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DISCUSSIONS ON HARDY-TYPE INEQUALITIES VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL INEQUALITY

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Summary. In this paper, we have shown that in the article entitled "New Riemann-Liouville generalizations for some inequalities of Hardy type" the Theorem 3.3 is false and that a necessary condition on the order α must be in Theorem 3.1 and Theorem 3.2. Additionally, the correct Theorem 3.3 and a new result with negative parameter are given; these results will prove using the Hölder inequality, and reverse Hölder inequality.

1 INTRODUCTION

Hardy-type inequalities and inverse Hardy-type inequalities have an important place in analysis and its applications, they have given rise to various extensions and generalizations in recent years, see for example [1]-[2]. So much paper appeared in different form of calculus mathematics, with functions of two variables [3], through the Steklov operator [4], on time scales [5]-[6]. In 2016, the authors have obtained some new results about Hardy type inequalities via Riemann-Liouville fractional integral operators [7]. In Theorem 3.1, the passing from the inequality (3.15) to inequality (3.16) and in Theorem 3.2, the passing from the inequality (3.27) to inequality (3.28), for $a < t \leq x \leq b$ we have

$$x - t \leq b - t\,,$$

if we take the order $0 < \alpha < 1$, then

$$\int_a^b (x-t)^{\alpha-1}\,dt \geq \int_a^b (b-t)^{\alpha-1}\,dt,$$

if $\alpha \geq 1$, we get

$$\int_a^b (x-t)^{\alpha-1}\,dt \leq \int_a^b (b-t)^{\alpha-1}\,dt.$$

So, it is necessary to mentioned that $\alpha \geq 1$ in Theorem3.1, Corollary3.1 and Theorem3.2 Now, let show the Theorem3.3 is not correct. In the inequalities (3.35) and (3.36) the following formulas are given:

Let f, g be positive functions on [a; b] and for $\alpha > 0$, $a < t \leq x \leq b$, we have the integral

$$\int_a^b (a-t)^{\alpha-1}\, g^{-q}(t) f^{p}(t) dt.$$

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So, we deduce that $(\mathrm{a}-\mathrm{t})<0$, then
for $0<\alpha<1$, we set $\alpha=\frac{1}{2}$, hence we get $(\mathrm{a}-\mathrm{t})^{\alpha-1}=(\mathrm{a}-\mathrm{t})^{-\frac{1}{2}}$ (contradiction) for $\alpha \geq 1$, we set $\alpha=\frac{3}{2}$, hence we get $(\mathrm{a}-\mathrm{t})^{\alpha-1}=(\mathrm{a}-\mathrm{t})^{\frac{1}{2}}$ (contradiction).

This implies that there existed errors in the proof of the inequalities (3.35) and (3.36) and that the Theorem 3.3 is not necessarily true. The correct versions of Theorem 3.3 will be proved in Section 2.

2 MAIN RESULTS

We present the correct version of the Theorem 3.3 according the right Riemann-Liouville fractional integral operator of order $\alpha>0$, for a continuous function f on $[a,b]$ defined as

$$j_{a^+}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad a<x\leq b. \qquad (1)$$

Theorem 2.1. Let $f\geq 0$ and $g>0$ on $[a;\ b]\subseteq[0;\ +\infty[$ such that g is non-decreasing. Then, for all $0<p<1,\ q>0,\ 0<\alpha<\frac{1}{1-p}$, we have

$$\int_a^b \frac{\left(j_{a^+}^{\alpha} f(x)\right)^p}{g^q(x)}dx \geq \frac{\Gamma^{1-p}(\alpha+1)}{(\alpha(1-p)+1)g^q(b)}$$
$$\left[(b-a)^{\alpha(1-p)+1} j_{a^+}^{\alpha} f^p(b) - j_{a^+}^{\alpha}\left(f^p(b)(b-a)^{\alpha(1-p)+1}\right)\right]. \qquad (2)$$

Proof. (i) For $0<p<1$, Apply the reverse Hölder inequality for $\frac{1}{p}+\frac{1}{p'}=1$, we get

$$\int_a^b \frac{\left(j_{a^+}^{\alpha} f(x)\right)^p}{g^q(x)}dx = \int_a^b g^{-q}(x)\left(\int_a^x \left[\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\frac{1}{p'}}\left[\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\frac{1}{p}} f(t)dt\right)^p dx$$

$$\geq \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)}\int_a^b g^{-q}(x)(x-a)^{\alpha(p-1)}\left(\int_{\mathbf{0}}^{\boldsymbol{x}} (x-t)^{\alpha-1} f^p(t)dt\right)dx,$$

since g is non-decreasing function on $[t,b]$ and by using Fubini Theorem, we obtain

$$\int_a^b \frac{\left(j_{a^+}^{\alpha} f(x)\right)^p}{g^q(x)}dx \geq \frac{\Gamma^{1-p}(\alpha+1)}{g^q(b)}\int_a^b \frac{f^p(t)}{\Gamma(\alpha)}(b-t)^{\alpha-1}\left(\int_{\boldsymbol{t}}^{\boldsymbol{b}} (x-a)^{\alpha(p-1)}dx\right)dt,$$

$$= \frac{\Gamma^{1-p}(\alpha+1)}{(\alpha(1-p)+1)g^q(b)}\int_a^b \frac{f^p(t)}{\Gamma(\alpha)}(b-t)^{\alpha-1}\left((b-a)^{\alpha(p-1)+1}-(t-a)^{\alpha(p-1)+1}\right)dt.$$

Therefore

$$\int_a^b \frac{\left(j_{a^+}^{\alpha} f(x)\right)^p}{g^q(x)}dx \geq \frac{\Gamma^{1-p}(\alpha+1)}{(\alpha(1-p)+1)g^q(b)}\left[\frac{(b-a)^{\alpha(p-1)+1}}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1} f^p(t)dt\right.$$

$$-\frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1} f^p(t)(t-a)^{\alpha(p-1)+1}dt\Bigg]$$

$$=\frac{\Gamma^{1-p}(\alpha+1)}{(\alpha(1-p)+1)g^q(b)}\left[(b-a)^{\alpha(p-1)+1} j_{a^+}^{\alpha} f^p(b) - j_{a^+}^{\alpha}(f^p(b)(b-a)^{\alpha(p-1)+1})\right].$$

This ends the proof.

Now we present a new result according the left Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[a,b]$ defined as

$$j_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad a \le x < b. \qquad (3)$$

We need the following reverse Hölder inequality (see [8]):

If $p < 0$, $\frac{1}{p}+\frac{1}{p'} = 1$, $\Phi, \Psi > 0$ and $\Phi \in L_p(a,b)$, $\Psi \in L_{p'}(a,b)$, then

$$\int_a^b \Phi(t)\,\Psi(t)\,dt \ge \left(\int_a^b \Phi^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_a^b \Psi^{p'}(t)dt\right)^{\frac{1}{p'}}. \qquad (4)$$

Theorem 2. 2. Let f ≥ 0 and $g > 0$ on [a; b] $\subseteq [0; +\infty[$ such that g is non-increasing. Then, for all $p < 0$, $\frac{1}{1-p} < \alpha$, we have

$$\int_a^b \frac{\left(j_{b^-}^{\alpha} f(x)\right)^p}{g(x)}dx \ge \frac{\Gamma^{1-p}(\alpha+1)}{(\alpha(1-p)-1)g(a)}$$

$$\left[j_{b^-}^{\alpha}\left(f^p(a)(b-a)^{\alpha(p-1)+1}\right) - (b-a)^{\alpha(p-1)+1} j_{b^-}^{\alpha} f^p(a)\right]. \qquad (5)$$

Proof. (i) For $p<0$, Apply the reverse Hölder inequality (4), we get

$$j_{b^-}^{\alpha} f(x) = \int_x^b \left[\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\frac{1}{p'}} \left[\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\frac{1}{p}} f(t)dt$$

$$\ge \left(\frac{1}{\Gamma(\alpha+1)}(b-x)^{\alpha}\right)^{\frac{1}{p'}} \left(\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f^p(t)dt\right)^{\frac{1}{p}}.$$

Since p < 0, we get

$$\int_a^b \frac{\left(j_{b^-}^{\alpha} f(x)\right)^p}{g(x)}dx \le \frac{\Gamma^{1-p}(\alpha+1)}{\Gamma(\alpha)}\int_a^b g^{-1}(x)(b-x)^{\alpha(p-1)}\left(\int_x^b (t-x)^{\alpha-1} f^p(t)dt\right)dx,$$

Since g is non-increasing function on $[a;t]$ and by using Fubini Theorem, we obtain

$$\int_a^b \frac{\left(j_{b^-}^{\alpha} f(x)\right)^p}{g(x)}dx \le \frac{\Gamma^{1-p}(\alpha+1)}{g(a)}\int_a^b \frac{f^p(t)}{\Gamma(\alpha)}(t-a)^{\alpha-1}\left(\int_a^t (b-x)^{\alpha(p-1)}dx\right)dt,$$

$$=\frac{\Gamma^{1-p}(\alpha+1)}{(\alpha(1-p)-1)g(a)}\int_a^b \frac{f^p(t)}{\Gamma(\alpha)}(t-a)^{\alpha-1}\left((b-t)^{\alpha(p-1)+1}-(b-a)^{\alpha(p-1)+1}\right)dt.$$

This gives us that

$$\int_a^b \frac{\left(j_{b^-}^{\alpha} f(x)\right)^p}{g(x)}dx \leq \frac{\Gamma^{1-p}(\alpha+1)}{(\alpha(1-p)-1)g(a)}\left[\frac{1}{\Gamma(\alpha)}\int_a^b (t-a)^{\alpha-1}f^p(t)(b-t)^{\alpha(p-1)+1}dt - \frac{(b-a)^{\alpha(p-1)+1}}{\Gamma(\alpha)}\int_a^b (t-a)^{\alpha-1}f^p(t)dt\right]$$

$$=\frac{\Gamma^{1-p}(\alpha+1)}{(\alpha(1-p)-1)g(a)}\left[j_{b^-}^{\alpha}\left(f^p(a)(b-a)^{\alpha(p-1)+1}\right)-(b-a)^{\alpha(p-1)+1}j_{b^-}^{\alpha} f^p(a)\right].$$

So, the proof of Theorem 3.2. is complete.

Putting $\alpha = 1$ in above Theorem 2.2, we obtain the following Corollary.

Corollary 2.1. Let $f \geq 0$ and $g > 0$ on $[a;\ b] \subseteq [0;\ +\infty[$ such that g is non-increasing and $F(x) = \int_x^b f(t)dt$. Then, for all $p < 0$, we have

$$-p\int_a^b \frac{F^p(x)}{g(x)}dx \geq \frac{1}{g(a)}\left(\int_a^b (b-x)^p f^p(x)dx-(b-a)^p\int_a^b f^p(x)dx\right). \quad (6)$$

Remark4.1 The inequality (6) for negative parameter coincides with inequality (4.26) in [5].

3 CONCLUSIONS

In this paper, we present the correction of Theorem3.3 with the right-sided Riemann-Liouville operator with positive parameter 0<p<1, we also introduced a new version using the left-sided Riemann-Liouville operator with the negative parameter. These results have been obtained by applying the Hölder's inequality and the reverse Hölder's inequality.

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