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ON A SUM OVER PRIMITIVE SEQUENCES OF FINITE DEGREE

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Summary. A sequence of strictly positive integers is said to be primitive if none of its terms divides the others and is said to be homogeneous if the number of prime factors of its terms counted with multiplicity is constant. In this paper, we construct primitive sequences A of degree d, for which the Erdős's analogous conjecture for translated sums is not satisfied.

1 INTRODUCTION

A sequenceA of strictly positive integers is said to be primitive if there is no term of Awhich divides any other. We can see directly that the set of primes $P = (p_n)_{n\geq 1}$is primitive. We define the degree of an integer, to be the number of prime factors counted with multiplicity and the degree of a sequence A is defined as the maximum degree of its terms. Erdős [1] showed that for any primitive sequence $A \neq \{1\}$, the series $\sum_{a\in A} \frac{1}{a\log a}$ converges. Later, in [2], he conjectured that if $A \neq \{1\}$, is a primitive sequence, then

$$\sum_{a\in A} \frac{1}{a\log a} \leq \sum_{p\in P} \frac{1}{p\log p}.$$

Based on the primitive sequences A of finite degree, in [3], Zhang proved this conjecture when the degree of $A$is at most 4 and in [4], he proved it for the particular case of primitive sequences when the degree of its terms is constant. In [5] the authors simplified the proof of [3] and Laib [6] improved this result up to degree 5. Recently, in [7], the authors studied translated sums of the form:

$$S(A,x) = \sum_{a\in A} \frac{1}{a\,(\log a + x)}, \qquad x \in \mathbb{R}$$

and they constructed primitive sequences A of degree 2, such that $S(A,x) > S(P,x)$ for all $x \geq 81$and in [8]the authors prove that $S(A,x) \gg S(P,x)$ for x large enough. In this note, we present a general case for any degree d, that is, we prove the following:

Theorem. Let $d \geq 2$ be an integer, $x_0 = \frac{dd!e^{d+1}}{(d+1)^{d-1}-d!}$ and let k_0 be the greatest integer such that $p_{k_0} \leq e^{e^{d+1}}$. Then for any $k \geq k_0$ and any primitive sequence

$$B_d^k = \{{p_1}^{\alpha_1}{p_2}^{\alpha_2} \dots {p_k}^{\alpha_k},\ \alpha_1 \dots \alpha_k \in \mathbb{N},\ \alpha_1 + \cdots + \alpha_k = d\} \cup \{p_n \mid p_n \in P, n > k\}$$

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we have $S(B_d^k, x) > S(P, x)$ for $x \geq x_0$.

2 MAIN RESULTS

Lemma 2.1. [9] For $x \geq 3275$ there exists a prime number p such that

$$x < p < x\left(1 + \frac{1}{2ln^2x}\right).$$

Lemma 2.2. For any integer $n > 1$, we have

$$n! \leq n^n e^{1-n}\sqrt{n}, \quad (1)$$

$$2.5n^n e^{1-n}\sqrt{n} < n! \leq n^{n-1}, \quad (2)$$

$$n! \leq 2(n+1)^{n-2}, \quad (3)$$

$$n! < n^{n-2} (n \geq 5). \quad (4)$$

Proof. For $n = 2$, the inequalities (1) and (2) is verified, for $n > 2$, it comes from the inequality [10]

$$n^n e^{-n}\sqrt{2\pi n} e^{\frac{1}{12n+1}} < n! < n^n e^{-n}\sqrt{2\pi n} e^{\frac{1}{12n}},$$

and we can prove (3) and (4) by induction.

Lemma 2.3. Let $n \geq 2$ be an integer and x be a reel number such that $x \geq n-1$. The function

$$x \mapsto f_n(x) = \frac{nn!\, e^x}{x^{n-1} - n!}$$

reaches its minimum x_n in the interval $]n-1, n+1]$, moreover $x_2 = 2$, $x_3 = \sqrt{7} + 1$, $x_4 \simeq 4.298$ and $x_n < n$ for $n \geq 5$.

Proof. Let $n \geq 2$ be an integer and let f_n be the function defined on the interval $I =]n - 1; +\infty[$

$$f_n(x) = \frac{nn!\, e^x}{x^{n-1} - n!}$$

f is differentiable on I and

$$f'_n(x) = \frac{nn!\, e^x(x^{n-1} - (n-1)x^{n-2} - n!)}{(x^{n-1} - n!)^2}.$$

For $x > n-1$, put $g_n(x) = x^{n-1} - (n-1)x^{n-2} - n!$

then

$$g'_n(x) = x^{n-1} - (n-1)x^{n-2} > 0\,, x \in I,$$

hence g_n increases on I. On the other hand, since g_n is continuous then by lemma 2.2, we have

$$\lim_{x \to n-1} g_n(x) = -n! < 0,$$

$$g_n(n) = n^{n-2} - n! > 0 \;\; for\; n \geq 5,$$

$$g_n(n+1) = 2(n+1)^{n-2} - n! \geq 0,$$

therefore, there exists only one root $x_n \in]n-1, n+1]$, where for $n \geq 5$ $x_n \in]n-1, n]$, such that $f'_n(x_n) = 0$. Since $g_n(x) < 0$ for $x < x_n$ and $g_n(x) > 0$ for $x > x_n$ then f_n strictly decreases on $]n-1, x_n]$ and strictly increases on $[x_n, +\infty[$, so we have

$$f_n(x) \geq f_n(x_n) \; where \; x_n \in]n-1, n+1].$$

It is clear that, for $n = 2,3,4$ the equation $x^{n-1} - (n-1)x^{n-2} - n! = 0$ gives $x_2 = 2$, $x_3 = \sqrt{7} + 1$ and $x_4 \simeq 4.298$. This completes the proof.

Lemma 2.4. For any integer $d \geq 2$, there exists a prime p such that

$$e^{e^{x_d}} < p \leq e^{e^{d+1}}, \tag{5}$$

moreover $\max\left\{p : p \in \left]e^{e^{x_d}}, e^{e^{d+1}}\right[\right\} > e^{e^d}$, where $(x_d)_{d\geq 2}$ is the sequence defined in lemma 2.3.

Proof. The inequality (5) is easy to verify for $d = 2,3,4$. By lemma 2.3, we have, for $d \geq 5$

$$d - 1 \leq x_d \leq d, \tag{6}$$

therefore $e^{e^{x_d}} > 3275$, then from lemma 1.1 there exists a prime p such that

$$e^{e^{x_d}} < p \leq e^{e^{x_d}}\left(1 + \frac{1}{2e^{2x_d}}\right).$$

From (6) we get $4 \leq x_d \leq d$, then $1 + \frac{1}{2e^{2x_d}} < 2$ and $e^{e^{x_d}} < e^{e^d}$, thus $\left(1 + \frac{1}{2e^{2x_d}}\right) < 2e^{e^d} < e^{e^{d+1}}$. Since $4e^{e^d} < \left(e^{e^d}\right)^2 < e^{e^{d+1}}$, then according to the Bertrand's postulate there exists a prime number in $\left[2e^{e^d}, \; 4e^{e^d}\right]$, thus, the greatest prime number in $\left[e^{e^{x_d}}, e^{e^{d+1}}\right]$ is greater than e^{e^d}. Which finishes the proof.

Lemma 2.5. [7] For any integer $k \geq 1$ and any integer $d \geq 2$, we define

$$A_d^k = \{p_1{}^{\alpha_1} p_2{}^{\alpha_2} \dots p_k{}^{\alpha_k}, \; \alpha_1, \dots, \alpha_k \in \mathbb{N}, \; \alpha_1 + \dots + \alpha_k = d\}$$

then we have the disjoint union

$$A_d^{k+1} = A_d^k \cup \{a p_{k+1} : a \in A_{d-1}^{k+1}\}.$$

Lemma 2.6. [11] For any real number$x > 1$, we have

$$\sum_{p\in P, \; p \leq x} \frac{1}{p} > loglogx.$$

Lemma 2.7. Let $d \geq 2$ and let k' be the integer such that $p_{k'} \geq exp\; exp\;(d)$. For any real number $x > 0$ the sequence $\left(S\left(A_d^k, x\right)\right)_{k\geq k'}$ strictly increases.

Proof. For any integer $k \geq 1$ and any integer $d \geq 2$, the multinomial formula ensures that

$$\sum_{a\in A_d^k}\frac{1}{a}=\sum_{\alpha_1+\cdots+\alpha_k=d}\frac{1}{{p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_k}^{\alpha_k}}$$
$$\geq\sum_{\alpha_1+\cdots+\alpha_k=d}\frac{(1/p_1)^{\alpha_1}}{\alpha_1!}\dots\frac{(1/p_k)^{\alpha_k}}{\alpha_k!}$$
$$=\frac{1}{d!}\left(\sum_{n=1}^{k}\frac{1}{p_n}\right)^d$$

therefore

$$\sum_{a\in A_d^k}\frac{1}{a}\geq\frac{1}{d!}\left(\sum_{n=1}^{k}\frac{1}{p_n}\right)^d. \tag{7}$$

Put $A_d^k=\{p_n/p_n\in P,n>k\}$, then from lemma 2.5 we have

$$B_d^{k+1}=A_d^{k+1}\cup A^{k+1}=A_d^k\cup\left\{ap_{k+1}:a\in A_{d-1}^{k+1}\right\}\cup A^{k+1}$$

so,

$$S\left(B_d^{k+1},x\right)=S\left(B_d^k,x\right)+E$$

where

$$E=\frac{1}{p_{k+1}}\left(S\left(A_{d-1}^{k+1},logp_{k+1}+x\right)-\frac{1}{logp_{k+1}+x}\right).$$

Since p_{k+1}^{d-1} is the greatest element of A_{d-1}^{k+1}, we have

$$S\left(A_{d-1}^{k+1},logp_{k+1}+x\right)=\sum_{a\in A_{d-1}^{k+1}}\frac{1}{a(loga+logp_{k+1}+x)}$$
$$\geq\sum_{a\in A_{d-1}^{k+1}}\frac{1}{a((d-1)logp_{k+1}+logp_{k+1}+x)}$$
$$\geq\frac{1}{dlogp_{k+1}+x}\sum_{a\in A_{d-1}^{k+1}}\frac{1}{a}$$

and by lemma 2.6 we obtain

$$\sum_{a\in A_{d-1}^{k+1}}\frac{1}{a}\geq\frac{1}{(d-1)!}\left(\sum_{n=1}^{k+1}\frac{1}{p_n}\right)^{d-1}$$

$\geq\frac{1}{(d-1)!}(loglogp_{k+1})^{d-1}$

$$\geq \frac{d^{d-1}}{(d-1)!}$$

$$\geq \frac{d^{d-1}}{d!} d \; for \; k \geq k',$$

and according to lemma 2.2 we have $d! \leq d^{d-1}$, then

$$\sum_{a \in A_{d-1}^{k+1}} \frac{1}{a} \geq d \; for \; k \geq k',$$

which implies

$$S\left(A_{d-1}^{k+1}, logp_{k+1} + x\right) - \frac{1}{logp_{k+1} + x} > \frac{1}{dlogp_{k+1} + x} - \frac{1}{logp_{k+1} + x}$$

$$= \frac{dx - x}{(dlogp_{k+1} + x)(logp_{k+1} + x)} > 0$$

thus $S\left(B_d^{k+1}, x\right) - S\left(B_d^k, x\right) > 0$. Which ends the proof.

Proof of theorem.
From [7], for any integer $k \geq 1$ and any integer $d \geq 2$, we have

$$\sum_{a \in B_d^k} \frac{1}{a(loga + x)} = \sum_{a \in A_d^k \cup A^k} \frac{1}{a(loga + x)} = \sum_{a \in A_d^k} \frac{1}{a(loga + x)} + \sum_{a \in A^k} \frac{1}{a(loga + x)}$$

$$\geq \frac{1}{dlogp_k + x} \sum_{a \in A_d^k} \frac{1}{a} + \sum_{n > k} \frac{1}{p_n(logp_n + x)}.$$

Using (7) and lemma 2.6, we get

$$\sum_{a \in A_d^k} \frac{1}{a} > \frac{(loglogp_k)^{d-1}}{d!} \sum_{n=1}^{k} \frac{1}{p_n},$$

therefore

$$\sum_{a \in B_d^k} \frac{1}{a(loga + x)} \geq \frac{x(loglogp_k)^{d-1}}{d!\,(dlogp_k + x)} \sum_{n=1}^{k} \frac{1}{xp_n} + \sum_{n > k} \frac{1}{p_n(logp_n + x)}$$

$$\geq \frac{x(loglogp_k)^{d-1}}{d!\,(dlogp_k + x)} \sum_{n=1}^{k} \frac{1}{p_n(logp_n + x)} + \sum_{n > k} \frac{1}{p_n(logp_n + x)}.$$

To obtain the inequality required in theorem, we must choose k and x so that

$$\frac{x(loglogp_k)^{d-1}}{d!\,(dlogp_k + x)} > 1. \tag{8}$$

Since the function

$$x \mapsto h_{k,d}(x) = \frac{x(loglogp_k)^{d-1}}{d!\,(dlogp_k + x)} \quad for\ d \geq 2, k > 1$$

strictly increases for $x > 0$, let $x_0$the smallest value for which the inequality (8) is verified. That is

$$\frac{(loglogp_k)^{d-1} - d!}{dd!\,logp_k} > \frac{1}{x_0}. \qquad (9)$$

Since $x_0 > 0$, we need to find k such that $(loglogp_k)^{d-1} - d! > 0$, then by lemma 2.2, we just take $loglogp_k > d$, and if we put $loglogp_k = z,$ then (9) becomes

$$\frac{dd!\,e^z}{z^{d-1} - d!} < x_0.$$

Now, we must choose z so that the number $\frac{dd!e^z}{z^{d-1}-d!}$ is the smallest possible. According to lemma 2.3, the function

$$x \mapsto f_d(x) = \frac{dd!\,e^z}{z^{d-1} - d!}$$

reaches its minimum x_d in

$$]d-1, d+1],$$

then we can take $z \in]x_d,\ d+1[$ and $x_0 = \frac{dd!e^{d+1}}{(d+1)^{d-1}-d!},$ from lemma 2.4, there exists a prime integer p_k such that

$$x_d < loglogp_k < d + 1.$$

Choose $p_{k_0} = \max\{p_k : loglogp_k \in]x_d,\ d+1]\}$ and $z = loglogp_{k_0}$, then we obtain $S(B_d^{k_0}, x) > S(P, x)$ for $x \geq x_0$. Finally, by lemma 2.4, we have $e^{e^d} \leq p_{k_0} \leq e^{e^{d+1}}$ and from lemma 2.7, we get for $k \geq k_0$, $x \geq x_0$,

$$S(B_d^{k_0}, x) > S(P, x).$$

And the proof is achieved.

3 CONCLUSIONS

In this work, we obtain a generalization of result introduced in [7], concerning primitive sequences of finite degree, thus, if we take $d = 2$ in the theorem we get $S(B_2^k,\ x) > S(B_1^k,\ x)$, for $k \geq 27775592$ and $x \geq 81$, which apply to improve the results of [3,6]. Since for x is sufficiently large, we have $S(B_d^k, x) > S(P, x)$, so we can ask if it is true: for any $d \geq 1$ there exists $k_0$such that$S(B_{d+1}^k, x) > S(B_d^k, x), k \geq k_0, x > 0$.

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