SOME ASPECTS OF NEYMAN TRIANGLES AND DELANNOY ARRAYS

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Summary. This note considers some number theoretic properties of the orthonormal Neyman polynomials which are related to Delannoy numbers and certain complex Delannoy numbers.

1 INTRODUCTION

Rayner and Best point out that “the concept of smooth goodness of fitness tests was introduced in Neyman (1937)” [22]. Goodness of fit concepts in general usually go back to Karl Pearson [20]. Rayner [21] further pointed out that Jerzy Neyman’s smooth alternative of order \( k \) to the uniform distribution on \((0,1)\) has probability density for

\[
h(y, \theta) = \exp \left\{ \sum_{i=1}^{k} \theta_i \pi_i(y) - K(\theta) \right\}, \quad 0 < y < 1, \quad k = 1, 2, ...
\]  

(1.1)

where \( K(\theta) \) is a normalising constant and the \( \pi_i(y) \) are orthonormal polynomials (Freeman) related to the Legendre polynomials.

It is the purpose of this note to consider some number theoretic properties of the \( \pi_i(y) \) polynomials \((i = 0, 1, 2, 3, 4 \) in Rayner) which, for convenience, we label as Neyman polynomials. In Deveci and Shannon [9] complex-type \( k \)-Fibonacci numbers are defined and the relationships between the \( k \)-step Fibonacci numbers and the complex-type \( k \)-Fibonacci numbers are provided together with miscellaneous properties of the complex-type \( k \)-Fibonacci numbers. In addition, they studied the complex-type \( k \)-Fibonacci sequence modulo \( m \). Finally, they obtained the period of the complex-type \( 2 \)-Fibonacci sequences in the Dihedral group \( D_{2n} \), \((n \geq 2)\).

In this paper, we define the complex-type Delannoy numbers and then give the relationships between the Delannoy numbers and the complex-type Delannoy numbers. Furthermore, we study the complex-type Delannoy sequence modulo \( m \).

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Key words and Phrases: Neyman polynomials, Legendre polynomials, Delannoy numbers, Fibonacci numbers, Tribonacci triangles.
2 NEYMAN POLYNOMIALS

Rayner elsewhere lists the first five such polynomials and we add some more in order to build up a picture of patterns. To help with this we have slightly modified some aspects of his notation as in Bera and Ghosh [3]:

\[
\begin{align*}
\pi_0(y) &= \sqrt{1(1)} \\
\pi_1(y) &= \sqrt{3(2y - 1)} \\
\pi_2(y) &= \sqrt{5(6y^2 - 6y + 1)} \\
\pi_3(y) &= \sqrt{7(20y^3 - 30y^2 + 12y - 1)} \\
\pi_4(y) &= \sqrt{9(70y^4 - 140y^3 + 90y^2 - 20y + 1)} \\
\pi_5(y) &= \sqrt{11(252y^5 - 630y^4 + 560y^3 - 210y^2 + 30y - 1)} \\
\pi_6(y) &= \sqrt{13(924y^6 - 2772y^5 + 3150y^4 - 1680y^3 + 420y^2 - 42y + 1)}.
\end{align*}
\]

Blinov and Lemeshko [4] have set out corresponding Legendre polynomials as, in effect,

\[
\begin{align*}
p_0(y) &= \sqrt{1(1)} \\
p_1(y) &= \sqrt{3(2y)} \\
p_2(y) &= \sqrt{5(6y^2 - 0.5)} \\
p_3(y) &= \sqrt{7(20y^3 - 3y)} \\
p_4(y) &= \sqrt{9(70y^4 - 15y^2 + 0.375)}.
\end{align*}
\]

3 NEYMAN TRIANGLE

We assemble the absolute values of the polynomial coefficients into a triangle, as the row sums are all unity if we include the signed values of the coefficients. The row sums are in the right-most column, and the pertinent OIES references [23] are in the bottom row.

<table>
<thead>
<tr>
<th></th>
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<th>1</th>
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<tbody>
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<td>1</td>
<td>13</td>
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<td>30</td>
<td>12</td>
<td>63</td>
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<td>70</td>
<td>140</td>
<td>90</td>
<td>20</td>
</tr>
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<td>252</td>
<td>630</td>
<td>560</td>
<td>210</td>
</tr>
<tr>
<td>924</td>
<td>2772</td>
<td>3150</td>
<td>1680</td>
</tr>
<tr>
<td>A000984</td>
<td>A002457</td>
<td>A002544</td>
<td>A007744</td>
</tr>
</tbody>
</table>

Table 1: Neyman triangle

The leading diagonals in this table generate the sequence \{1,2,7,26,101,404,1645,\ldots\} which does not seem to be in OEIS, but the anti-diagonals can related to OEIS sequences in Table 2(a).
The patterns are clearer when we express the Neyman anti-diagonals as multiples of the first element in each row, as in Table 2 (b). The leading diagonal here yields a known sequence (A005809) as do the anti-diagonals (A001519), the odd Fibonacci numbers as a bisection of the Fibonacci sequence, but we shall not pursue these here.

Table 2(a): Anti-diagonals in Neyman triangle

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
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<td>56</td>
<td>72</td>
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<tr>
<td>6</td>
<td>30</td>
<td>90</td>
<td>210</td>
<td>420</td>
<td>756</td>
<td>1260</td>
<td>1980</td>
<td>A033487</td>
</tr>
<tr>
<td>20</td>
<td>140</td>
<td>560</td>
<td>1680</td>
<td>4200</td>
<td>9240</td>
<td>18480</td>
<td>34320</td>
<td>A105939</td>
</tr>
<tr>
<td>70</td>
<td>630</td>
<td>3150</td>
<td>11550</td>
<td>34650</td>
<td>90090</td>
<td>210210</td>
<td>450450</td>
<td>70xA000581</td>
</tr>
</tbody>
</table>

Table 2(b): Anti-diagonals in Neyman triangle

The leading diagonals in Table 2(a) generate the sequence \{1,3,13,63,321,1683,8989,…\} [A001850] the elements of which are the Central Delannoy numbers [2], so called because they constitute the central anti-diagonal in the infinite square Delannoy array [A008288] in Table 3. The leading anti-diagonal here is A005809.

Table 3: Square Delannoy array

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>1</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
<td>13</td>
<td>25</td>
<td>41</td>
<td>61</td>
<td>85</td>
<td>113</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>7</td>
<td>25</td>
<td>63</td>
<td>129</td>
<td>231</td>
<td>377</td>
<td>575</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>9</td>
<td>41</td>
<td>129</td>
<td>321</td>
<td>681</td>
<td>1289</td>
<td>2241</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>11</td>
<td>61</td>
<td>231</td>
<td>681</td>
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<td>3653</td>
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</tr>
<tr>
<td>6</td>
<td>1</td>
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<td>19825</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>15</td>
<td>113</td>
<td>575</td>
<td>2241</td>
<td>7183</td>
<td>19825</td>
<td>48639</td>
</tr>
</tbody>
</table>

The leading diagonals in this array generate the Pell numbers \{1,2,5,12,29,…\}, and, in the sense of this paper, Alladi and Hoggatt [1] further related these numbers to Tribonacci triangles. When this array is turned clockwise through 45° we have the Pell triangle.

We also see regular intersections (as common elements) among the row and column sequences, which is a topic worth exploring as in Stein [24] who found it necessary to examine the intersection of Fibonacci sequences in order to answer the question of whether every member of a variety is a quasigroup given that every finite member is [25].

The Central Delannoy numbers \{a_n\}, \(n \geq 0\), can be expressed as
and
\[ a_n = \pi_n(2) \quad \text{in terms of the Neyman numbers, which would appear to be new. This suggests we consider in turn} \]
\[ \frac{\pi_n(3)}{\sqrt{n}} = \{1,5,37,305,2641,23525,\ldots\} \]
which is A006442, the expansion of \((x^2 - 10x + 1)^{-\frac{1}{2}}\), which is also related to the Delannoy numbers. Likewise A084768 is
\[ \frac{\pi_n(4)}{\sqrt{n}} = \{1,7,73,847,10321,129367,1651609,\ldots\} \]
and so on.

4 THE COMPLEX-TYPE DELANNOY NUMBERS

Now we define a new sequence that we call the complex-type Delannoy sequence \(\{D'(m,n)\}\) as follows:
\[
D'(m,n) = \begin{cases} 
1 & \text{if } m = 0 \text{ or } n = 0, \\
i \cdot D'(m-1,n) + i \cdot D'(m,n-1) - D'(m-1,n-1) & \text{otherwise.}
\end{cases}
\]
Note that when \(m = n = a\), the complex-type Delannoy sequence \(\{D'(m,n)\}\) is reduced to the central complex-type sequence \(\{D'(a,a)\}\).

A table for the values of the complex-type Delannoy numbers is given by below:

<table>
<thead>
<tr>
<th>( n \downarrow m \rightarrow )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2i-1</td>
<td>-3</td>
<td>-2i-1</td>
<td>1</td>
<td>2i-1</td>
<td>-3</td>
<td>-2i-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-3</td>
<td>-8i+1</td>
<td>13</td>
<td>16i+1</td>
<td>-19</td>
<td>-24i+1</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-2i-1</td>
<td>13</td>
<td>34i-1</td>
<td>-63</td>
<td>-98i-1</td>
<td>141</td>
<td>194i-1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>16i+1</td>
<td>-63</td>
<td>-160i+1</td>
<td>321</td>
<td>560i+1</td>
<td>-895</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2i-1</td>
<td>-19</td>
<td>-98i-1</td>
<td>321</td>
<td>802i-1</td>
<td>-1683</td>
<td>-3138i-1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>-3</td>
<td>-24i+1</td>
<td>141</td>
<td>560i+1</td>
<td>-1683</td>
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<td>8989</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>-2i-1</td>
<td>29</td>
<td>194i-1</td>
<td>-895</td>
<td>-3138i-1</td>
<td>8989</td>
<td>22146i-1</td>
</tr>
</tbody>
</table>

Table 4: Square complex-type Delannoy numbers
From the definitions of the Delannoy numbers and the complex-type Delannoy numbers, we derive the following relations:

**i.** For $m,n \geq 1$

$$D\left(m,n\right) = \begin{cases} 2(i)^n \cdot D\left(m-1,n-1\right) - D\left(m-1,n-1\right), & n \equiv 1 \pmod{4}, \\ 2(i)^{n+1} \cdot D\left(m-1,n-1\right) - D\left(m-1,n-1\right), & n \equiv 2 \pmod{4}, \\ 2(i)^{n+2} \cdot D\left(m-1,n-1\right) - D\left(m-1,n-1\right), & n \equiv 3 \pmod{4}, \\ 2(i)^{n+3} \cdot D\left(m-1,n-1\right) - D\left(m-1,n-1\right), & n \equiv 0 \pmod{4}. \end{cases}$$

**ii.** For $m,n \geq 0$, $D\left(m,n\right) = D\left(n,m\right)$.

**iii.** For $m,n \geq 0$, $D\left(n+1,n\right) = D\left(n,n+1\right) = (-1)^n \cdot D\left(n,n\right)$.

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence.

The research on the conformity of a single term, $a_n \pmod{p}$, has a long history forming most known Pascal's oldest fractal problem, which was originally created by the parities of binomial coefficients $\binom{n}{k}$; see for example, [5,6,7,8,10,12,14,16,17,18,29,30]. We now extend the concept to the complex-type Delannoy numbers.

Consider the sequence

$$\{D'\left(m,n\right)\} = \{D'\left(0,n\right), D'\left(1,n\right), D'\left(2,n\right)\ldots\}$$

where $n$ is a fixed positive integer and $m = 0,1,2,\ldots$.

If we reduce the sequence $\{D'\left(m,n\right)\}$ modulo $\alpha$, taking least nonnegative residues, then we can get the repeating sequence, denoted by

$$\{D'\left(m,n\right)(\alpha)\} = \{D'\left(0,n\right)(\alpha), D'\left(1,n\right)(\alpha), D'\left(2,n\right)(\alpha)\ldots\}$$

where $D'\left(u,n\right)(\alpha)$ is used to mean the $u$th element of the sequence $\{D'\left(m,n\right)(\alpha)\}$ modulo $\alpha$ for the positive integer constant $n$.

We note here that the sequence $\{D'\left(m,n\right)(\alpha)\}$ has the same recurrence relation as in (1).

**Theorem 4.1.** The sequence $\{D'\left(m,n\right)(\alpha)\}$ is periodic.

**Proof.** It is clear that sequence $\{D'\left(m,1\right)(\alpha)\}$ is a constant sequence. Since the sequence $\{D'\left(m,1\right)(\alpha)\}$ is a constant sequence; that is, since it consists only the repetitions of a constant subsequence, we can say that the sequence $\{D'\left(m,2\right)(\alpha)\}$ is also a periodic sequence, using the recurrence relation in the sequence $\{D'\left(m,n\right)(\alpha)\}$. Similarly, since the
sequences \( \{D^i(m,1)(\alpha)\} \) and \( \{D^i(m,2)(\alpha)\} \) are periodic; that is, they consist only the repetitions of constant sub-sequences, the sequence \( \{D^i(m,n)(\alpha)\} \) is also periodic. By a similar idea, we get the repeating sequences \\
\( \{D^i(m,1)(\alpha)\}, \{D^i(m,2)(\alpha)\}, \ldots, \{D^i(m,n-1)(\alpha)\} \) are periodic; that is, they consist only the repetitions of constant sub-sequences, using the recurrence relation in the sequence \( \{D^i(m,n)(\alpha)\} \). Thus, this implies that the sequence \( \{D^i(m,n)(\alpha)\} \) is periodic.

Example 2.1. We have \\
\( \{D^i(m,3)(3)\} = \{1, i-1, i-1, 0, i-1, 0, 2i-1, 0, 2i-1, i-1, i-1, \ldots\} \) \\
and its terms repeat so we get \\
\( L(D^i(m,3)(3)) = 12 \), where the period of the sequence \( \{D^i(m,n)(\alpha)\} \) is denoted by \( L(D^i(m,n)(\alpha)) \).

Conjecture 4.1. Let \( p \) be prime, let \( n \) be a fixed positive integer and \( m = 0, 1, 2, \ldots \). If \( u \) is the smallest positive integer such that \\
\( L(D^i(m,n)(p^{u+1})) \neq L(D^i(m,n)(p^u)) \), then \\
\( L(D^i(m,n)(p^u)) = p^{-u} \cdot L(D^i(m,n)(p^u)) \).

Theorem 4.2. Let \( \alpha_1 \) and \( \alpha_2 \) be positive integers with \( \alpha_1, \alpha_2 \geq 2 \), then \\
\( L(D^i(m,n)(\text{lcm}(\alpha_1,\alpha_2))) = \text{lcm}[L(D^i(m,n)(\alpha_1)), L(D^i(m,n)(\alpha_2))] \).

Proof. Let \( \text{lcm}(\alpha_1,\alpha_2) = \alpha \). Then, \\
\( D^i(m,n)[L(D^i(m,n)(\alpha))] = D^i(m,n)[L(D^i(m,n)(\alpha)) + 1] \) \\
\( = \cdots = D^i(m,n)[L(D^i(m,n)(\alpha)) + n - 1] \equiv 0 \mod \alpha \) \\
and \\
\( D^i(m,n)[L(D^i(m,n)(\alpha_k))] = D^i(m,n)[L(D^i(m,n)(\alpha_k)) + 1] \) \\
\( = \cdots = D^i(m,n)[L(D^i(m,n)(\alpha_k)) + n - 1] \equiv 0 \mod \alpha_k \) \\
for \( k = 1, 2 \). Using the least common multiple operation this implies that \\
\( D^i(m,n)[L(D^i(m,n)(\alpha))] = D^i(m,n)[L(D^i(m,n)(\alpha)) + 1] \) \\
\( = \cdots = D^i(m,n)[L(D^i(m,n)(\alpha)) + n - 1] \equiv 0 \mod \alpha_k \)
for \( k = 1,2 \). So we have \( L(D'(m,n)(\alpha_1))L(D'(m,n)(\alpha_2)) \) and \( L(D'(m,n)(\alpha_2))L(D'(m,n)(\alpha_1)) \), which means that \( \text{lcm}\left[L(D'(m,n)(\alpha_1)), L(D'(m,n)(\alpha_2))\right] \) divides \( L(D'(m,n)(\text{lcm}(\alpha_1, \alpha_2))) \). We also know that

\[
D'(m,n)\left[\text{lcm}\left(L(D'(m,n)(\alpha_1)), L(D'(m,n)(\alpha_2))\right)\right] = D'(m,n)\left[\text{lcm}\left(L(D'(m,n)(\alpha_1)), L(D'(m,n)(\alpha_2))\right)\right]+1
\]

\[
= \ldots = D'(m,n)\left[\text{lcm}\left(L(D'(m,n)(\alpha_1)), L(D'(m,n)(\alpha_2))\right)\right]+n-1 = 0(\text{mod} \alpha).
\]

Then,

\[
D'(m,n)\left[\text{lcm}\left(L(D'(m,n)(\alpha_1)), L(D'(m,n)(\alpha_2))\right)\right] = D'(m,n)\left[\text{lcm}\left(L(D'(m,n)(\alpha_1)), L(D'(m,n)(\alpha_2))\right)\right]+1
\]

\[
= \ldots = D'(m,n)\left[\text{lcm}\left(L(D'(m,n)(\alpha_1)), L(D'(m,n)(\alpha_2))\right)\right]+n-1 = 0(\text{mod} \alpha).
\]

and it follows that \( L(D'(m,n)(\text{lcm}(\alpha_1, \alpha_2))) \) divides \( \text{lcm}\left[L(D'(m,n)(\alpha_1)), L(D'(m,n)(\alpha_2))\right] \). Therefore, we have the following conclusions. □

**Corollary 4.1.** Let \( v \) and \( u \) be positive integers. If \( n = 2^v \), then \( L(D'(m,n)(2^n)) = 2^{u-v-1} \) for \( u + 2 \geq v \).

**Corollary 4.2.** Let \( n \) be a positive integer and \( u \) a positive integer such that \( u \geq 2 \). Then \( L(D'(m,n)(2^n)) = 2^{u-1} \).

5 CONCLUDING COMMENTS


REFERENCES


O. Deveci and Anthony G. Shannon


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