* row and 2*nd* column entries with the matrices in above equation, then we acquire

$$K_n = \alpha^n + \beta^n.$$

Now, for the generalized bivariate Fibonacci and Lucas matrix polynomials, we give the summations by determining principles.

Theorem 2.7 For $j \geq m > 0$ and $n \geq 1$, we have

$$\sum_{i=0}^{n-1} \mathrm{H}_{mi+j} = \frac{(-q)^m \mathrm{H}_{mn+j-m} - \mathrm{H}_{mn+j} + \mathrm{H}_j - (-q)^m \mathrm{H}_{j-m}}{(-q)^m - K_m + 1} \tag{2.5}$$

and

$$\sum_{i=0}^{n-1} \mathrm{K}_{mi+j} = \frac{(-q)^m \mathrm{K}_{mn+j-m} - \mathrm{K}_{mn+j} + \mathrm{K}_j - (-q)^m \mathrm{K}_{j-m}}{(-q)^m - K_m + 1}. \tag{2.6}$$

Proof. We will consider the proof over the generalized bivariate Lucas matrix polynomials and will disregard the part of Fibonacci. From Theorem 2.5, we have

$$\begin{aligned} \sum_{i=0}^{n-1} \mathrm{K}_{mi+j} &= \sum_{i=0}^{n-1} \left(A_2 \alpha^{mi+j} + B_2 \beta^{mi+j} \right) \\ &= A_2 \alpha^j \left(\frac{\alpha^{mn} - 1}{\alpha^m - 1} \right) + B_2 \beta^j \left(\frac{\beta^{mn} - 1}{\beta^m - 1} \right). \end{aligned}$$

Here, by simplification the last equation, is implied equation (2.6), as claimed.

3 RELATIONSHIPS BETWEEN NEW MATRIX POLYNOMIALS

The following proposition presents us that there exist some relations between the generalized bivariate Fibonacci and Lucas matrix polynomials.

Proposition 3.1 We obtain the following identities:

i. $\mathrm{K}_n = \mathrm{H}_{n+1} + q\mathrm{H}_{n-1}$,

ii. $\mathrm{K}_0 \mathrm{H}_n = \mathrm{H}_n \mathrm{K}_0 = \mathrm{K}_n$,

iii. $\mathrm{K}_0 \mathrm{K}_n = \mathrm{K}_n \mathrm{K}_0 = p\mathrm{K}_n + 2q\mathrm{K}_{n-1}$,

where $n \in \mathrm{N}$.

Proof. The proof can be easily done by the equations (1.2),(1.3), Definition 2.1 and Theorem 2.2.

Theorem 3.2 The following identities are satisfied:

i. $\mathrm{H}_m \mathrm{H}_n = \mathrm{H}_{m+n}$,

ii. $\mathrm{H}_m \mathrm{K}_n = \mathrm{K}_n \mathrm{H}_m = \mathrm{K}_{m+n}$,

iii. $\mathrm{K}_m \mathrm{K}_n = p\mathrm{K}_{m+n} + 2q\mathrm{K}_{m+n-1}$, where $m, n \in \mathrm{N}$.

Proof.
i. From Theorem 2.5 with its assumptions, we can have

$$\begin{aligned}\mathrm{H}_m\mathrm{H}_n &= \left(A_1\alpha^m + B_1\beta^m\right)\left(A_1\alpha^n + B_1\beta^n\right)\\ &= A_1^2\alpha^{m+n} + A_1B_1\alpha^m\beta^n + B_1A_1\beta^m\alpha^n + B_1^2\beta^{m+n}.\end{aligned}$$

Here, since $\alpha+\beta = p$ and $\alpha\beta = -q$, a simple matrix calculations imply that $A_1^2 = A_1$, $B_1^2 = B_1$ and

$$A_1B_1 = B_1A_1 = [0].$$

Then we obtain

$$\mathrm{H}_m\mathrm{H}_n = A_1\alpha^{m+n} + B_1\beta^{m+n} = \mathrm{H}_{m+n}.$$

ii. Here, we will just prove the correctness of the $\mathrm{H}_m\mathrm{K}_n = \mathrm{K}_{m+n}$ since the other can be performed nearly the same. Now, by Proposition 3.1-ii., we get

$$\mathrm{H}_m\mathrm{K}_n = \mathrm{H}_m\mathrm{H}_n\mathrm{K}_0.$$

Finally, by the above i. and again Proposition 3.1-ii, we obtain

$$\mathrm{H}_m\mathrm{K}_n = \mathrm{H}_{m+n}\mathrm{K}_0 = \mathrm{K}_{m+n}.$$

iii. By Theorem 3.2-i., Proposition 3.1-ii. and iii., we have

$$\begin{aligned}\mathrm{K}_m\mathrm{K}_n &= \mathrm{K}_0\mathrm{H}_m\mathrm{H}_n\mathrm{K}_0\\ &= \mathrm{K}_{m+n}\mathrm{K}_0\\ &= p\mathrm{K}_{m+n} + 2q\mathrm{K}_{m+n-1}\end{aligned}$$

as desired.

In [13, 20], the authors obtain the relations for the generalized bivariate Fibonacci and Lucas polynomials. Even so, comparing matrix entries and then utilizing Theorems 2.2 and 3.2, we find the next result.

Corollary 3.3 We have the following equalities for the generalized bivariate Fibonacci and Lucas polynomials:

i. $H_{m+1}H_n + qH_mH_{n-1} = H_{m+n}$,

ii. $H_{m+1}K_n + qH_mK_{n-1} = K_{m+n}$,

iii. $K_{m+1}K_n + qK_mK_{n-1} = K_{m+n}$.

We get the powers of the generalized bivariate Fibonacci and Lucas matrix polynomials in the following theorems.

Theorem 3.4 The following identities are satisfied:

i. $\mathrm{H}_n^m = \mathrm{H}_{mn}$,

ii. $\mathrm{H}_{n+1}^m = \mathrm{H}_1^m \mathrm{H}_{mn}$,

iii. $\mathrm{H}_{n-r}\mathrm{H}_{n+r} = \mathrm{H}_n^2 = \mathrm{H}_2^n$,
where $r, m, n \in \mathrm{N}$ and $n \geq r$.

Proof.
i.We can write $\mathrm{H}_n^m = \mathrm{H}_n \mathrm{H}_n \ldots \mathrm{H}_n$ (m-times). Then, we get H_{mn} by Theorem 3.2-i.
ii. Similar to i., we obtain

$$\begin{aligned} \mathrm{H}_{n+1}^m &= \mathrm{H}_{n+1}\mathrm{H}_{n+1}\ldots \mathrm{H}_{n+1} \\ &= \mathrm{H}_{m(n+1)} \\ &= \mathrm{H}_m \mathrm{H}_{mn}. \end{aligned}$$

From this Theorem i. , we can write $\mathrm{H}_m = \mathrm{H}_1^m$. Then, we acquire

$$\mathrm{H}_{n+1}^m = \mathrm{H}_1^m \mathrm{H}_{mn}.$$

iii. The proof can be done similarly to ii..

Theorem 3.5 The equalities always hold:

$$\mathrm{K}_{n-r}\mathrm{K}_{n+r} = \mathrm{K}_n^2 \quad \text{and} \quad \mathrm{K}_n^m = \mathrm{K}_0^m \mathrm{H}_{mn},$$

where $r, m, n \in \mathrm{N}$ and $n \geq r$.

Proof. Firstly, we are using Theorem 2.5. Thus we have

$$\mathrm{K}_{n-r}\mathrm{K}_{n+r} - \mathrm{K}_n^2 = \left(A_2\alpha^{n-r} + B_2\beta^{n-r}\right)\left(A_2\alpha^{n+r} + B_2\beta^{n+r}\right) - \left(A_2\alpha^n + B_2\beta^n\right)^2,$$

where A_2, B_2 and α, β are as given in Theorem 2.5. By applying the elementary calculations, we get

$$\mathrm{K}_{n-r}\mathrm{K}_{n+r} - \mathrm{K}_n^2 = A_2B_2\alpha^{n-r}\beta^{n+r} + B_2A_2\beta^{n-r}\alpha^{n+r} - 2A_2B_2\alpha^n\beta^n.$$

Ultimately, by $A_2B_2 = B_2A_2 = [0]_{3\times 3}$, we obtain $\mathrm{K}_{n-r}\mathrm{K}_{n+r} = \mathrm{K}_n^2$, as claimed.

In the second case of the proof, let us take the right-hand side of the equality $\mathrm{K}_n^m = \mathrm{K}_0^m \mathrm{H}_{mn}$. By Theorem 3.4-i., we get

$$\mathrm{K}_0^m \mathrm{H}_{mn} = \underbrace{\mathrm{K}_0\mathrm{K}_0\ldots\mathrm{K}_0}_{m \text{ times}}\ \underbrace{\mathrm{H}_n\mathrm{H}_n\ldots\mathrm{H}_n}_{m \text{ times}}.$$

By iterating usage of Proposition 3.1-ii., we finally obtain

$$\mathrm{K}_0^m \mathrm{H}_{mn} = \mathrm{K}_0\mathrm{H}_n\mathrm{K}_0\mathrm{H}_n\ldots\mathrm{K}_0\mathrm{H}_n = \mathrm{K}_n\mathrm{K}_n\ldots\mathrm{K}_n = \mathrm{K}_n^m.$$

The result is obtained.

4 CONCLUSIONS

In this paper, we define the generalized bivariate Fibonacci and Lucas matrix polynomials and have a major chance to crosscheck new properties on these matrix polynomials. Hence, we enlarge the recent studies in the literature. That is,

• By giving $p(x,y)=s, q(x,y)=t$ in $H_n(x,y)$ and $K_n(x,y)$ of the results in Sections 2 and 3, we obtain the (s,t)-Fibonacci and (s,t)-Lucas matrix sequences which these results may be found in [4,5].

• By giving $p(x,y)=x, q(x,y)=y$ in $H_n(x,y)$ and $K_n(x,y)$ of the results in Sections 2 and 3, we obtain the bivariate Fibonacci and bivariate Lucas matrix polynomials.

• By giving $p(x,y)=2s, q(x,y)=t$ in $H_n(x,y)$ and $K_n(x,y)$ of the results in Sections 2 and 3, we obtain the (s,t)-Pell and (s,t)-Pell-Lucas matrix sequences which these results may be found in [9].

• By giving $p(x,y)=2x, q(x,y)=y$ in $H_n(x,y)$ and $K_n(x,y)$ of the results in Sections 2 and 3, we obtain the bivariate Pell and bivariate Pell-Lucas matrix polynomials.

• By giving $p(x,y)=s, q(x,y)=2t$ in $H_n(x,y)$ and $K_n(x,y)$ of the results in Sections 2 and 3, we obtain the (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas matrix sequences which these results may be found in [22].

• By giving $p(x,y)=x, q(x,y)=2y$ in $H_n(x,y)$ and $K_n(x,y)$ of the results in Sections 2 and 3, we obtain the bivariate Jacobsthal and bivariate Jacobsthal-Lucas matrix polynomials which these results may be found in [25].

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