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CONSTRUCTION OF EXACT SOLUTIONS OF SOME EQUATIONS OF HYPERBOLIC TYPE CONTAINING DISCONTINUITY MOVING ON A NON UNIFORM BACKGROUND

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Summary. Exact solutions for the quasilinear transport equation and a system of shallow water equations that contain discontinuities propagating along an inhomogeneous background are constructed in this paper. These solutions can be used as test problems for verification of newly created software packages and numerical methods. The influence of the limiter on the order of approximation using the Galerkin discontinuous method was studied. To calculate the order of approximation, the exact solutions constructed in this paper were used.

1 INTRODUCTION

Equations and systems of equations such as conservation laws [1-4] arise in many practical applications, and therefore their numerical solution is of considerable interest [5-7]. For this purpose, more advanced numerical methods and algorithms are developed. The Galerkin method with discontinuous basis functions [8,9] is quite often used recently. This method has proved itself to solve a wide class of applied problems of mathematical physics with a complex geometry of the investigated object and a multiscale structure of the studied processes. This method has a number of advantages inherent in both finite-element and finite-difference approximations. As you know, there are two approaches to improve the accuracy of the solution. The first approach is to grind the grid in areas of existing solution features (hp-adaptation), the second - to increase the order of accuracy of the scheme. The application of the Galerkin discontinuous method makes it possible to use both approaches simultaneously [10,11]

An important component of the computational algorithm development process is the verification stage, with particular interest in the behavior of numerical solutions in regions containing strong and weak discontinuities. For this, it is necessary to have examples of exact solutions. The construction of which is described in numerous works.

The problems described by the system of shallow water equations are of great practical importance, such as the destruction of the hydroelectric dam, the emergence of large sea waves such as tsunami in shallow water, currents in the atmosphere, and others. Analytic solutions for these problems are constructed [12-22], however, either piecewise constant initial data [12-17] or solutions that exclude discontinuities [18,19] are considered. At the same time, the most important goal is to investigate the accuracy of computational techniques on model problems in which discontinuities propagate along a non-uniform background. The behavior of a numerical solution obtained using difference schemes of high order of accuracy in the flow behind the discontinuity region was studied [23], in which it was shown that the order of approximation in this region drops to the first. In order to eliminate this defect, a

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combined scheme is proposed in [24], which allows maintaining a high order of accuracy of the scheme. However, there is a very limited amount of test problems that have an exact solution with a discontinuity propagating along an inhomogeneous background. In this paper, the authors construct exact discontinuous solutions for the quasilinear transport equation and the system of shallow water equations, using the characteristic approach. A method for determining the time instant after which the characteristics intersect occurs is described in detail. This makes it possible to guarantee the correctness of the exact solution constructed before the specified time. In the second part of the paper, we describe the application of the discontinuous Galerkin method [9] and study the order of approximation of the exact solutions obtained in the first part of this article by numerical solutions.

2 QUASILINEAR TRANSPORT EQUATION

The simplest example of a quasilinear transport equation is the Hopf equation [25], by means of which the motion of a gas of noninteracting particles can be described. In the one-dimensional case, it has the following form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{1}$$

The general solution of equation (1) is the functional dependence $G(x-ut,u)=0$. In particular, we can consider a family of solutions of the

$$u(x,t) = \frac{x+A}{t+B} \tag{2}$$

Equation (1) admits the existence of discontinuous solutions. We construct this solution using two different solutions of the form (2):

$$u = \begin{cases} u_L = (x+A)/(t+B),\ x \le x_f, \\ u_R = (x+C)/(t+D),\ x > x_f, \end{cases} \tag{3}$$

where $x_f(t)$ is the position of the front of the shock wave, the expression for which must be found in order to obtain the final exact solution. According to [26-28], we obtain:

$$\frac{dx_f}{dt} = \frac{1}{2}\left(\frac{x_f+A}{t+B} + \frac{x_f+C}{t+D}\right) \tag{4}$$

or after integration

$$x_f(t) = C_1\sqrt{(t+B)(t+D)} + \frac{(A-C)t + AD - CB}{B-D} \tag{5}$$

where C_1 is the integration constant, which can be determined by specifying the initial condition $x_f(0) = x_0$.

We give an example of a constructed exact solution (5) containing a discontinuity propagating along an inhomogeneous background, concretizing the values of the constants.

Let's set $A=1$, $B=2$, $C=0$, $D=1$ and $x_0=1/2$. In this case $C_1=\sqrt{2}/4$. The corresponding profiles of u at different times are given in Fig. 1.

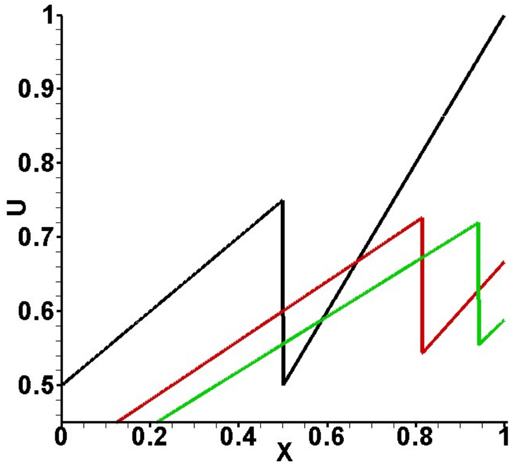


Figure 1. The exact solution of the Hopf equation at $t=0.0$ (black line), 0.5 (red line) and 0.8 (green line).

3 SHALLOW WATER EQUATIONS

Let us turn to the system of shallow water equations, for which it is possible to carry out a similar procedure of constructing an exact discontinuous solution. This system of equations can be obtained from the system of equations of hydrodynamics in the approximation, when the length of gravitational waves is large in comparison with the depth of the liquid [1]:

$$\begin{cases} \dfrac{\partial h}{\partial t}+\dfrac{\partial hu}{\partial x}=0, \\ \dfrac{\partial hu}{\partial t}+\dfrac{\partial}{\partial x}\left(hu^2+\dfrac{gh^2}{2}\right)=0, \end{cases} \tag{6}$$

where h is the depth, u is the flow velocity in the horizontal direction, the gravitational constant g let's set equal to 1. The system of equations (6) can be rewritten in the following form:

$$\begin{cases} \dfrac{\partial R^+}{\partial t}+\left(u+\sqrt{h}\right)\dfrac{\partial R^+}{\partial x}=0, \\ \dfrac{\partial R^-}{\partial t}+\left(u-\sqrt{h}\right)\dfrac{\partial R^-}{\partial x}=0, \end{cases} \tag{7}$$

where $R^{\pm}=u\pm 2\sqrt{h}$ are Riemann invariants for the system (6).

Consider the domain $-\infty < x < \infty$. As a background solution in the region $x \geq x_0$ we set a solution in the form of a simple centered wave, for which the invariant R^- remains constant. Let's set

$$R_r^- = \beta \tag{8}$$

In this case h and u are defined by:

$$u_r = \beta + 2\sqrt{h_r},\ \sqrt{h_r} = \frac{1}{3}\left(\beta + \frac{x+A}{t+B}\right) \tag{9}$$

where $u_r = u\left(x \geq x_0, t\right)$, $h_r = h\left(x \geq x_0, t\right)$. We assume that in the region $x < x_0$ the other invariant R^+ remains constant. Let's set

$$R_l^+ = \alpha \tag{10}$$

The corresponding solution of the system (6) can also be given in the form of a simple centered wave:

$$u_l = \alpha - 2\sqrt{h_l},\ \sqrt{h_l} = \frac{1}{3}\left(\alpha - \frac{x+C}{t+D}\right) \tag{11}$$

We require the fulfillment of the Hugoniot conditions on the discontinuity:

$$\begin{cases} h_l^* u_l^* - h_r^* u_r^* = W\left(h_l^* - h_r^*\right), \\ h_l^*\left(u_l^*\right)^2 + \dfrac{\left(h_l^*\right)^2}{2} - h_r^*\left(u_r^*\right)^2 - \dfrac{\left(h_r^*\right)^2}{2} = W\left(h_l^* u_l^* - h_r^* u_r^*\right) \end{cases} \tag{12}$$

where W is the velocity of the discontinuity, $h_{l,r}^*$ and $u_{l,r}^*$ are the values of the corresponding functions to the right and left of the discontinuity. Eliminating W from equations (12), you can get

$$\frac{h_l^* + h_r^*}{2}\left(h_l^* - h_r^*\right)^2 = h_l^* h_r^*\left(u_l^* - u_r^*\right)^2 \tag{13}$$

taking into account (8) and (10)

$$\frac{h_l^* + h_r^*}{2}\left(h_l^* - h_r^*\right)^2 = h_l^* h_r^*\left(\alpha - \beta - 2\sqrt{h_l^*} - 2\sqrt{h_r^*}\right)^2 \tag{14}$$

From (14) it is possible to determine the quantities h_l^* and u_l^* on the discontinuity.

To agree the solutions (9) and (11) to the left and right of the discontinuity, the constants in (11) must be determined with the following condition taken into account:

$$\frac{C + x_0}{D} = \alpha - 3\sqrt{h_l^*} \tag{15}$$

obtained at the initial moment $t = 0$.

On the basis of the following simple considerations, we can now draw some conclusions about the roots of equation (13). Thus, in the case of a discontinuity moving to the right, equation (13) with allowance for (10) can be rewritten in the form:

$$-2\sqrt{y} + (1 - y)\sqrt{\frac{1}{2}(y^{-1} + 1)} = \frac{u_r^* - \alpha}{\sqrt{h_r^*}} \tag{16}$$

relatively dimensionless variable $y = h_l^* / h_r^*$. The left-hand side of (16) is a monotonically decreasing function with respect to the variable $y > 0$, which takes all values on the real line $(-\infty, +\infty)$,which follows from the negativity of the variable. This means that for any values of the right-hand side, equation (16) has a unique solution. We note that only solutions that satisfy the inequality $y > 1$ are physicly correct, which is ensured if and only if $u_r^* + 2\sqrt{h_r^*} < \alpha$. The investigation of the roots of the polynomial (14) is generally given in Appendix A.

The exact solution at any time t in the region bounded from the left by the characteristic $x_0 + \left(u_l^* - \sqrt{h_l^*}\right)t$ (see Fig. 2), will be determined by the functions (11), and the shock front bounded by the right $x_f(t)$ - functions (9).

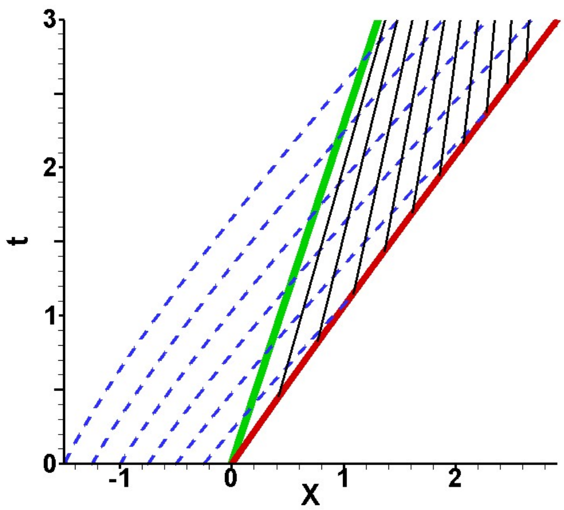


Figure 2. Trajectory of motion of the shock wave front (red line) and family of characteristics: a green line is a characteristic with a slope $u - \sqrt{h}$, which emerges from the point of the initial position of the discontinuity, continuous thin lines - a field of characteristics emerging from the shock wave trajectory, on which the value $R^- = u_l^* - 2\sqrt{h_l^*}$, dashed thin lines - a field of characteristics on which the value of the invariant $R^+ = u_l + 2\sqrt{h_l}$, and by which this value is transferred to the front of the shock wave.

In the region between the indicated characteristic and the discontinuity, in which the value R^+ will remain constant, the solution will still be a simple wave, which, however, will no longer be centered. In this interval, the exact solution is determined using the method of characteristics. Let us describe this procedure in more detail.

Values before the rupture front h_r^* and u_r^* at each instant of time are determined by the formulas (9). Solving equation (14) we find the values h_l^* and u_l^* as functions of x_f and t, where x_f is a position of the rupture front. From (12) we determine the velocity of the discontinuity $W\left(x_f,t\right)$. Trajectory of the discontinuity motion $x_f\left(t\right)$ is defined as a solution of equation

$$\frac{dx_f}{dt}=W\left(x_f,t\right),\ x_f\left(0\right)=x_0 \tag{17}$$

From each point of this trajectory $x_f\left(t\right)$, t release the characteristic on which the value is stored $R^-=u_l^*-2\sqrt{h_l^*}$. In this case, these characteristics will have a constant slope, i.e. will be straight lines. Finally, we determine the point of intersection x^* straight line $t=T$ and characteristics. Function values u and h to a given point are carried by characteristics in accordance with the values of the Riemann invariants.

We perform the above procedure for specific values of the constants. Let's set $\beta=0$, $\alpha=3/2$, $x_0=0$, $A=1/2$, $B=1$, $D=1$, and C is determined in accordance with (15). Equation (14) can be rewritten in the form

$$P\left(\xi\right)=0 \tag{18}$$

where P is polynomial with constant coefficients, $\xi=\sqrt{h}$. The graph of the dependence is given in Fig. 3.

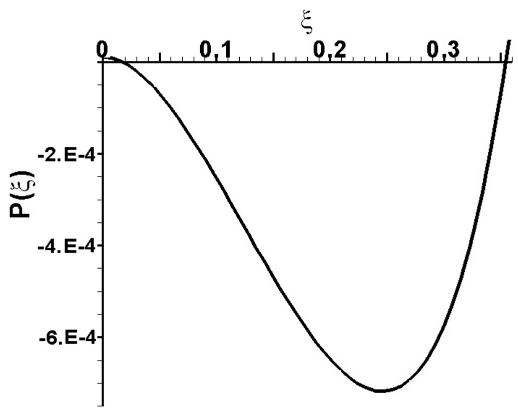
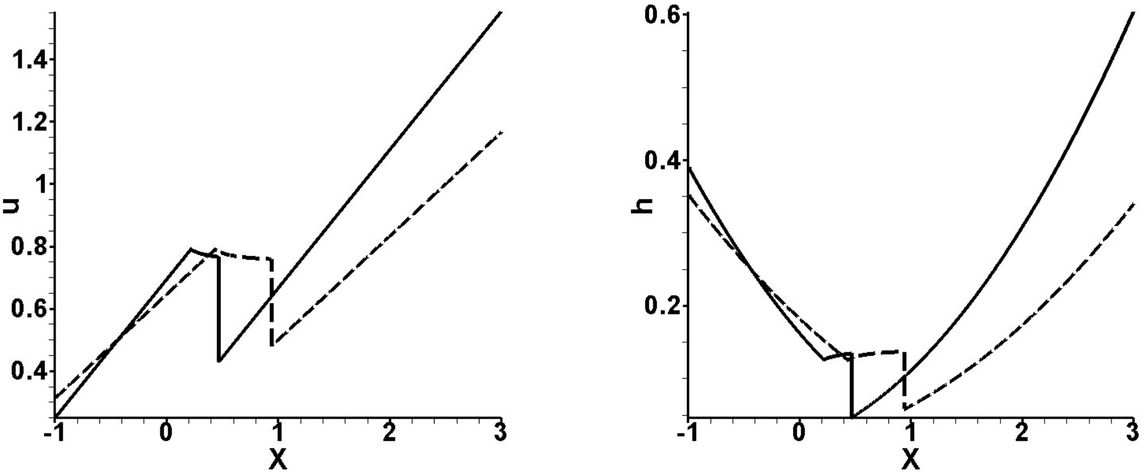


Figure 3. The function $P\left(\xi\right)$.

By building a series of Sturm for $P(\xi)$ and applying the Sturm theorem [29], we can determine that equation (17) has two roots on the positive semiaxis, one of which corresponds to a shock compression wave ($h_l^* > h_r^*$), another - a shock wave of rarefaction ($h_l^* < h_r^*$). From the physical considerations, we choose the first one. For this case, the system of characteristics is shown in Fig. 2, and the evolution of the exact solution is shown in Fig. 4.



a) b)

Figure 4. The exact solution of the system of shallow water equations at times 0.5 (solid line) and 1.0 (dashed line).

It should be noted that the method of characteristics is applicable only when the characteristics do not intersect in the considered region. The condition for the intersection of two infinitely close characteristics at time T emerging from points $(x_f(t), t)$ and $(x_f(t+\Delta t), t+\Delta t)$ with different angular coefficients a_1 and a_2 respectively, is given by

$$a_1(t_c - t) + x_f(t) = a_2(t_c - t - \Delta t) + x_f(t + \Delta t) \tag{19}$$

allowing to find the moment of their intersection

$$t_c = t + (a_2 - a_1)^{-1}[a_2 \Delta t - x_f(t + \Delta t) + x_f(t)] \tag{20}$$

The expression for the angular coefficient of an arbitrary characteristic with allowance for the constancy of the invariant $R^+ = \alpha$ is defined by

$$a = u_l^* - \sqrt{h_l^*} = \alpha - 3\sqrt{h_l^*} \tag{21}$$

In turn, the quantity h_l^* is given by means of equation (16) as an implicit continuously differentiable function depending on the arguments u_r^* and h_r^*, which in accordance with (9) are continuously differentiable functions of two variables x and t. Therefore, according to (19), the expression for the angular characteristic coefficient can be regarded as a continuously differentiable function $a = a(x,t)$. In view of the latter circumstance $a_1 = a(x_f(t),t)$ and $a_2 = a(x_f(t+\Delta t), t+\Delta t)$. Passing to the limit as $\Delta t \to 0$ on the right-hand side of (20) and taking (16) into account, we obtain the instant of intersection of infinitely close characteristics emerging from the point $(x_f(t),t)$

$$t_c = t + \left(\frac{\partial a}{\partial t} + W(x_f(t),t) \frac{\partial a}{\partial x} \right)^{-1} [a(x_f(t),t) - W(x_f(t),t)] \tag{22}$$

Introducing the notation

$$X = \sqrt{\frac{1}{2}\left(\frac{1}{h_l^*} + \frac{1}{h_r^*}\right)},\ Y = 4X^2 + \frac{h_l^* - h_r^*}{(h_r^*)^2},\ Z = 4X^2\sqrt{h_l^*} + 4X - \frac{h_l^* - h_r^*}{h_l^* \sqrt{h_l^*}}, \tag{23}$$

expressions for the partial derivatives of the function $a = a(x,t)$ can be found

$$\frac{\partial a}{\partial x} = -\frac{3}{2} Z^{-1} \left(Y \frac{\partial h_r^*}{\partial x} - 4X \frac{\partial u_r^*}{\partial x} \right),\ \frac{\partial a}{\partial t} = -\frac{3}{2} Z^{-1} \left(Y \frac{\partial h_r^*}{\partial t} - 4X \frac{\partial u_r^*}{\partial t} \right). \tag{24}$$

Relations (22) - (24) for given values of the constants A, B, C and D, which were defined above, allow us to numerically determine the minimum value of the right-hand side of (22) $t_c^* \approx 3.22$ on the set of all characteristics issuing from points $(x_f(t),t)$, that specifies the applicability boundary of the method of characteristics in the studied problem.

4 DISCONTINIOUS GALERKIN METHOD

We use the exact solutions obtained in the previous sections of the Hopf equation and the system of shallow water equations to determine the order of approximation when using the Galerkin discontinuous method for their numerical solution. In this section we briefly describe the essence of the approach using the example of a generalized hyperbolic system of quasilinear equations of the type of conservation laws:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0 \tag{25}$$

where $\mathbf{U}(x,t)$ is vector of variables, and $\mathbf{F}(\mathbf{U})$ are defined streaming vector functions containing m components. For system (25) we set the Cauchy problem with initial data

$$\mathbf{U}(x,0) = \mathbf{U}_0(x), \tag{26}$$

Suppose that the Cauchy problem (25) - (26) has a unique generalized solution $\mathbf{U}(x,t)$, limited with $t > 0$. Let's set $\mathbf{U}^n(x) = \mathbf{U}(x,t_n)$ a numerical solution of this problem corresponding to the instant of time t_n.

To apply the Galerkin discontinuous method on a uniform grid, we define the following system of basis functions

$$\varphi_{i,k}(x)=\begin{cases}\phi_k(x), & x\in[x_i,x_{i+1}],\\ 0, & x\notin[x_i,x_{i+1}],\end{cases} \quad (27)$$

where $\phi_k(x)=\left((x-x_i^c)/\Delta\right)^k$, $x_i^c=(x_i+x_{i+1})/2$. Then on each time layer in each space cell $I_i=[x_i,x_{i+1}]$ an approximate solution of the system of equations (25) will be sought in the form of a polynomial of degree p

$$\mathbf{U}_i^n(x)=\mathbf{U}_i(x,t_n)=\sum_{k=0}^{p}\mathbf{U}_{ik}^n\phi_k(x) \quad (28)$$

with time-dependent coefficients $\mathbf{U}_{ik}^n=\mathbf{U}_{ik}(t_n)$.

Multiplying (25) by the basis function and performing integration over x on the interval I_i, we obtain the following expression

$$\int_{I_i}\frac{\partial}{\partial t}\mathbf{U}(x,t)\phi_l dx-\int_{I_i}\mathbf{F}(\mathbf{U})\phi_l' dx+\mathbf{F}(\mathbf{U})\phi_l\Big|_{x_i}^{x_{i+1}}=0 \quad (29)$$

on the basis of which Galerkin's discontinuous method is constructed. Replacing in the first two terms of equation (29) the function $\mathbf{U}(x,t)$ on function $\mathbf{U}_i^n(x)$, and in the third term - differential flows $\mathrm{F}(\mathbf{U}(x_j,t))$, where $j=i,i+1$, on numerical flows $\mathbf{F}_j^n=\mathbf{\Phi}\left(\mathbf{U}_{j-1}^n\left(x_j^-\right),\mathbf{U}_j^n\left(x_j^+\right)\right)$, in which $x_j^{\pm}=x_j\pm 0$, we get

$$\frac{\partial}{\partial t}\int_{I_i}\mathbf{U}_i^n\phi_l dx-\int_{I_i}\mathbf{F}\left(\mathbf{U}_i^n\right)\phi_l' dx+\mathbf{F}_{i+1}^n\phi_l\left(x_{i+1}^-\right)-\mathbf{F}_i^n\phi_l\left(x_i^+\right)=0 \quad (30)$$

In this paper we use the Rusanov-Lax-Friedrichs numerical flows [30,31], in which the function $\mathbf{\Phi}(\mathbf{x},\mathbf{y})$ is determined by the formula

$$\mathbf{\Phi}(\mathbf{x},\mathbf{y})=\frac{1}{2}\left(\mathbf{f}(\mathbf{x})+\mathbf{f}(\mathbf{y})-A(\mathbf{x},\mathbf{y})(\mathbf{y}-\mathbf{x})\right),\ A(\mathbf{x},\mathbf{y})=\max_m\left(\left|\lambda_m(\mathbf{x})\right|,\left|\lambda_m(\mathbf{y})\right|\right), \quad (31)$$

where λ_m are eigenvalues of the Jacobi matrix $\dfrac{\partial\mathbf{F}(\mathbf{U})}{\partial\mathbf{U}}$ of set (25).

From (30), taking (28) into account, we obtain a system of ordinary differential equations

$$\frac{d\mathbf{U}_{ki}^n}{dt}=\mathbf{A}_i^{-1}\mathbf{R}_i,\quad \mathbf{R}_i=\int_{I_i}\mathbf{F}\left(\mathbf{U}_i^n\right)\phi_l' dx-\mathbf{F}_{i+1}^n\phi_l\left(x_{i+1}^-\right)+\mathbf{F}_i^n\phi_l\left(x_i^+\right) \quad (32)$$

to calculate the coefficients $\mathbf{U}_{ki}^n$, where $\mathbf{A}_i^{-1}$ is an inverse matrix for matrix $\mathbf{A}_i=\left(a_{kl}^i\right)$, whose coefficients are determined by the formula $a_{kl}^{i} = \int\limits_{I_i} \phi_k(x)\phi_l(x)dx$. The system (32) is solved by the explicit Runge-Kutta method of the third order, in which the time step τ is chosen from the stability condition of Courant

$$\tau = \frac{z\Delta}{\max\limits_{m,j,n} |\lambda_m(\mathbf{U}_{j+1/2}^n)|} \tag{33}$$

where $z \in (0,1)$ is safety factor.

The following calculations use polynomials (28) of the first order, i.e. $p = 1$.

To ensure the monotony of the numerical solution obtained by this method, it is necessary to introduce flow limiters, especially if the solution contains strong discontinuities. In this paper we apply the Cockburn limiter [9], which is widely used in applied multidimensional calculations conducted on grids of arbitrary structure. In the case when the solution (28) is sought in the form of linear x functions

$$\mathbf{U}_i^n(x) = \mathbf{U}_{i0}^n + \mathbf{U}_{i1}^n \frac{x - x_i^c}{\Delta} \tag{34}$$

the action of this limiter leads to the fact that the vector coefficient $\mathbf{U}_{i1}^n$ in the formula (34) is replaced by the quantity

$$V_{i1}^n = M\left[U_{i1}^n, \alpha\left(U_{i+1,0}^n - U_{i0}^n\right), \alpha\left(U_{i0}^n - U_{i-1,0}^n\right)\right] \tag{35}$$

where U_{j0}^n – the corresponding components of the vectors $\mathbf{U}_{j0}^n$, $\alpha \in [1,2]$– heuristic parameter, chosen as a result of test calculations, M is a $\min\mathrm{mod}$ operator, the action of which is determined by formula

$$M\left[u_1, u_2, u_3\right] = s \min\left(|u_1|, |u_2|, |u_3|\right), \tag{36}$$

where $s = sign(u_i)$ provided that all numbers u_i have the same sign and $s = 0$ otherwise.

5 THE ORDER OF APPROXIMATION OF DISCONTINUOUS GALERKIN METHOD ON EXACT SOLUTIONS

We consider a sequence of difference solutions of the Cauchy problem (25) - (26) obtained using the numerical scheme based on the discontinuous Galerkin method on uniform grids with spatial steps Δ and $\Delta/2$. It is possible to calculate the approximation order achieved having the exact solution obtained in Sections 2 and 3:

$$r_{1i} = \log_2\left(\frac{\left\|u_\Delta - u_{ex}\right\|_{I_i}}{\left\|u_{\Delta/2} - u_{ex}\right\|_{I_i}}\right), \tag{37}$$

where $\|\varphi\|_{I_i} = \left(\int\limits_{x_i^c-\Delta/2}^{x_i^c+\Delta/2} \varphi^2(x)\,dx \right)^{1/2}$, $I_i = \left[x_i^c - \Delta/2, x_i^c + \Delta/2\right]$ is coarse grid, x_i^c is cell center I_i.

Integral, included in the definition of the norm $\|\varphi\|_{I_i}$, is calculated analytically, as the exact solution of the Hopf equation is a fractional linear function (3), and the numerical solution obtained by the discontinuous Galerkin method is a linear polynomial (28). For the system of shallow water equations, it is impossible to write out an exact solution in the form of an analytic function in the region between the characteristics emerging from the discontinuity point at the initial time instant. Therefore, we shall consider local norm convergence:

$$r_{2i} = \log_3\left(\frac{\left\| u_\Delta\left(x_i^c\right) - u_{ex}\left(x_i^c\right)\right\|}{\left\| u_{\Delta/3}\left(x_i^c\right) - u_{ex}\left(x_i^c\right)\right\|} \right), \tag{38}$$

where $\|\varphi(x)\| = |\varphi(x)|$.

Fig. 5-8 represent numerical solutions of the Hopf equation and the system of shallow water equations with and without the use of the limiting procedure in the discontinuous Galerkin method on grids of various dimension. In Fig. 5 and 7, the numerical solution practically coincides with the exact solution, with the exception of a narrow region, near the discontinuities. In more detail these areas are presented in Fig. 6 and 8. As the grid pitch decreases, the numerical solution converges to the exact one, which is clearly seen in Fig. 6 and 8. It can also be noted that without using the limiting procedure in a numerical solution near the discontinuity, strong enough oscillations are observed, suppression of which occurs when limiters are used to ensure the monotonicity of the solution.

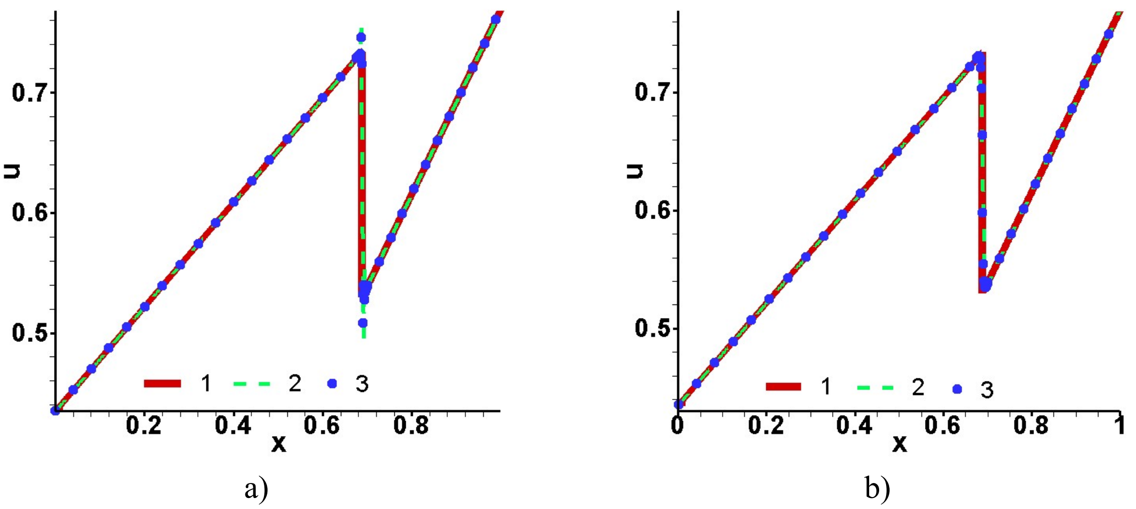


Figure 5. Profiles of exact and numerical solutions of the Hopf equation at time moment $t = 0.5$. The exact solution on the graph corresponds to number 1. Numerical solutions were obtained using the discontinuous Galerkin method (a) without applying the restriction procedure on meshes containing 400 (2) and 800 (3) cells, (b) - the same with the application of the limiting procedure.

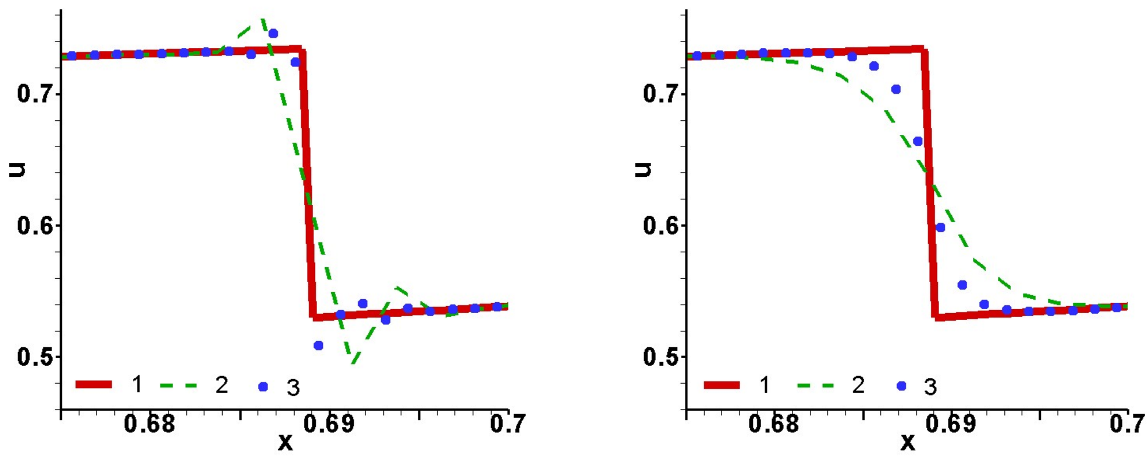


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Figure 6. Profiles of exact and numerical solutions of the Hopf equation at an instant $t = 0.5$ in the vicinity of the discontinuity. The exact solution on the graph corresponds to number 1. Numerical solutions were obtained using the Galerkin discontinuous method (a) without applying the restriction procedure on meshes containing 400 (2) and 800 (3) cells, (b) - the same with the application of the limiting procedure.

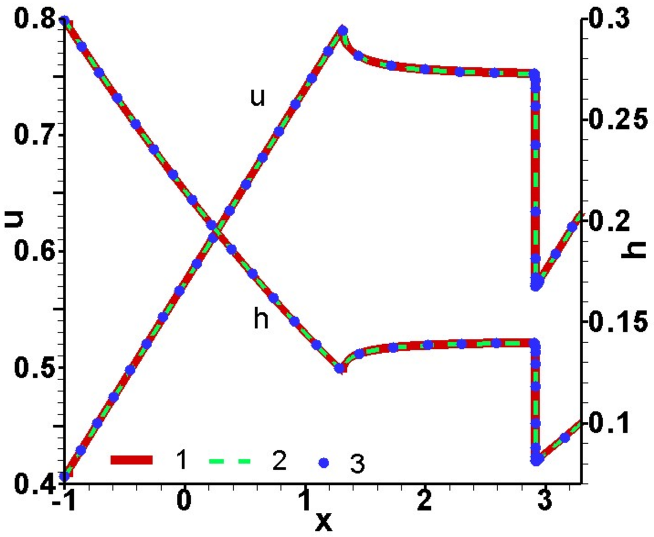
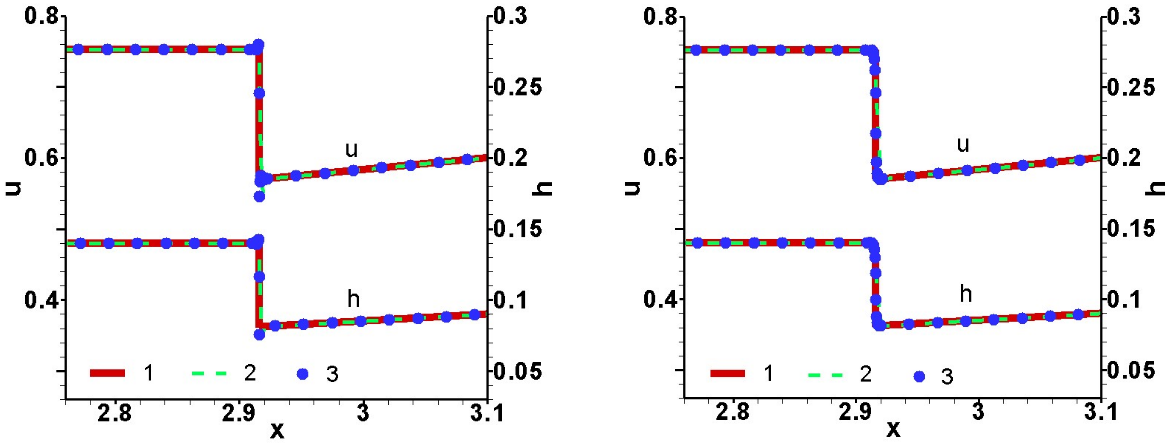


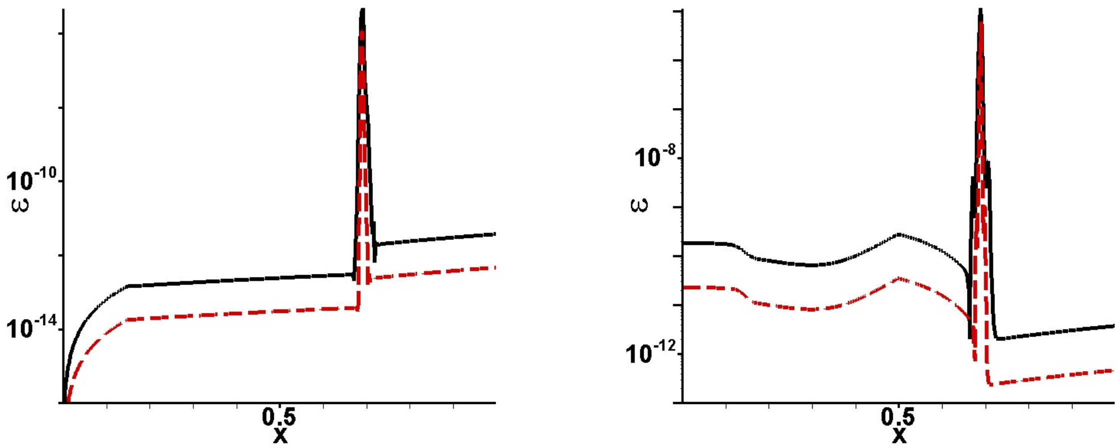
Figure 7. Profiles of exact and numerical solutions of the system of shallow water equations at a time moment $t = 3.0$. The exact solution on the graph corresponds to number 1. Numerical solutions were obtained using the discontinuous Galerkin method with the use of the limiting procedure on grids containing 3000 (2) and 9000 (3) cells.



a) b)

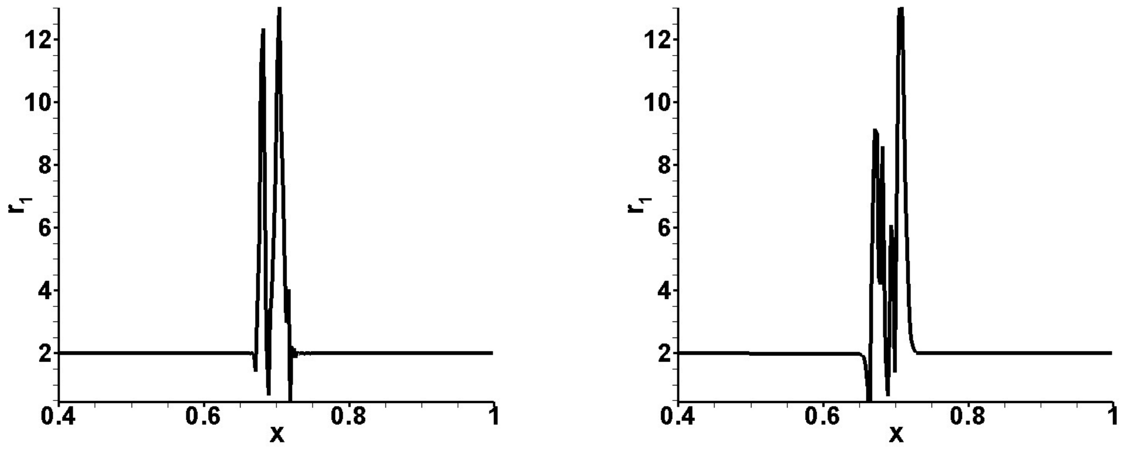
Figure 8. The profiles of the exact and numerical solutions of the system of shallow water equations at the instant of time $t = 3.0$ in the vicinity of the discontinuity. The exact solution on the graph corresponds to number 1. Numerical solutions were obtained using the discontinuous Galerkin method (a) without applying the limiting procedure on grids containing 3000 (2) and 9000 (3) cells, (b) - the same with the use of the limiting procedure.

In Fig. 9 and 10 are the graphs of the discrepancy between the numerical solution of the Hopf equation and the approximation order calculated according to (37), respectively. In spite of the fact that, in the case of applying the limiting operator, the error in the numerical solution in the region beyond the gap increases by 3 orders of magnitude (see Fig. 9), the approximation order value remains equal to two, as in the case without the limiter [32,33].



a) b)

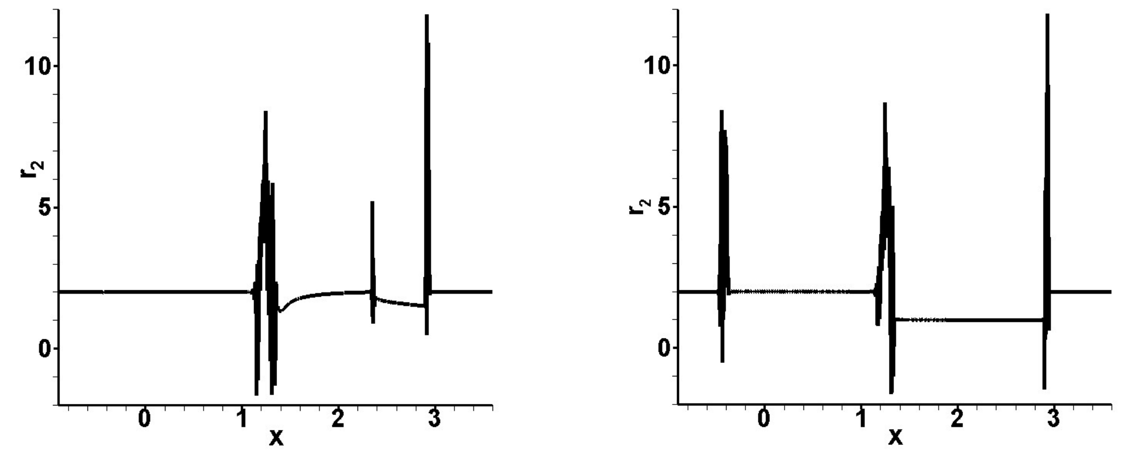
Figure 9. The discrepancy between the numerical solution of the Hopf equation obtained using the discontinuous Galerkin method (a) without the use of the limiting procedure and (b) with it on meshes containing 400 (solid line) and 800 (dashed line) cells.



a) b)

Figure 10. The order of approximation of the exact solution of the Hopf equation using the Galerkin discontinuous method (a) without applying the limiting procedure and (b) with limiting.

The reason for this result is that in the nonlinear transport equation there is only one invariant that is transported by characteristics and is not affected by the shock wave. In contrast to the Hopf equation, the system of shallow water equations (6) contains two invariants and is more suitable for investigating the accuracy of the numerical method. In Fig. 11 shows the order of local convergence of the numerical solution to the exact one, calculated according to (38).



a) b)

Figure 11. The order of local convergence of numerical solutions obtained by using the discontinuous Galerkin method (a) without applying the limiting procedure and b) using a limiter on meshes containing 3000 cells and 9000 cells to an exact solution.

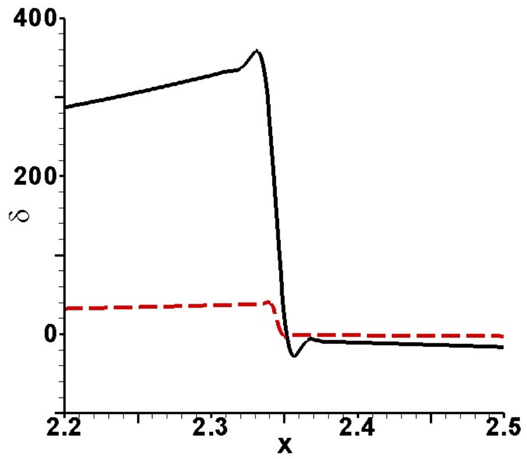


Figure 12. Deviations of the numerical solution from the exact one in the vicinity of the point $x \approx 2.3$ in the calculation without the use of a limiter; solid line corresponds to $A\cdot(h_\Delta - h_{ex})$, dashed line – $A\cdot(h_{\Delta/3} - h_{ex})$, where $A = 10^{10}$, h_Δ – the solution obtained on the grid of 3000 cells, $h_{\Delta/3}$ - the solution obtained on the grid of 9000 cells, h_{ex} - the exact solution.

In calculations without using a limiter in the entire region, a second order of accuracy of the solution was obtained (see Fig. 11a). The jumps of orders at the point $x \approx 1.3$ correspond to the gluing of a solution of the type (9) and the solution between the characteristics emerging from the discontinuity point at the initial instant of time, and at the point $x \approx 3.0$ - to the position of the front of the shock wave. At the point $x \approx 2.3$, the intersection of numerical and exact solutions occurs. This can be clearly seen in Fig. 12, which shows the deviation of the numerical solution on different grids from the exact one. At the point $x \approx 2.3$ these functions change sign and, accordingly, in its vicinity the order of accuracy is not defined. The behavior of the order in the vicinity of the point $x \approx -0.44$ is shown on Fig. 11b when using the limiting operator can be explained in the similar way. However, the general behavior of the order is somewhat different. It can be seen from the calculations that the solution in the region of smoothness ahead of the front of the shock wave and in the solution region of type (9) has a second order of accuracy irrespective of the use of the limiting operator. In the solution region behind the front of the shock wave, but located between the characteristics emerging from the point of discontinuity at the initial instant of time, the accuracy of the solution drops to the first. An interesting fact is that in the region [0,1] a second order of accuracy is observed. In this region, both invariants are carried over the characteristics from the initial data, while in the region of reduced orders of accuracy the invariant is transferred from the front of the shock wave. Thus, it can be argued that the use of the limiting operator negatively affects the accuracy of the solution obtained only in the region of the impact of the shock wave [1.5, 3].

6 CONCLUSIONS

In this paper, exact discontinuous solutions are constructed for the quasilinear transport equation, and also for the system of shallow water equations using the method of characteristics. The results of calculations obtained by programs implementing the discontinuous Galerkin method showed good agreement of the numerical solutions with the constructed exact solutions, confirming the possibility of using them as test tasks for verification of program complexes and numerical methods. Using the Hopf equation as an example, it was shown in the paper that the use of hyperbolic equations containing only one Riemann invariant is not sufficient to study the order of approximation of numerical methods. To this end, it is necessary to use more complex systems, for example, a system of shallow water equations. In this example, it was shown that the introduction of the limiter reduces the order of accuracy of the discontinuous Galerkin method in the regions of influence of strong discontinuities.

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APPENDIX A

Let us introduce a new notation, namely, $\sqrt{h_l^*} = y$, $\sqrt{h_r^*} = x$, $\alpha - \beta = p$ In this case (14) is rewritten in the form:

$$P = x^2 y^2 (p - 2x - 2y)^2 - \frac{1}{2}\left(x^2 + y^2\right)\left(x^2 - y^2\right)^2, \tag{A1}$$

where y is the chosen dependent variable, x and p are independent parameters.

As is known, the classical Sturm sequence [29] is defined only for polynomials with real coefficients. Degenerations and rearrangements of the solution set are possible in the case of the dependence of the coefficients on the parameters. However, for this problem it is possible to construct the Sturm sequence in general form. After an elaborate work we can get $S = [s_1, \ldots, s_7]$, where

$$s_1 = -x^6 + \left(2x^2p^2 - 8px^3 + 9x^4 + \left(16x^3 - 8px^2 + \left(9x^2 - y^2\right)y\right)y\right)y^2,$$

$$s_2 = \left(2x^2p^2 - 8px^3 + 9x^4 + \left(-12px^2 + 24x^3 + \left(18x^2 - 3y^2\right)y\right)y\right)y,$$

$$s_3 = 3x^4 + \left(-4p^2 + 16px - 18x^2 + (-24x + 12p - 9y)y\right)y^2,$$

$$s_4 = -12(2x - p)x^4 + \Big(-18x^2(2x - p)^2 +$$

$$+\Big(-4(2x - p)(2p + x)(-2p + 9x) + \left(-24x^2 + 36p^2 - 144px\right)y\Big)y\Big)y,$$

$$s_5 = -6x^4 p\left(-29px^2 + 180x^3 + p^3 - 8xp^2\right) + \Big(-9x^2 p\Big(-490px^3 + 312x^4$$

$$+295x^2p^2-80p^3x+8p^4\Big)-$$

$$-p\Big(-3456px^4-573x^2p^3+2024x^3p^2-8p^5+96xp^4+1728x^5\Big)y\Big)y, \qquad \text{(A2)}$$

$$s_6=\Big(34992x^{10}+347760px^9+371556x^8p^2-2068672p^3x^7+3534048x^6p^4-$$

$$-3060540p^5x^5+1563205x^4p^6-501504x^3p^7+100464x^2p^8-11520xp^9+$$

$$+576p^{10}\Big)y+54\Big(648x^7+7196px^6+14616x^5p^2-28532p^3x^4+17850x^3p^4-$$

$$-5817x^2p^5+1008p^6x-72p^7\Big)x^4,$$

$$s_7=1.$$

Substituting the specific values of x and p in (A2), one can determine the number of nondegenerate roots on a given interval of variation y, as the difference between the number of sign changes in the Sturm sequences corresponding to the edges of the interval. Degenerate cases require particular consideration.

For the case considered in the text of the paper, namely, $\alpha=3/2$, $\beta=0$ ($p=3/2$), (A1) has the following graph of the dependence on the parameter x – see Fig. 13.

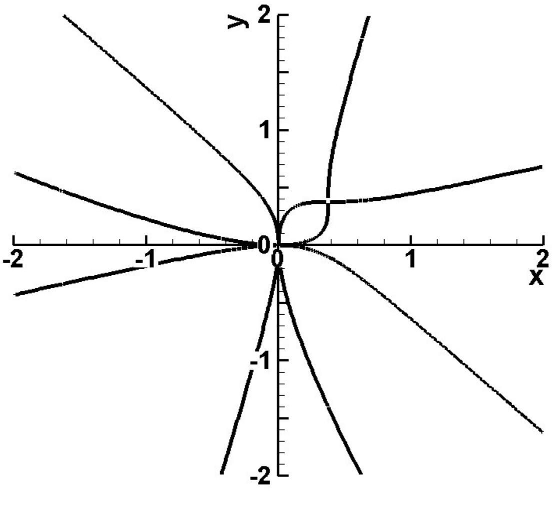

Figure 13. The dependency (A1) in the case of $p=3/2$.

From Fig. 13 is obvious that there is at least one degenerate case. Calculations show that this happens when value $x=p/4$. Asymptotic analysis shows that there are no other degenerate cases.

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