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ON THE CONSTRUCTION OF A FAMILY OF ANOMALOUS-DIFFUSION FOKKER–PLANCK−KOLMOGOROV'S EQUATIONS BASED ON THE SHARMA–TANEJA–MITTAL ENTROPY FUNCTIONAL

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Summary. A logical scheme for constructing thermodynamics of anomalous stochastic systems based on the nonextensive two-parameter (κ, ς) -entropy of Sharma–Taneja–Mittal (SHTM) is considered. Thermodynamics within the framework (2 - q) -statistics of Tsallis was constructed, which belongs to the STM family of statistics. The approach of linear nonequilibrium thermodynamics to the construction of a family of nonlinear equations of Fokker−Planck−Kolmogorov (FPK), is used, correlated with the entropy of the STM, in which the stationary solution of the diffusion equation coincides with the corresponding generalized Gibbs distribution obtained from the extremality (κ, ς) - entropy condition of a non-additive stochastic system. Taking into account the convexity property of the Bregman divergence, it was shown that the principle of maximum equilibrium entropy is valid for (κ, ς) - systems, and also was proved the H - theorem determining the direction of the time evolution of the non-equilibrium state of the system. This result is extended also to non-equilibrium systems that evolve to a stationary state in accordance with the nonlinear FPK equation. The method of the ansatz- approach for solving non-stationary FPK equations is considered, which allows us to find the time dependence of the probability density distribution function for non-equilibrium anomalous systems. Received diffusive equations FPK can be used, in particular, at the analysis of diffusion of every possible epidemics and pandemics. The obtained diffusion equations of the FPK can be used, in particular, in the analysis of the spread of various epidemics and pandemics.

1 INTRODUCTION

As it has now become clear, the statistical mechanics of Boltzmann−Gibbs−Shannon and standard thermodynamics are not completely universal theories, since they have limited areas of applicability. In physics and information theory, as well as in other natural sciences that use statistical methods, numerous examples of complex (anomalous) systems are known, which are characterized by the effects of long-range interaction, the fractal nature of the phase space, non-Markov random processes, and nonlocal correlations between individual elements of the aggregate system. The complex space-time structure of these systems leads to a violation of the principle of additivity for such important thermodynamic quantities as entropy or internal energy. Research in the field of statistical mechanics and thermodynamics of non-additive systems has recently become a subject of considerable interest, which is explained both by the novelty of the general theoretical problems arising here and by the importance of practical applications.

The beginning of a systematic study in this direction is associated with the work of K. Tsallis

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[1-3], in which a one-parameter entropy functional was introduced, which depends on the so-called deformation parameter and is nonadditive for a set of independent systems. Numerous reviews of new results obtained in the study of anomalous physical phenomena in the framework of Tsallis statistics are available in the scientific literature (see, for example, [4]).

At the same time, the definition of the one-parameter Tsallis entropy is not the only example of deformed entropy [5-6]. Numerous non-logarithmic entropies, in particular, the one-parameter Renyi entropy [7], the two-parameter Sharma−Mittal entropy [8-10], the one-parameter entropy entropy of Kaniadakis [11-16], two-parameter entropy Sharma−Taneja−Mittal [17-19], etc. The range of applications of various parametric entropies is currently constantly expanding, covering various areas in science, such as cosmology and cosmogony, plasma theory and quantum mechanics, special and general relativity, stochastic dynamics and fractals, geophysics, biomedicine and many others.

Unfortunately, generalized thermodynamics based on some nonextensive statistics are intended mainly to describe the equilibrium states of physical systems, and are not quite applicable to the description of their nonequilibrium states [20]. At the same time, one of the basic phenomenological equations of statistical mechanics, describing, in particular, the dynamic evolution of a nonequilibrium system, is the nonlinear diffusion Fokker–Planck–Kolmogorov equation. Recently, an effective approach has been developed that allows one to construct nonlinear FPK equations in such a way that their stationary solutions are consistent with the corresponding equilibrium (canonical) probability density distributions obtained from the condition of extremality of entropies for the systems under consideration [21-23].

This approach, which links the entropy of a system with nonlinear FPK equations describing the evolution of nonequilibrium phenomena, is one of the useful applications of nonextensive statistical mechanics. The use of diffusion equations correlated with the entropy method allows one to find the time dependence of the probability density distribution function for nonequilibrium nonextensive systems.

Power-law FPK equations have found application in various fields of science, such as astrophysics, plasma physics, hydrodynamics, biophysics, etc. (see, for example, [5, 24-29]). In addition, they were used to simulate energy propagation in highly nonlinear disordered lattices [30]. Nonlinear diffusion and FPK equations are also closely related to nonlinear versions of the Schrödinger, Dirac, and Klein–Gordon equations [31], which admit complex soliton-like analytical solutions, as well as to nonlinear wave equations having exponential plane wave solutions modulated by q- Gaussians [32]. Nonlinear diffusion processes are also important when studying the distribution of biological populations [33,34]. In particular, the nonlinear diffusion equations of FPK based on kappa statistics can be used in epidemiology to predict the spread of epidemics and pandemics.

It should be noted that despite the wide variety of studies in these scientific areas, they all have much in common due to the cooperative interaction between the individual subsystems of the considered aggregate system. These interactions lead to a decrease in a large number of degrees of freedom of many-body systems and, thus, allow a low-dimensional description in terms of the nonlinear nonstationary Fokker–Planck–Kolmogorov equation, which reveals the dynamics underlying many observed physical phenomena.

This paper shows how the approach of linear nonequilibrium thermodynamics to the construction of FPK equations can be applied to a relatively large category of entropies, which are special cases of the two-parameter Sharma−Taneja−Mittal entropy.

2 SOME DEFINITIONS AND STATISTICAL PROPERTIES OF ENTROPY HARMA-TANEDJA-MITTAL

In two-parameter statistical mechanics of STM for continuous characteristics of anomalous (κ,ς)-systems with probabilistic normalization

$$\int p(\boldsymbol{r})\,d\Gamma = 1\,, \qquad (0 \le p(\boldsymbol{r}) < \infty) \tag{1}$$

for the probability density of the state $p(\boldsymbol{r}) \equiv p^{\mathrm{STM}}(\boldsymbol{r})$ of the system in the phase space $\boldsymbol{r} := \left\{x_1,...x_n; v_1,...v_n\right\}$ of the Gibbs physical statistical ensemble (describing the dynamics of the microstate of a chaotic system), the dimensionless (κ,ς)-information entropy is given by the following functional [18, 35]

$$S_{\{\kappa,\varsigma\}}(p) := -\int p^{1+\varsigma}(\boldsymbol{r})\frac{p^{\kappa}(\boldsymbol{r}) - p^{-\kappa}(\boldsymbol{r})}{2\kappa}d\Gamma\,. \tag{2}$$

Here the region of integration coincides with the entire $6n$-dimensional phase space, and the dimensionless element $d\Gamma$ of the phase space is written in the modern form $d\Gamma := \left\{n!h^{3n}\right\}^{-1}d\boldsymbol{r}$, where $h = 2\pi\hbar$ is Planck's constant; entropy indices κ and ς (deformation parameters) are real numbers satisfying inequalities $|\kappa| < 1$ and $\varsigma > 0$.

Entropy (2) can also be rewritten in the following equivalent forms:

$$S_{\{\kappa,\varsigma\}}(p) := -\int p(\boldsymbol{r})\ln_{\{\kappa,\varsigma\}}\left[p(\boldsymbol{r})\right]d\Gamma = -\left\langle \ln_{\{\kappa,\varsigma\}}\left[p(\boldsymbol{r})\right]\right\rangle. \tag{3}$$

When writing (3), the so-called "deformed" logarithm Sharma−Mittal was used [8]

$$\ln_{\{\kappa,\varsigma\}}(x) := \frac{x^{\varsigma+\kappa} - x^{\varsigma-\kappa}}{2\kappa}\,, \quad (\forall x > 0), \tag{4}$$

as well as the following method for obtaining the average value for any physical quantity $\mathrm{T}_k(\boldsymbol{r})$, characterizing the microstate of the system, namely

$$\boldsymbol{E}[\mathrm{T}] = \langle \mathrm{T}_k(\boldsymbol{r})\rangle := \int p(\boldsymbol{r})\mathrm{T}_k(\boldsymbol{r})d\Gamma\,. \tag{5}$$

Due to the deformed logarithm in the expression for the entropy, the STM statistics describe events that are practically unattainable in simple systems characterized by the Boltzmann–Gibbs statistics.

Two-parameter entropy (2) combines the statistics of Boltzmann−Gibbs, Tsallis, Kanyadakis, and some others and, by manipulating two deformation parameters κ and ς, defines them as some limiting one-parameter cases of one family (see, for example, [10,19, 36]).

If $\varsigma = \pm|\kappa| = (1-q)/2$, then definition (2) implies the Tsallis $(2-q)$-entropy $S_{\{\kappa,\varsigma\}}(p) \to S_{2-q} := \int \frac{p^{2-q}(\boldsymbol{r}) - p(\boldsymbol{r})}{q-1} d\Gamma$ [3]; at $\varsigma = 0$ is the Kaniadakis entropy $S_{\{\kappa\}} := \int \frac{p^{\kappa+1}(\boldsymbol{r}) - p^{-\kappa+1}(\boldsymbol{r})}{2\kappa} d\Gamma$ [12-14], and at $\varsigma = \sqrt{1+\kappa^2} - 1 > 0$, $q_A = \varsigma + |\kappa| + 1$ is the Abe entropy [37]. On the basis of these entropies, in particular, the corresponding statistical thermodynamics are constructed [19,36]. In the weak-coupling limit, when $\varsigma \to 0$ and $\kappa \to 0$, the entropy $S_{\kappa,\varsigma}$ goes over into the classical formula $S_{\mathrm{BG}} := -\int p(\boldsymbol{r}) p(\boldsymbol{r}) d\Gamma$ of Gibbs statistics; indeed, for $\kappa \to 0$ we have $p^{\pm|\kappa|} = \mathrm{e}^{\pm|\kappa|\ln p} \to 1 \pm |\kappa| \ln p$: and, therefore,

$$S_{\{\kappa\to 0,\varsigma=0\}} = \lim_{\kappa\to 0}\left\{-\int p(\boldsymbol{r}) \frac{p(\boldsymbol{r})^{\kappa} - p(\boldsymbol{r})^{-\kappa}}{2\kappa} d\Gamma\right\} = S_{\mathrm{BG}}.$$

Equilibrium probability distribution. The modified equilibrium Gibbs distribution in the STM statistics can be obtained by maximizing the (κ,ς)-entropy (3) under the following additional conditions: given the total energy of the system

$$\mathrm{E}_{\{\kappa,\varsigma\}} := \langle \varepsilon \rangle_{\{\kappa,\varsigma\}} = \int \varepsilon(\boldsymbol{r}) p(\boldsymbol{r}) d\Gamma \tag{6}$$

and preserving the probabilistic normalization (1) of the distribution $p(\boldsymbol{r})$. Here $\varepsilon(\boldsymbol{r})$ is the internal energy of the system in the state $\boldsymbol{r}$, which is determined by the mathematical model of the studied physical processes.

Using the Jaynes [38] methodology, standard in information theory and statistical mechanics, to obtain the extreme distribution from the variational principle, we introduce the functional

$$\mathrm{L}(p) := -\int p(\boldsymbol{r}) \ln_{\{\kappa,\varsigma\}}\left[p(\boldsymbol{r})\right] d\Gamma + \beta\mu \int p(\boldsymbol{r}) d\Gamma - \beta \int \varepsilon(\boldsymbol{r}) p(\boldsymbol{r}) d\Gamma \tag{7}$$

and find its unconditional extremum. This β, $\beta\mu$ is the essence of the Lagrange multipliers; μ – "chemical potential". In accordance with Lagrange's theorem, the probability distribution $p^{eq}(\boldsymbol{r})$ "maximizing" the entropy $S_{\{\kappa,\varsigma\}}(p)$ under the indicated constraints is determined from the condition:

$$\frac{\delta \mathrm{L}(p)}{\delta p} = -(1+\varsigma) \ln_{\{\kappa,\varsigma\}}\left[p(\boldsymbol{r})\right] - u_{\{\kappa,\varsigma\}}\left[p(\boldsymbol{r})\right] - \beta\left[\varepsilon(\boldsymbol{r}) - \mu\right] = 0. \tag{8}$$

This implies

$$\lambda \ln_{\{\kappa,\varsigma\}}\left(\frac{p^{eq}(\boldsymbol{r})}{\alpha}\right) = -\beta\left[\varepsilon(\boldsymbol{r}) - \mu\right], \tag{9}$$

where

$$\lambda := \frac{(1+\varsigma-\kappa)^{\frac{\varsigma+\kappa}{2\kappa}}}{(1+\varsigma+\kappa)^{\frac{\varsigma-\kappa}{2\kappa}}}, \quad \alpha := \left(\frac{1+\varsigma-\kappa}{1+\varsigma+\kappa}\right)^{\frac{1}{2\kappa}}, \tag{10}$$

moreover,

$$\alpha = \exp_{\kappa,-\varsigma}(-1/\lambda), \quad (1+\varsigma\pm\kappa) = \lambda\alpha^{-\varsigma\mp\kappa}. \tag{10*}$$

Expression (9) can be written in the form of the following generalized Gibbs distribution in statistics (GMD)

$$p^{eq}(\boldsymbol{r},\beta) = \alpha\exp_{\{\kappa,\varsigma\}}\left\{-\frac{\beta}{\lambda}\left[\varepsilon(\boldsymbol{r})-\mu\right]\right\}, \tag{11}$$

which in the limit $\kappa,\varsigma\to 0$ reduces to the microcanonical Gibbs distribution of classical statistics [39].

In relations (8), (9), and (11), the following functions important in the statistical mechanics of STM appear:

[1] Function conjugate to the deformed logarithm (4) [18, 40]

$$u_{\{\kappa,\varsigma\}}(x) := x^{\varsigma}\frac{x^{\kappa}+x^{-\kappa}}{2}, \quad (\kappa,\varsigma)\in \mathrm{R}\,. \tag{12}$$

[2] The Kanyadakis exponent $\exp_{\{\kappa,\varsigma\}}(x)$, which is the inverse of the (κ,ς)-logarithm

$$\ln_{\{\kappa,\varsigma\}}\left[\exp_{\{\kappa,\varsigma\}}(x)\right] = \exp_{\{\kappa,\varsigma\}}\left[\ln_{\{\kappa,\varsigma\}}(x)\right] = x\,. \tag{13}$$

Here are some useful properties of the generalized functions introduced above, which will be used below:

$$\ln_{\{\kappa,\varsigma\}}(xy) = u_{\{\kappa,\varsigma\}}(x)\ln_{\{\kappa,\varsigma\}}(y) + u_{\{\kappa,\varsigma\}}(y)\ln_{\{\kappa,\varsigma\}}(x), \tag{14}$$

$$\ln_{\{\kappa,\varsigma\}}(x) = -\ln_{\{\kappa,-\varsigma\}}\left(\frac{1}{x}\right), \quad u_{\{\kappa,\varsigma\}}(x) = u_{\{\kappa,-\varsigma\}}\left(\frac{1}{x}\right), \tag{15}$$

$$\lambda\ln_{\{\kappa,\varsigma\}}\left(\frac{x}{\alpha}\right) = u_{\{\kappa,\varsigma\}}(x) + (1+\varsigma)\ln_{\{\kappa,\varsigma\}}(x), \tag{16}$$

$$\frac{d}{dx}\left[x\ln_{\{\kappa,\varsigma\}}(x)\right] = \lambda\ln_{\{\kappa,\varsigma\}}\left(\frac{x}{\alpha}\right), \tag{17}$$

$$u_{\{\kappa,\varsigma\}}\left(\frac{1}{\alpha}\right) = \frac{1+\varsigma}{\lambda}, \quad \ln_{\{\kappa,\varsigma\}}\left(\frac{1}{\alpha}\right) = \frac{1}{\lambda}, \tag{18}$$

$$x\frac{d}{dx}\left[\lambda\ln_{\{\kappa,\varsigma\}}\left(\frac{x}{\alpha}\right)\right] = \frac{d}{dx}\left[xu_{\{\kappa,\varsigma\}}(x)\right] + \varsigma\frac{d}{dx}\left[x\ln_{\{\kappa,\varsigma\}}x\right], \tag{19}$$

$$\frac{d}{dx}\left[xu_{\{\kappa,\varsigma\}}(x)\right]=(1+\varsigma)u_{\{\kappa,\varsigma\}}(x)+\kappa^2\ln_{\{\kappa,\varsigma\}}(x)=\lambda u_{\{\kappa,\varsigma\}}\left(\frac{x}{\alpha}\right). \tag{20}$$

Equilibrium thermodynamics can be constructed on the basis of nonextensive (κ,ς)- entropy and taking into account the equilibrium distribution (11) [19,41]. Let's consider some of the elements of this thermodynamics.

Statistical thermodynamics relations for (κ,ς)-systems. Using the equilibrium distribution (11), it is easy to obtain the expression:

$$(1+\varsigma)\ln_{\{\kappa,\varsigma\}}\left[p^{eq}(\boldsymbol{r})\right]+u_{\{\kappa,\varsigma\}}\left[p^{eq}(\boldsymbol{r})\right]+\beta\left[\varepsilon(\boldsymbol{r})-\mu\right]=0\,, \tag{21}$$

averaging which with the help of the distribution $p^{est}(\boldsymbol{r})$ we obtain the extreme value of the (κ,ς) -entropy

$$S^{eq}_{\{\kappa,\varsigma\}}=\frac{1}{1+\varsigma}\left[\mathrm{I}^{\,eq}_{\{\kappa,\varsigma\}}+\beta\left(\mathrm{E}^{eq}_{\{\kappa,\varsigma\}}-\mu\right)\right]. \tag{22}$$

Further, in those cases when this does not cause ambiguity, the index of the thermodynamic parameters will be omitted.

The function $\mathrm{I}^{\,eq}_{\kappa,\varsigma}$, that plays an important role in the Sharma−Taneja−Mittal statistics is given by the expression

$$\mathrm{I}_{\{\kappa,\varsigma\}}(p^{eq}):=\int p^{eq}\,u_{\{\kappa,\varsigma\}}(p^{eq})d\Gamma=-(1+\varsigma)\left\langle\ln_{\{\kappa,\varsigma\}}(p^{eq})\right\rangle+\lambda\left\langle\ln_{\{\kappa,\varsigma\}}\left(p^{eq}/\alpha\right)\right\rangle=$$

$$=(1+\varsigma)S^{eq}_{\{\kappa,\varsigma\}}-\beta(\mathrm{E}^{eq}_{\{\kappa,\varsigma\}}-\mu)\,. \tag{23}$$

Further, in those cases when this does not cause ambiguity, the index of the thermodynamic parameters will be omitted.

Differentiating expression (3) for $S_{\{\kappa,\varsigma\}}$ with respect to internal energy $\mathrm{E}_{\{\kappa,\varsigma\}}$, taking into account (11) and (17), as a result, we obtain

$$\frac{dS_{\{\kappa,\varsigma\}}}{d\mathrm{E}_{\{\kappa,\varsigma\}}}=-\int\frac{d}{dp(\boldsymbol{r})}\left\{p(\boldsymbol{r})\ln_{\{\kappa,\varsigma\}}\left[p(\boldsymbol{r})\right]\right\}\frac{dp(\boldsymbol{r})}{d\mathrm{E}_{\{\kappa,\varsigma\}}}d\Gamma=$$

$$=-\lambda\int\ln_{\{\kappa,\varsigma\}}\left(\frac{p(\boldsymbol{r})}{\alpha}\right)\frac{dp(\boldsymbol{r})}{d\mathrm{E}_{\{\kappa,\varsigma\}}}d\Gamma=\int\beta\left[\varepsilon(\boldsymbol{r})-\mu\right]\frac{dp(\boldsymbol{r})}{d\mathrm{E}_{\{\kappa,\varsigma\}}}d\Gamma\,. \tag{24}$$

When using (1) and (6), this equality is reduced to a generalization of the definition of inverse temperature:

$$dS_{\{\kappa,\varsigma\}}/d\mathrm{E}_{\{\kappa,\varsigma\}}=\beta:=T^{-1}\,. \tag{25}$$

It is important to keep in mind that the STM entropy and the corresponding statistical mechanics preserve a number of properties of the classical Boltzmann–Gibbs theory [39]. In particular, the Legendrian structure of thermostatics was investigated in the works cited above by introducing several thermodynamic potentials. It was found that introduced by the relation

$$\boldsymbol{Z}_{\{\kappa,\varsigma\}}(\beta) := \exp_{\{\kappa,\varsigma\}}\left[\frac{1}{1+\varsigma}\left(\mathrm{I}_{\{\kappa,\varsigma\}} - \beta(\varsigma \mathrm{E}_{\{\kappa,\varsigma\}} + \mu)\right)\right] \quad (26)$$

analogue of classical partition function, satisfies the Legendre relation

$$\frac{d}{d\beta}\ln_{\{\kappa,\varsigma\}}(\mathrm{Z}_{\{\kappa,\varsigma\}}) = -\mathrm{E}_{\{\kappa,\varsigma\}} \ . \quad (27)$$

In this case, the (κ,ς)-entropy and the generalized statistical sum are related by the relation

$$S_{\{\kappa,\varsigma\}} = \ln_{\{\kappa,\varsigma\}}(\boldsymbol{Z}_{\{\kappa,\varsigma\}}) + \beta \mathrm{E}_{\{\kappa,\varsigma\}} . \quad (28)$$

A combination of expressions (22) and (26) leads to the basic thermodynamic identity

$$S_{\{\kappa,\varsigma\}} = \ln_{\{\kappa,\varsigma\}}(\boldsymbol{Z}_{\{\kappa,\varsigma\}}) + \beta \mathrm{E}_{\{\kappa,\varsigma\}} = \beta\left(\mathrm{E}_{\{\kappa,\varsigma\}} - \mathrm{F}_{\{\kappa,\varsigma\}}\right), \quad (29)$$

where free energy is defined by the equality

$$\mathrm{F}_{\{\kappa,\varsigma\}}(p) := -\beta^{-1}\ln_{\{\kappa,\varsigma\}}(\boldsymbol{Z}_{\{\kappa,\varsigma\}}) . \quad (30)$$

Thus, the (κ,ς) -free energy satisfies the Legendre structural transformations generalized by the following relations [19]

$$\mathrm{F}_{\{\kappa,\varsigma\}}(p) = \mathrm{E}_{\{\kappa,\varsigma\}}(p) - \frac{S_{\{\kappa,\varsigma\}}(p)}{\beta}, \quad \frac{d}{d\beta}(\beta \mathrm{F}_{\kappa,\varsigma}) = \mathrm{E}_{\kappa,\varsigma} \ . \quad (31)$$

3 BREGMAN'S DIVERGENCE. GENERALIZED *H*-THEOREM

Let us now show that during a spontaneous transition between an arbitrary state of a system with a distribution $p(\boldsymbol{r},t)$ and an equilibrium state with a distribution $p^{eq}(\boldsymbol{r})$, the entropy of the system can only decrease, i.e. $S_{\{\kappa\varsigma\}}[p^{eq}(\boldsymbol{r})] \geq S_{\{\kappa\varsigma\}}[p(\boldsymbol{r},t)]$.

To this end, we introduce into consideration the so-called Bregman divergence [42,43]

$$\mathrm{B}_{\{\kappa,\varsigma\}}[u:p] = S_{\{\kappa,\varsigma\}}(p) - S_{\{\kappa,\varsigma\}}(u) + \int \frac{\partial S_{\{\kappa,\varsigma\}}(p)}{\partial p}(u-p)\,d\Gamma \geq 0 \ , \quad (32)$$

which belongs to the most significant statistical characteristics of a nonextensive dynamic (κ,ς) -system [44]. As a functional, it determines the measure of statistical ordering in the microstates of a system with a distribution $p(\boldsymbol{r})$ relative to a state with a distribution $u(\boldsymbol{r})$. Expression (32) is a functional for two normalized distributions $\int p(\boldsymbol{r},t)\,d\Gamma = \int u(\boldsymbol{r},t)\,d\Gamma = 1$.

Various properties of the general form of Bregman divergence can be found in [43]. Here we note that the quantity $\mathrm{B}_{\{\kappa,\varsigma\}}[u:p]$ is a real, positive, convex (in the first argument) functional. In addition, since for $p(\boldsymbol{r})=u(\boldsymbol{r})$ equality $\mathrm{B}_{\{\kappa,\varsigma\}}[p:p]=0$ holds, the Bregman divergence is a Lyapunov function [i)].

Taking into account property (16) for the deformed logarithm $\ln_{\{\kappa,\varsigma\}}$, expression (32) can be given the following form:

$$\mathrm{B}_{\{\kappa,\varsigma\}}[u:p]=S_{\{\kappa,\varsigma\}}(p)-S_{\{\kappa,\varsigma\}}(u)-\lambda\int(u-p)\ln_{\{\kappa,\varsigma\}}(p/\alpha)d\Gamma\geq 0\,. \tag{33}$$

Now let the distribution $u(\boldsymbol{r})$ be equilibrium, for which the expression is valid:

$$-\lambda\ln_{\{\kappa,\varsigma\}}[p^{eq}(\boldsymbol{r})/\alpha]=\beta\,[\varepsilon(\boldsymbol{r})-\mu]\equiv z(\boldsymbol{r})\,, \tag{34}$$

and the distribution $p\equiv p(\boldsymbol{r},t)$ characterizes an arbitrary (but close to equilibrium) state of the system. In addition, suppose that for both distributions the equality holds (the so-called Gibbs condition)

$$\int p(\boldsymbol{r},t)\,z(\boldsymbol{r})d\Gamma=\int p^{eq}(\boldsymbol{r})\,z(\boldsymbol{r})d\Gamma\,. \tag{35}$$

Let us show that in this case the equality

$$(\varsigma+1)\int(p^{eq}-p)\ln_{\{\kappa,\varsigma\}}p^{eq}d\Gamma=\int(p-p^{\,eq})u_{\{\kappa,\varsigma\}}(p)d\Gamma\,. \tag{36}$$

Indeed, combining (11) and (16), we find the equality

$$(1+\varsigma)\ln_{\{\kappa,\varsigma\}}p^{eq}=-u_{\{\kappa,\varsigma\}}(p^{eq})-z(\boldsymbol{r})\,, \tag{37}$$

using which together with (35), we obtain (36).

On the other hand, when using definition (34) and formulas (1) and (6) (with replacement $p\Rightarrow u$), we get the expression:

$$S_{\{\kappa,\varsigma\}}(p):=\int p\ln_{\{\kappa,\varsigma\}}(p)d\Gamma=-\mathrm{B}_{\{\kappa,\varsigma\}}[p:p^{eq}]+$$
$$+\int p^{eq}\ln_{\{\kappa,\varsigma\}}(p^{eq})\,d\Gamma-\lambda\int(p-p^{eq})\ln_{\{\kappa,\varsigma\}}(p^{eq}/\alpha)\,, \tag{38}$$

which, taking into account formula (16), takes the form:

$$S_{\{\kappa,\varsigma\}}(p)=-\mathrm{B}_{\{\kappa,\varsigma\}}[p:p^{eq}]+\int p^{eq}\ln_{\{\kappa,\varsigma\}}(p^{eq})\,d\Gamma-$$
$$-\int(p-p^{eq})\,u_{\{\kappa,\varsigma\}}(p^{eq})d\Gamma-(1+\varsigma)\int(p-p^{eq})\ln_{\{\kappa,\varsigma\}}(p^{eq})\,. \tag{39}$$

Finally, from (36) and (39) the desired result follows [44]

[i)] Recall that the Lyapunov function is a function of a definite sign that vanishes at the equilibrium point of the system. An equilibrium state is an attractor when the time derivative of the Lyapunov function has a sign opposite to that of the function itself.

$$S_{\{\kappa,\varsigma\}}[p(\boldsymbol{r},t)] = S_{\{\kappa,\varsigma\}}[p^{eq}(\boldsymbol{r})] - \mathrm{B}_{\{\kappa,\varsigma\}}[p : p^{eq}] \le S_{\{\kappa,\varsigma\}}[p^{eq}(\boldsymbol{r})] . \quad (40)$$

Here the Bregman divergence appears as a negative contribution to the Sharma−Taneja−Mittal entropy and manifests itself as negentropy [45]. It follows from (40) that an increase in the entropy of the system up to its maximum value in equilibrium occurs together with the loss of idiscrimination information, that is, there is a joint increase in statistical disorder and a decrease in the statistical ordering of microstates of a nonextensive system.

Gibbs' theorem and *H*-theorem. Since, according to the convexity property [43] the Bregman divergence $\mathrm{B}_{\{\kappa,\varsigma\}}[p^{eq} : p]$ is a sign-definite Lyapunov function, in order for the state of complete equilibrium $p^{eq}(\boldsymbol{r})$ to be stable, the following inequality must be satisfied

$$\frac{d}{dt}\mathrm{B}_{\{\kappa,\varsigma\}}[p : p^{eq}] = -\frac{d}{dt}\left\{S_{\{\kappa,\varsigma\}}[p(\boldsymbol{r},t)] - S_{\{\kappa,\varsigma\}}[p^{eq}(\boldsymbol{r})]\right\} \le 0 . \quad (41)$$

Thus, as the system tends to an equilibrium state in time evolution, the magnitude of the Bregman divergence decreases. Moreover, from (41) follows the *H*-theorem for open nonequilibrium nonextensive (κ,ς)-systems

$$\frac{d}{dt}S_{\{\kappa,\varsigma\}}[p(\boldsymbol{r},t)] > 0 , \quad (42)$$

which is valid when approaching the state of complete statistical equilibrium. This theorem states that the Sharma−Taneja−Mittal entropy of the system continuously increases in the direction of equilibrium, where it reaches a final value and becomes maximum. It is in this way that a spontaneous transition from an arbitrary nonequilibrium state to an equilibrium state occurs, at which the degree of disorder of the (κ,ς) -system increases and at equilibrium reaches its maximum value.

4 RELATIONSHIP OF THE FOKKER−PLANK−KOLMOGOROV EQUATION ENTROPY SYSTEM

Anomalous diffusion phenomena are very common in nature and can be adequately described using the nonlinear Fokker−Planck−Kolmogorov equations, which have found wide application in various natural scientific fields, such as astrophysics, plasma physics, quantum mechanics, general and special theories of relativity, nonlinear hydrodynamics, biophysics, etc. The phenomena considered in them have a common physical mechanism arising due to the cooperative interaction between individual subsystems of the overall system. Cooperative interactions lead to a decrease in a large number of degrees of freedom of systems of many bodies and, thereby, connect individual subsystems through the process of self-organization into synergistic objects. Such synergistic systems allow low-dimensional descriptions in terms of nonlinear FPK equations, which are characterized by specific types of nonlinear diffusion contributions.

Such contributions can be associated, in particular, with non-extensive statistical mechanics. In the scientific literature, situations where diffusion contributions are written as a degree of probability density have been studied in most detail (see, for example, [21,24, 46-54]).

Recently, the method of constructing the FPK equation for any nonextensive physical system, associated with the local production of its entropy, has become widespread. This method was developed on the basis of linear nonequilibrium thermodynamics by T. Frank [22,55] and its content is detailed in the monograph [23].

The essence of this method is as follows: The starting point is the local continuity equation for the probability density $p(\boldsymbol{r},t)$ in the phase space

$$\frac{\partial}{\partial t}p(\boldsymbol{r},t)+\nabla_{\boldsymbol{r}}\boldsymbol{J}(\boldsymbol{r},t)=0\,, \tag{43}$$

which takes place both in physical space $\boldsymbol{q}$ and in the vector space of velocities $\boldsymbol{v}$. In this case, the nonlinear flow of probability is given by the relation

$$\boldsymbol{J}(\boldsymbol{r},t):=-p(\boldsymbol{r},t)\nabla_{\boldsymbol{r}}\left(\frac{\delta\mathrm{F}\ (p)}{\delta p}\right). \tag{44}$$

where the quantity $\nabla_{\boldsymbol{r}}\left(\delta\mathrm{F}\ (p)/\delta p\right)$ is the thermodynamic force and $\mathrm{F}\ (p)$ is the free energy for the problem under consideration [23].

Further, when constructing the FPK equation on the basis of the STM entropy functional, for simplicity, we restrict ourselves to considering the classical stochastic Markov process in the velocity space $\boldsymbol{v}$, which is described by the distribution function $p(\boldsymbol{v},t)$, information entropy $S_{\{\kappa,\varsigma\}}(p)$ and average energy $\mathrm{E}_{\{\kappa,\varsigma\}}(p)$. Then the functional is given by the expression [23]

$$\mathrm{F}_{\{\kappa,\varsigma\}}(p):=\mathrm{E}_{\{\kappa,\varsigma\}}(p)-DS_{\{\kappa,\varsigma\}}(p)\,, \tag{45}$$

in which

$$\mathrm{E}_{\{\kappa,\varsigma\}}(p)=\int\varepsilon(\boldsymbol{v})\,p_{\{\kappa,\varsigma\}}(\boldsymbol{v})d\boldsymbol{v}\,,\quad S_{\{\kappa,\varsigma\}}(p)=-\int p_{\{\kappa,\varsigma\}}(\boldsymbol{v})\ln_{\{\kappa,\varsigma\}}[p_{\{\kappa,\varsigma\}}(\boldsymbol{v})]d\boldsymbol{v}\,, \tag{46}$$

$D=D_{\{\kappa,\varsigma\}}(\boldsymbol{v},t)$ − diffusion coefficient (or noise intensity coefficient), which plays the role of temperature in the velocity space (in the general case $D\neq 1/\beta$); $\varepsilon(\boldsymbol{v})=\frac{m}{2}\boldsymbol{v}^2$ − kinetic energy of a particle (hereinafter we will assume that $m=1$).

Taking into account formula (17), when calculating the variational derivative $\delta\mathrm{F}\ (p)/\delta p$, we obtain:

$$\frac{\delta}{\delta p}\mathrm{F}\ (p)=\varepsilon(\boldsymbol{v})+D\left\{\lambda\ln_{\{\kappa,\varsigma\}}\left[\frac{p(\boldsymbol{v})}{\alpha}\right]\right\}. \tag{47}$$

Accordingly, for the flow of probability, taking into account formula (19), we will have

$$\boldsymbol{J}_{\{\kappa,\varsigma\}}(\boldsymbol{v},t)=-p(\boldsymbol{v},t)\nabla_{\boldsymbol{v}}\left(\frac{\delta\mathrm{F}\ (p)}{\delta p}\right)=$$

$$= -p(v,t)\nabla_v \varepsilon(v) - D_{\{\kappa,\varsigma\}} p(v,t)\nabla_v \left[\lambda \ln_{\{\kappa,\varsigma\}}\left(\frac{p(v,t)}{\alpha}\right)\right] = \qquad (48)$$

$$= F(v)p(v,t) - D_{\kappa,\varsigma}\nabla_v \left\{p(v,t)u_{\{\kappa,\varsigma\}}\left[p(v,t)\right]\right\} - \varsigma D_{\{\kappa,\varsigma\}}\nabla_v \left\{p(v,t)\ln_{\{\kappa,\varsigma\}}\left[p(v,t)\right]\right\},$$

and when using relation $\lambda\alpha^{-\varsigma\mp\kappa} = 1+\varsigma\pm\kappa$ and definition (4) for the function $\ln_{\kappa,\varsigma}(x)$, we obtain a different representation of the probability flow

$$J_{\{\kappa,\varsigma\}}(v,t) = F(v)\,p(v,t) - D_{\{\kappa,\varsigma\}} p(v,t)\nabla_v \left[\lambda \ln_{\{\kappa,\varsigma\}}\left(\frac{p(v,t)}{\alpha}\right)\right] =$$

$$= F(v)p(v,t) - D_{\{\kappa,\varsigma\}}\left[p(v,t)\,\nabla_v\left(\frac{\lambda\alpha^{-(\kappa+\varsigma)}}{2\kappa}p(v,t)^{\varsigma+\kappa} - \frac{\lambda\alpha^{(\kappa-\varsigma)}}{2\kappa}p(v,t)^{\varsigma-\kappa}\right)\right]$$

$$= F(v)\,p(v,t) - D_{\{\kappa,\varsigma\}}\left[p(v,t)\nabla_v\left(\frac{\varsigma+\kappa+1}{2\kappa}p(v,t)^{\varsigma+\kappa} - \frac{\varsigma-\kappa+1}{2\kappa}p(v,t)^{\varsigma-\kappa}\right)\right]. \qquad (49)$$

Here $F(v) = -\nabla_v \varepsilon(v) = -v$ is the linear drift coefficient. Thus, the nonlinear power-law FPK equation in the SHTM kinetics has the form:

$$\frac{\partial p(v,t)}{\partial t} = -\nabla_v\left[F(v)p(v,t)\right] +$$

$$+ D_{\{\kappa,\varsigma\}}(t)\nabla_v\left[p(v,t)\nabla_v\left(\frac{\varsigma+\kappa+1}{2\kappa}p(v,t)^{\varsigma+\kappa} - \frac{\varsigma-\kappa+1}{2\kappa}p(v,t)^{\varsigma-\kappa}\right)\right]. \qquad (50)$$

Here and below, for simplicity, we will assume that the diffusion coefficient $D_{\{\kappa,\varsigma\}}$ depends only on time.

If a parameter $\varsigma = 0$, then from (48) and (19) it follows

$$J_{\{\kappa\}}(v,t) = -F(v)\,p_{\{\kappa\}}(v,t) - D_{\{\kappa\}}(t)\nabla_v\left[p_{\{\kappa\}}(v,t)u_{\{\kappa\}}(v,t)\right], \qquad (51)$$

and, accordingly, the FPK equation in the Kanyadakis statistics takes the form:

$$\frac{\partial}{\partial t}p(v,t) = -\nabla_v\left[F(v)\,p(v,t)\right] + D_{\{\kappa\}}(t)\nabla_v^2\left[\frac{p(v,t)^{1+\kappa} + p(v,t)^{1-\kappa}}{2}\right]. \qquad (52)$$

If the parameters κ, ς and q are related by the relation $\varsigma = \pm|\kappa| = (1-q)/2$, then definition (48) implies the following expression for the probability flow

$$J_q(v,t) = F(v)\,p(v,t) - D_{\{2-q\}}(t)\nabla_v\left[p(v,t)^{2-q}\right]. \qquad (53)$$

Then for the FPK equation in the $(2-q)$ -formalism of the Tsallis statistics we obtain

$$\frac{\partial}{\partial t}p(\boldsymbol{v},t)=-\nabla_{\boldsymbol{v}}\left[F(\boldsymbol{v})p(\boldsymbol{v},t)\right]+D_{\{2-q\}}(t)\nabla_{\boldsymbol{v}}^2\left[p(\boldsymbol{v},t)^{2-q}\right]. \tag{54}$$

If $q=1$, then equation (53) describes the Ornstein−Uhlenbeck processes [56].

5 ELEMENTS OF (*2 - q*) - TSALLIS FORMALISM

Next, we will consider methods for finding solutions to the obtained nonstationary Fokker−Planck−Kolmogorov equations using the example of the one-dimensional power equation (54) valid in the Tsallis $(2-q)$ -formalism. For this purpose, let us recall some elements of this formalism obtained in the framework of the statistical mechanics of the STM. First of all, we find the deformed equilibrium Gibbs distribution in the $(2-q)$-statistics. If $\varsigma=\pm|\kappa|=(1-q)/2$, then the definition (2) of entropy $S_{\{\kappa,\varsigma\}}$ implies the following expression for the Tsallis $(2-q)$ – entropy

$$S_{\{\kappa,\varsigma\}}(p)\to S_{\{2-q\}}:=-\int\frac{p^{2-q}(v)-p(v)}{1-q}dv=-\int p(v)\ln_{\{q\}}[p(v)]dv\,. \tag{55}$$

Since in this case

$$\lambda:=\frac{(1+\varsigma-\kappa)^{\frac{\varsigma+\kappa}{2\kappa}}}{(1+\varsigma+\kappa)^{\frac{\varsigma-\kappa}{2\kappa}}}\Rightarrow 1,\quad \alpha:=\left(\frac{1+\varsigma-\kappa}{1+\varsigma+\kappa}\right)^{\frac{1}{2\kappa}}\Rightarrow\alpha':=\left(2-q\right)^{\frac{1}{q-1}}, \tag{56}$$

$$\ln_{\{\kappa,\varsigma\}}(x):=\frac{x^{\varsigma+\kappa}-x^{\varsigma-\kappa}}{2\kappa}\Rightarrow\ln_{\{q\}}(x):=\frac{x^{1-q}-1}{1-q}, \tag{57}$$

then formula (9) for the equilibrium distribution $p^{eq}(v)$ in $(2-q)$- statistics takes the form:

$$\ln_{\{q\}}\left(\alpha'^{-1}p^{eq}(v)\right)=-\beta\left(\varepsilon(v)-\mu\right), \tag{58}$$

or

$$p^{eq}(v)=\alpha'\exp_{\{q\}}\left\{-\beta\left[\varepsilon(v)-\mu\right]\right\}. \tag{58*}$$

Here

$$\ln_{\{q\}}(x):=\frac{x^{1-q}-1}{1-q},\quad \exp_{\{q\}}(x):=\begin{cases}\left[1+(1-q)x\right]^{1/(1-q)}, & \text{if}\quad 1+(1-q)x>0,\\ 0, & \text{if}\quad 1+(1-q)x<0\end{cases}$$

— the deformed logarithm and the Tsallis exponent, respectively; q − the deformation parameter.

Equivalent equilibrium distribution function. Further, another representation of the equilibrium probability distribution function (58) of the system in equilibrium will be used. To obtain it, let us average (58) over the distribution $p^{eq}(v)$; as a result we will have

$$\left\langle \ln_{\{q\}}\left[(\alpha')^{-1}p^{eq}(v)\right]\right\rangle = -\beta(\mathrm{E}_{\{q\}}-\mu), \tag{59}$$

where $\mathrm{E}_{\{q\}}=\int \varepsilon(v)p^{eq}(v)dv$ is the average energy of the system. On the other hand, taking into account the formulas $\ln_{\{q\}}(x/y)=y^{q-1}\left[\ln_{\{q\}}(x)-\ln_{\{q\}}(y)\right]$ and $\ln_{\{q\}}\alpha'=1/(q-2)$, one can obtain

$$\left\langle \ln_{\{q\}}\left[(\alpha')^{-1}p^{eq}(v)\right]\right\rangle = \left(2-q\right)\int p^{eq}\left[\ln_{\{q\}}(p^{eq})-1)\right]dv = 1-\left(2-q\right)S^{eq}_{\{2-q\}}. \tag{60}$$

From (59) and (60) it follows

$$\left(2-q\right)S^{eq}_{\{2-q\}}-1=\beta(\mathrm{E}_{\{q\}}-\mu). \tag{61}$$

Using the Tsallis exponent property now [57]

$$\exp_{\{q\}}(x+y)=\exp_{\{q\}}(x)\cdot\exp_{\{q\}}\left[\frac{y}{1+(1-q)x}\right], \tag{62}$$

then another expression (equivalent to (58 *)) can be obtained for the equilibrium probability distribution function in the Tsallis ($2-q$) -statistics

$$p^{eq}(v)=\alpha'\exp_{\{q\}}\left\{-\beta\varepsilon(v)+\beta\mu\right\}=\alpha'\exp_{\{q\}}\left\{1-\left(2-q\right)S^{eq}_{\{2-q\}}-\beta\left[\varepsilon(v)-\mathrm{E}_{\{q\}}\right]\right\}=$$

$$=\alpha'\exp_{\{q\}}\left[1-\left(2-q\right)S^{eq}_{\{2-q\}}\right]\cdot\exp_{\{q\}}\frac{-\beta\left[\varepsilon(v)-\mathrm{E}_{\{q\}}\right]}{\left(2-q\right)\left[1-(1-q)S^{eq}_{\{2-q\}}\right]},$$

or in final form

$$p^{eq}(v)=\frac{1}{Z_{\{2-q\}}}\exp_{\{q\}}\left\{-\beta_{\{2-q\}}\left[\varepsilon(v)-\mathrm{E}_{\{q\}}\right]\right\}. \tag{63}$$

The following notation is introduced here for the normalization constant $Z_{\{2-q\}}$ and the inverse physical temperature $\beta_{\{2-q\}}$:

$$Z^{-1}_{\{2-q\}}:=\alpha'\exp_{\{q\}}\left[1-\left(2-q\right)S^{eq}_{\{2-q\}}\right]=\int\exp_q\left\{-\beta_{\{2-q\}}\left[\varepsilon(v)-\mathrm{E}_{\{q\}}\right]\right\}dv\ , \tag{64}$$

$$\beta_{\{2-q\}}:=\frac{\beta}{\left(2-q\right)\left[1-(1-q)S^{eq}_{\{2-q\}}\right]}. \tag{65}$$

The value $Z^{-1}_{\{2-q\}}$, taking into account the properties $\ln_{\{q\}}(x/y)=y^{q-1}(\ln_{\{q\}}x-\ln_{\{q\}}y)$ and $1/\exp_{\{q\}}(-x)=\exp_{\{2-q\}}(x)$ [10] and formulas $(\alpha')^{1-q}=(2-q)^{-1}$ and $\ln_{\{q\}}\alpha'^{-1}=1$, can be transformed to the following form:

$$\ln_{\{q\}}\left[\boldsymbol{Z}_{\{2-q\}}^{-1}\right]=(\alpha')^{1-q}\left[1-\left(2-q\right)S_{\{2-q\}}^{eq}-\ln_{q}(\alpha')^{-1}\right]=$$

$$=\left(2-q\right)^{-1}\left[-\left(2-q\right)S_{\{2-q\}}^{eq}\right]=-S_{\{2-q\}}^{eq}\,.$$

Hence, for the quantity $Z_{\{2-q\}}$ we obtain several equivalent forms of notation

$$\boldsymbol{Z}_{\{2-q\}}=1/\exp_{\{q\}}\left(-S_{\{2-q\}}^{eq}\right)=\exp_{\{2-q\}}\left(S_{\{2-q\}}^{eq}\right)=$$

$$=\left[1+(q-1)S_{\{2-q\}}^{eq}\right]^{1/(q-1)}=\left[\int\left(p^{eq}\right)^{2-q}(v)\,dv\right]^{1/(q-1)}. \tag{66}$$

From (66) and (67) there also follow several representations for the parameter

$$\beta_{\{2-q\}}=\frac{\beta\left(2-q\right)^{-1}}{1-(1-q)S_{\{2-q\}}^{eq}}=\frac{\beta}{\left(2-q\right)}\boldsymbol{Z}_{\{2-q\}}^{1-q}=\frac{\beta}{\left(2-q\right)}\left[\int\left(p^{eq}\right)^{2-q}(v)\,dv\right]^{-1}. \tag{67}$$

Further, we will use the values known in the literature for the quantities $\mathrm{E}_{\{q\}}$ and $\boldsymbol{Z}_{\{2-q\}}$ [58]

$$\mathrm{E}_{\{q\}}=\int\tfrac{1}{2}v^{2}p^{eq}(v)dv=\frac{1}{5-3q}\left(\frac{\beta}{2-q}\right)^{\frac{2}{q-3}}\left[\sqrt{\frac{2\pi}{q-1}}\,\frac{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)}\right]^{\frac{2(1-q)}{q-3}}, \tag{68}$$

$$\boldsymbol{Z}_{\{2-q\}}=\left[\int\left(p^{eq}\right)^{2-q}(v)\,dv\right]^{\frac{1}{q-1}}=\left[\sqrt{\frac{\pi}{\beta(q-1)}}\,\frac{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)}\right]^{\frac{1}{q-1}}. \tag{69}$$

Let us now extend these results to a ($2-q$)-system that is not in an equilibrium state, but evolves into this state in accordance with the kinetic process described by the power-law nonlinear Fokker−Planck−Kolmogorov equation [59].

6. SOLUTION OF A NON-STATIONARY EQUATION FOKKER-PLANK-KOLMOGOROVA IN (*2-q*) - FORMALISM

The most famous private analytical methods for solving the nonstationary FPK equation for a one-dimensional probability density are [60]:

- variable separation method,
- Laplace transform method,
- method of transformation of variables,
- the method of applying the Green's function.

In this work, to find solutions to the above nonstationary FPK equations, we use the so-called ansatz method.

Let's consider the essence of this method. At the first step, an assumption is made that the solution of the considered equation has a specific form of a function (ansatz), which depends on a number of undefined parameters. Formally, the initial guess, based on some heuristic considerations, can be confirmed only after the final solution of the equation has been found. Then, in the second step, the test function is substituted into the equation to be solved, which leads to a system of differential equations for free parameters, which, as a rule, is much easier to solve than the original equation. The ansatz approach is often an effective method for solving anomalous FPK equations, when it is possible to substitute a trial function into them and then check the solution. It is important to bear in mind that a modification of the locally equilibrium Gibbs distribution, which maximizes the entropy of the system under consideration, can often be used as such a test function. As it was shown in a series of works [21, 23, 24] this distribution coincides with the stationary solution (having a quasi-Gibbs form) of the corresponding anomalously diffusion FPK equation, constructed from the point of view of a linear nonequilibrium thermodynamics [23].

Note that in Sec. 2, the *H*-theorem was proved for the (κ,ς) -system, which provides a spontaneous increase in the entropy of the STM of an arbitrary nonequilibrium state of the stochastic (κ,ς) -system in the direction of equilibrium, where it becomes maximum. At the same time, this process can be described by the power-law diffusion equation of the FPK, since these equations can be used to analyze irreversible systems are both far from equilibrium and near thermal equilibrium (i.e., near the maximum entropy).

Let us now illustrate the ansatz approach by the example of solving the nonlinear non-stationary one-dimensional diffusion equations of the FPK, written in the velocity space v

$$\frac{\partial}{\partial t}p(v,t)=\frac{\partial}{\partial v}\Big[v\,p(v,t)\Big]+D_{\{2-q\}}\frac{\partial^2}{\partial v^2}\Big[p(v,t)^{2-q}\Big]. \tag{70}$$

As usual, we will assume that the probability distribution, together with its first derivative, as well as the drift term $F(v)p(v,t)$, should vanish at infinity:

$$p(v,t)\Big|_{v\to\pm\infty}=0,\quad \frac{\partial}{\partial v}p(v,t)\Big|_{v\to\pm\infty}=0,\quad F(v)\,p(v,t)\Big|_{v\to\pm\infty}=0,\quad (\forall t).$$

As a test function, we use the function (compare with the locally equilibrium distribution (64))

$$p(v,t)=\frac{1}{\boldsymbol{Z}_{\{2-q\}}(t)}\exp_{\{q\}}\left\{-\beta_{\{2-q\}}(t)\Big[\tfrac{1}{2}v^2-\mathrm{E}_{\{q\}}(t)\Big]\right\}, \tag{71}$$

where, $\boldsymbol{Z}_{\{2-q\}}(t)$, $\beta_{\{2-q\}}(t)$ and $\mathrm{E}_{\{q\}}(t)$ are time-dependent free parameters to be defined. For this purpose, we substitute (71) into the FPK equation (70) and find a system of first-order differential equations, which the indicated free parameters must satisfy, if the test function (71) is indeed a solution to equation (70).

Introducing the notation

$$p(v,t)=\frac{1}{\mathbf{Z}_{\{2-q\}}(t)}\Big[Q(v,t)\Big], \tag{72}$$

where

$$Q(v,t):=\exp_{\{q\}}\left\{-\beta_{\{2-q\}}(t)\left[\tfrac{1}{2}v^2-\mathrm{E}_{\{q\}}(t)\right]\right\}$$

and applying the rule $d[\exp_{\{q\}}(x)]=[\exp_{\{q\}}(x)]^q dx$ of differentiation of the Tsallis exponent, we obtain the following intermediate relations:

$$\frac{\partial p(v,t)}{\partial t}=\frac{\partial}{\partial t}\left(\frac{Q(v,t)}{\mathbf{Z}_{\{2-q\}}(t)}\right)=-\frac{Q(v,t)}{\mathbf{Z}^2_{\{2-q\}}(t)}\frac{\partial \mathbf{Z}_{\{2-q\}}(t)}{\partial t}+\frac{1}{\mathbf{Z}_{\{2-q\}}(t)}\frac{\partial Q(v,t)}{\partial t}, \tag{73}$$

$$\frac{\partial (v\,p)}{\partial v}=\frac{v}{\mathbf{Z}_{\{2-q\}}(t)}\frac{\partial Q(v,t)}{\partial v}+\frac{Q(v,t)}{\mathbf{Z}_{\{2-q\}}(t)}\left[1-\frac{v}{\mathbf{Z}_{\{2-q\}}(t)}\frac{\partial \mathbf{Z}_{\{2-q\}}(t)}{\partial v}\right], \tag{74}$$

$$\frac{\partial^2}{\partial v^2}\left[p^{2-q}\right]=\frac{2-q}{\mathbf{Z}^{2-q}_{\{2-q\}}(t)}Q(v,t)^{-q}\left[Q(v,t)\frac{\partial^2 Q(v,t)}{\partial v^2}+(1-q)\left(\frac{\partial Q(v,t)}{\partial v}\right)^2\right], \tag{75}$$

where

$$\frac{\partial Q(v,t)}{\partial t}=Q(v,t)^q\left\{-\frac{\partial \beta_{\{2-q\}}(t)}{\partial t}\left(\tfrac{1}{2}v^2-\mathrm{E}_{\{q\}}(t)\right)+\beta_{\{2-q\}}(t)\frac{\partial \mathrm{E}_{\{q\}}(t)}{\partial t}\right\},$$

$$\frac{\partial Q(v,t)}{\partial v}=Q(v,t)^q\left(-v\,\beta_{\{2-q\}}(t)\right),$$

$$\frac{\partial^2 Q(v,t)}{\partial v^2}=\beta_{2-q}(t)Q(v,t)^q\left(-1+qv^2\beta_{2-q}(t)Q(v,t)^{q-1}\right).$$

Substituting now (73)-(75) into (70), we obtain the equation

$$\frac{\partial \ln\left[\mathbf{Z}_{\{2-q\}}(t)\right]}{\partial t}+1-D_{\{2-q\}}\,\beta(t)-v^2Q^{q-1}(v,t)\beta_{\{2-q\}}(t)\left(1-D_{\{2-q\}}\,\beta(t)\right)-$$

$$-Q^{q-1}(v,t)\beta_{2-q}(t)\left\{\frac{\partial \mathrm{E}_q(t)}{\partial t}-\frac{\partial \ln[\beta_{2-q}(t)]}{\partial t}\left(\tfrac{1}{2}v^2-\mathrm{E}_q(t)\right)\right\}=0\quad. \tag{76}$$

Here we used the notation (cf. (68))

$$\beta(t):=\left(2-q\right)\beta_{\{2-q\}}(t)\left[\mathbf{Z}_{\{2-q\}}(t)\right]^{q-1}, \tag{77}$$

reducing at $t\to 0$ to the expression

$$\beta(0) := (2-q)\beta_{\{2-q\}}(0)\left[\boldsymbol{Z}_{\{2-q\}}(0)\right]^{q-1}. \qquad (78)$$

We will further assume that at $t=0$ the unknown parameters $\boldsymbol{Z}_{\{2-q\}}(t)$, $\beta_{\{2-q\}}(t)$ and $\mathrm{E}_{\{q\}}(t)$ coincide with their analogs in formula (64), i.e. fair identities $\boldsymbol{Z}_{\{2-q\}}(0) \equiv \boldsymbol{Z}_{\{2-q\}}$, $\beta_{\{2-q\}}(0) \equiv \beta_{\{2-q\}}$, $\mathrm{E}_{\{q\}}(0) = \mathrm{E}_{\{q\}}$ and $\beta(0) = \beta$.

Let us now consider in detail the procedure for solving the nonstationary equation (76). Since the parameter $\boldsymbol{Z}_{\{2-q\}}(t)$ does not depend on velocity v, two first-order equations follow from (76)

$$\frac{\partial}{\partial t}\ln\left[\boldsymbol{Z}_{\{2-q\}}(t)\right] = -1 + D_{\{2-q\}}\,\beta(t), \qquad (79)$$

$$\frac{\partial \mathrm{E}_{\{q\}}(t)}{\partial t} + \mathrm{E}_{\{q\}}(t)\frac{\partial}{\partial t}\ln\left[\beta_{\{2-q\}}(t)\right] = \tfrac{1}{2}v^2\left[\frac{\partial}{\partial t}\ln\left[\beta_{\{2-q\}}(t)\right] - 2\left(1 - D_{\{2-q\}}\,\beta(t)\right)\right] = 0 \;. \qquad (80)$$

For a similar reason, two equations also follow from (80)

$$\frac{\partial \beta_{\{2-q\}}(t)}{\partial t} = 2\beta_{\{2-q\}}(t)\left[1 - D_{\{2-q\}}\beta(t)\right], \qquad (81)$$

$$\frac{\partial \mathrm{E}_{\{q\}}(t)}{\partial t} = -2\mathrm{E}_{\{q\}}(t)\left[1 - D_{\{2-q\}}\beta(t)\right]. \qquad (82)$$

Using (79) and (81), we can obtain $\frac{\partial}{\partial t}\ln\left[\boldsymbol{Z}_{\{2-q\}}(t)\right] - \frac{\partial}{\partial t}\ln \mathrm{E}^{1/2}_{\{q\}}(t) = 0$, whence follows the algebraic relation

$$\boldsymbol{Z}_{\{2-q\}}(t) = \left(\frac{\boldsymbol{Z}_{\{2-q\}}(0)}{\mathrm{E}^{1/2}_{\{q\}}(0)}\right)\mathrm{E}^{1/2}_{\{q\}}(t) = \left(\frac{\boldsymbol{Z}_{\{2-q\}}}{\mathrm{E}^{1/2}_{\{q\}}}\right)\mathrm{E}^{1/2}_{\{q\}}(t) \qquad (83)$$

− formula that allows you to determine a parameter $\boldsymbol{Z}_{\{2-q\}}(t)$, if the value $\mathrm{E}_{\{q\}}(t)$ is known. Carrying out a similar procedure with equations (81) and (82), we obtain, taking into account (68), the algebraic relation

$$\beta_{\{2-q\}}(t) = \left(\beta_{\{2-q\}}\mathrm{E}_{\{q\}}\right)\frac{1}{\mathrm{E}_{\{q\}}(t)} = \frac{\beta}{(2-q)}\left(\boldsymbol{Z}^{1-q}_{\{2-q\}}\mathrm{E}_{\{q\}}\right)\frac{1}{\mathrm{E}_{\{q\}}(t)}. \qquad (84)$$

We now turn to the derivation of the equation for the function $\mathrm{E}_{\{q\}}(t)$ and its solution. Combining (83), (84), (77) and (78), for the quantity $\beta(t)$ we have

$$\beta(t) := \left(2-q\right)\beta_{\{2-q\}} \frac{\left[Z_{\{2-q\}}(0)\right]^{q-1}}{\left[\mathrm{E}_{\{q\}}(0)\right]^{(q-3)/2}} \mathrm{E}_{\{q\}}^{(q-3)/2}(t) \equiv \left[\frac{\beta}{\mathrm{E}_{\{q\}}^{(q-3)/2}}\right] \mathrm{E}_{\{q\}}^{(q-3)/2}(t)\,. \tag{85}$$

Substituting (85) into (82), we obtain the following equation for $\mathrm{E}_{\{q\}}(t)$

$$\frac{\partial \ln\left[\mathrm{E}_{\{q\}}(t)\right]}{\partial t} = -2 + 2\frac{\beta D_{\{2-q\}}}{\mathrm{E}_{\{q\}}^{(q-3)/2}}\left[\mathrm{E}_{\{q\}}(t)\right]^{(q-3)/2}. \tag{86}$$

If we introduce a substitution $\mathrm{E}_{\{q\}}(t) = w(t)^{-1/\lambda}$ and accept the notation $\lambda \equiv \frac{q-3}{2}$ and $b_{\{q\}} \equiv \beta D_{\{2-q\}}\mathrm{E}_{\{q\}}^{(3-q)/2}$, then equation (86) can be written in the form of the well-known Riccati equation

$$\frac{\partial w(t)}{\partial t} = 2\lambda w(t) - 2b_1\lambda\,. \tag{87}$$

The solution to this equation has the form

$$w(t) = b_{\{q\}} + C\exp(-2\lambda t)\,, \tag{88}$$

where $b_{\{q\}}$ is a particular solution of equation (87), C is the constant of integration. This implies the following algebraic relation for the required parameter $\mathrm{E}_{\{q\}}(t)$

$$\mathrm{E}_{\{q\}}(t) = \left\{\beta D_{\{2-q\}}\mathrm{E}_{\{q\}}^{(3-q)/2} + C\exp\left[(q-3)t\right]\right\}^{2/(3-q)}. \tag{89}$$

Determining the constant $C = \mathrm{E}_{\{q\}}^{(3-q)/2}(1-\beta D_{\{2-q\}})$ from (89) (for $t=0$), we finally have

$$\mathrm{E}_{\{q\}}(t) = \mathrm{E}_{\{q\}}\left\{\beta D_{\{2-q\}} + (1-\beta D_{\{2-q\}})\exp\left[(q-3)t\right]\right\}^{2/(3-q)}. \tag{90}$$

Thus, using expressions (68) and (69) for the quantities $\mathrm{E}_{\{q\}}$ and $Z_{\{2-q\}}$, we can determine the sought parameters $Z_{\{2-q\}}(t)$, $\beta_{\{2-q\}}(t)$ and $\mathrm{E}_{\{q\}}(t)$ of the problem, from formulas (83), (84), and (90).

7 CONCLUSIONS

Investigations in the field of statistical mechanics and thermodynamics of nonextensive systems have recently acquired considerable general theoretical interest in connection with the manifestations of nonextensive properties in many anomalous physical phenomena and the importance of practical applications. The range of applications of various non-extensive parametric entropies is constantly expanding, covering various areas in science, such as cosmology and cosmogony, quantum mechanics and statistics, special and general relativity, stochastic dynamics and fractals, geophysics, biomedicine, and many others. Among nonextensive entropies, the two-parameter Sharma–Taneja–Mittal entropy occupies a special place, since it allows one to obtain distributions that are observed in various physical, natural, and artificial systems.

In the presented work, within the framework of statistics based on the (κ,ς)-entropy of the STM, it is shown how one can obtain the equilibrium thermodynamics of a nonextensive system and determine its features. The basic mathematical properties of doubly deformed logarithm and exponent are presented, as well as other related functions that are necessary in the construction of nonextensive equilibrium thermodynamics. Taking into account the convexity property of Bregman's divergence, it is shown that the principle of maximum equilibrium entropy of the STM, the Legendrian structure of the theory, and the H-theorem describing the evolution of a chaotic system in time are preserved for (κ,ς)-systems.

An important aspect related to the derivation of nonlinear power-law FPK equations correlated with the Sharma–Taneja–Mittal entropy is analyzed. In this case, the resulting diffusion equations are written in such a way that their stationary solutions are probability distributions that maximize the entropy of the STM for nonextensive (κ,ς)-systems. The method of the ansatz approach is considered for solving the nonlinear nonstationary one-dimensional FPK equations written in the $(2-q)$-formalism Tsallis statistic.

The nonlinear diffusion equations of FPK constructed in this way can be used to solve problems of probabilistic analysis in many areas of science. In particular, the statistical thermodynamic approach, as well as the power-law FPK based on kappa statistics, can be useful in studying the spread of epidemics and pandemics.

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