MATHEMATICA MONTISNIGRI
Vol XLII (2018)

SOME CHARACTERIZATIONS OF RIGHT WEAKLY PRIME Γ-HYPERIDEALS OF ORDERED Γ-SEMIHYPERGROUPS

SABER OMIDI[*] AND BIJAN DAVVAZ[†]

[*] Department of Mathematics
Yazd University
Yazd, Iran
e-mail: omidi.saber@yahoo.com

[†] Department of Mathematics
Yazd University
Yazd, Iran
Email: davvaz@yazd.ac.ir - Web page: https://pws.yazd.ac.ir/davvaz/

Summary. In this paper, we deal with ordered Γ-semihypergroups. In particular, we study right weakly prime Γ-hyperideals and maximal Γ-hyperideals in ordered Γ-semihypergroups. Moreover, we give some results on ordered Γ-semihypergroups.

1 INTRODUCTION AND PREREQUISITES

Hyperstructure theory was first introduced in Marty's classical paper [1]. Semihypergroup is the generalization of semigroup theory with the wide range of usages in theory of hyperstructures [2,3]. In [4], Heidari and Davvaz studied a semihypergroup $(S,\bullet)$ besides a binary relation $\leq$, where $\leq$ is a partial orderrelation such that satisfies the monotone condition. This structure is called an ordered semihypergroup. As a reference for more definitions and results on ordered semihypergroups we refer to [3,5,6]. Omidi and Davvaz [7] investigated on the relation N in ordered semihypergroups. We refer to [5] for a survey of some results on the pseudo orders of ordered semihypergroups. Omidi et al. [8] discussed quasi-Γ-hyperideals and hyperfilters in ordered Γ-semihypergroups. Tang et al. [9] studied fuzzy quasi-Γ-hyperideals in ordered Γ-semihypergroups. In 2016, Omidi and Davvaz [10,11,12] studied some properties of hyperideals and k-hyperideals in ordered semihyperrigs and hyperrings. The study of weakly prime ideals of ordered Γ-semigroups was started by the pioneering work of Kwon and Lee [13]. In 2013, Changphas [14] defined right prime ideals and maximal right ideals in ordered semigroups. Weakly prime ideals in involution po-Γ-semigroups discussed by Abbasi and Basar [15].

Let S be a non-empty set. A mapping $\bullet : S \times S \rightarrow P^*(S)$, where $P^*(S)$ denotes the family of all non-empty subsets of S, is called a hyperoperation on S. By a hypergroupoid we mean a non-empty set S endowed with a hyperoperation $\bullet$. In the above definition, if A and B are two non-empty subsets of S and $x \in S$, then we denote $A \bullet B$ the union of $a \bullet b$, where $a \in A$ and $b \in B$. Moreover, for every $x \in S$,

$$A \bullet x = A \bullet \{x\} \text{ and } x \bullet B = \{x\} \bullet B.$$

A hypergroupoid $(S,\bullet)$ is called a semihypergroup if for all x,y,z of S, we have

2010 Mathematics Subject Classification: 16Y99, 20N20.
Key words and Phrases: Ordered Semihypergroup, Right Weakly Prime Hyperideal, Maximal Hyperideal.

$$x\bullet(y\bullet z)=(x\bullet y)\bullet z,$$

which means that the union of $x\bullet u$, where $u\in y\bullet z$ is equal to the union of $v\bullet z$, where $v\in x\bullet y$.

A non-empty subset A of a semihypergroup $(S,\bullet)$ is called a subsemihypergroup of S if $A\bullet A$ is a subset of A. Let $(S,\bullet)$ be a semihypergroup. Then, S is called a hypergroup if it satisfies the reproduction axiom, for all $x\in S$, $x\bullet S=S=S\bullet x$. A non-empty subset K of S is a subhypergroup of S if $a\bullet K=K=K\bullet a$, for every $a\in K$.

Let S and Γ be two non-empty sets. Then, S is called a Γ-semihypergroup [16] if every $\gamma\in\Gamma$ is a hyperoperation on S, i.e., $x\gamma y$ is a subset of S for every $x,y\in S$ and for every $\alpha,\beta\in\Gamma$ and $x,y,z\in S$, we have

$$x\alpha(y\beta z)=(x\alpha y)\beta z.$$

A Γ-semihypergroup S is called commutative if for all $x,y\in S$ and $\gamma\in\Gamma$, we have $x\gamma y=y\gamma x$. For some properties of Γ-semihypergroups, readers can see [16]. The following concepts are adapted from [8,9].

An ordered semihypergroup $(S,\bullet,\leq)$ is a semihypergroup $(S,\bullet)$ together with a partial order $\leq$ that is compatible with the hyperoperation $\bullet$, meaning that for any $x,y,z\in S$,

$$x\leq y \text{ implies that } z\bullet x\leq z\bullet y \text{ and } x\bullet z\leq y\bullet z.$$

Here, $z\bullet x\leq z\bullet y$ means for any $a\in z\bullet x$ there exists $b\in z\bullet y$ such that $a\leq b$. The case $x\bullet z\leq y\bullet z$ is defined similarly.

Definition 1.1 An algebraic hyperstructure $(S,\Gamma,\leq)$ is called an ordered Γ-semihypergroup if (S,Γ) is a Γ-semihypergroup and $(S,\leq)$ is a partially ordered set such that for any $x,y,z\in S$ and $\gamma\in\Gamma$, we have

$$x\leq y \text{ implies that } z\gamma x\leq z\gamma y \text{ and } x\gamma z\leq y\gamma z.$$

Here, $z\gamma x\leq z\gamma y$ means for any $a\in z\gamma x$ there exists $b\in z\gamma y$ such that $a\leq b$. The case $x\gamma z\leq y\gamma z$ is defined similarly.

See to [8,9] for the examples of the ordered Γ-semihypergroups. For a non-empty subset A of an ordered Γ-semihypergroup S, we denote

$$(A]=\{x\in S \mid x\leq a \text{ for some } a\in A\}.$$

Definition 1.2 A non-empty subset I of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is called a left (resp. right) Γ-hyperideal of S if

(1) $S\Gamma I\subseteq I$ (resp. $I\Gamma S\subseteq I$);

(2) when $x\in I$ and $y\in S$ such that $y\leq x$, imply that $y\in I$.

Note that the condition (2) in Definition 1.1 is equivalent to $(I] \subseteq I$. A non-empty subset I of S is called a Γ-hyperideal of S if it is a right and left Γ-hyperideal of S. A Γ-hyperideal T of S is said to be proper if $T \neq S$.

Theorem 1.3 [9] Let $(S,\Gamma,\leq)$ be an ordered Γ-semihypergroup. Then,

(1) $A \subseteq (A]$ for any $A \subseteq S$.

(2) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.

(3) $(A]\Gamma(B] \subseteq (A\Gamma B]$ and $((A]\Gamma B]] = (A\Gamma B]$ for any $A,B \subseteq S$.

Now, we present two examples of ordered Γ-semihypergroups. We refer the readers to see more examples of ordered Γ-semihypergroups in [9,17].

Example 1.4 Let $S=[0,1]$ and $\Gamma=N$. For every $x,y \in S$ and $\gamma \in \Gamma$, we define $\gamma: S \times \Gamma \times S \rightarrow P^*(S)$ by $x\gamma y = [0, \frac{xy}{\gamma}]$. Then, γ is a hyperoperation. For every $x,y,z \in S$ and $\alpha,\beta \in \Gamma$, we have

$$(x\alpha y)\beta z = [0, \frac{xyz}{\alpha\beta}] = x\alpha(y\beta z).$$

This means that S is a Γ-semihypergroup [16]. Consider S as a poset with the natural ordering.

Thus, $(S,\Gamma,\leq)$ is an ordered Γ-semihypergroup.

Example 1.5 Let $S=\{a,b,c,d\}$ and $\Gamma=\{\gamma,\beta\}$ be the sets of binary hyperoperations defined as follows:

γ	a	b	c	d
a	a	$\{a,b\}$	$\{c,d\}$	d
b	$\{a,b\}$	b	$\{c,d\}$	d
c	$\{c,d\}$	$\{c,d\}$	c	d
d	d	d	d	d

β	a	b	c	d
a	a	$\{a,b\}$	$\{c,d\}$	d
b	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	d
c	$\{c,d\}$	$\{c,d\}$	c	d
d	d	d	d	d

Clearly, S is a Γ-semihypergroup. We have $(S,\Gamma,\leq)$ is an ordered Γ-semihypergroup where the order relation $\leq$ is defined by:

$$\leq := \{(a,a), (a,b), (b,b), (c,b), (c,c), (c,d), (d,b), (d,d)\}.$$

2 MAIN RESULTS

For the sake of simplicity, throughout this paper, we denote $I^2=I\Gamma I$. Let A be a non-empty subset of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$. We denote by $I(A)$ the Γ-hyperideal of S generated by A. One can easily prove that

$$I(A)=(A\cup S\Gamma A\cup A\Gamma S\cup S\Gamma A\Gamma S].$$

A non-empty subset I of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is called prime if for every $A,B\subseteq S$ such that $A\Gamma B\subseteq I$, we have $A\subseteq I$ or $B\subseteq I$.

Definition 2.1 A non-empty subset I of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is called weakly prime if for all Γ-hyperideals A,B of S such that $A\Gamma B\subseteq I$, we have $A\subseteq I$ or $B\subseteq I$. Also, I is called a weally prime Γ-hyperideal if I is a Γ-hyperideal which is weakly prime. A Γ-hyperideal I of S is said to be maximal if for any proper Γ-hyperideal K of S, $I\subseteq K$ implies that $I=K$.

Remark 2.2 It is easy to see that every prime Γ-hyperideal is weakly prime.

Theorem 2.3 Let $(S,\Gamma,\leq)$ be a commutative ordered Γ-semihypergroup. If P is a weakly prime Γ-hyperideal of S, then P is prime.

Proof. Assume that A,B are non-empty subsets of S such that $A\Gamma B\subseteq P$. We have

$$\begin{aligned} I(A)\Gamma I(B) &= (A\cup S\Gamma A\cup A\Gamma S\cup S\Gamma A\Gamma S]\Gamma(B\cup S\Gamma B\cup B\Gamma S\cup S\Gamma B\Gamma S] \\ &\subseteq (A\Gamma S\cup S\Gamma A\Gamma B] \\ &\subseteq I(A\Gamma B) \\ &\subseteq I(P)\subseteq (P]=P. \end{aligned}$$

Since P is a weakly prime Γ-hyperideal of S, we get $I(A) \subseteq P$ or $I(B) \subseteq P$. So, we have $A\subseteq P$ or $B\subseteq P$. Therefore, P is prime.

Lemma 2.4 Let $(S,\Gamma,\leq)$ be an ordered Γ-semihypergroup. If A and B are Γ-hyperideals of S, then $A\cup B$ and $A\cap B$ are Γ-hyperideals of S.

Proof. The proof is straightforward.

Theorem 2.5 Let P be a Γ-hyperideal of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$. Then P is weakly prime if and only if for all Γ-hyperideals A and B of S such that $(A\Gamma B]\cap(B\Gamma A]\subseteq P$, we have $A\subseteq P$ or $B\subseteq P$.

Proof. Suppose that P is a weakly prime Γ-hyperideal of S. Let A and B be Γ-hyperideals of S such that $(A\Gamma B]\cap(B\Gamma A]\subseteq P$. First, we show that $(A\Gamma B]$ is a Γ-hyperideal of S. Let $y\in S$ and $x\in(A\Gamma B]$. Then there exist $x\in(A\Gamma B]$, $a\in A$, $b\in B$ and $\alpha\in\Gamma$ such that $x\leq c\leq a\alpha b$. Since S is an ordered Γ-semihypergroup, we get

$$x\beta y \leq c\beta y \leq (a\alpha b)\beta y = a\alpha(b\beta y) \subseteq A\Gamma(B\Gamma S) \subseteq A\Gamma B,$$

where $\beta \in \Gamma$. Hence, $x\beta y \subseteq (A\Gamma B]$. Similarly, we have $y\beta x \subseteq (A\Gamma B]$. If $y \leq x$, then $y \leq x \leq z \in A\Gamma B$, and so $y \in (A\Gamma B]$. Therefore $(A\Gamma B]$ is a Γ-hyperideal of S. Similarly, we can prove that $(B\Gamma A]$ is a Γ-hyperideal of S. Thus,

$$(A\Gamma B]\Gamma(B\Gamma A] \subseteq (A\Gamma B]\Gamma S \subseteq (A\Gamma B] \text{ and } (A\Gamma B]\Gamma(B\Gamma A] \subseteq S\Gamma(B\Gamma A] \subseteq (B\Gamma A].$$

So, we have

$$(A\Gamma B]\Gamma(B\Gamma A] \subseteq (A\Gamma B] \cap (B\Gamma A] \subseteq P.$$

Since P is a weakly prime Γ-hyperideal of S, we get $(A\Gamma B] \subseteq P$ or $(B\Gamma A] \subseteq P$. By Theorem 1.3(1), we have $A\Gamma B \subseteq P$ or $B\Gamma A \subseteq P$. This implies that $A \subseteq P$ or $B \subseteq P$.

Conversely, assume that A and B are Γ-hyperideals of S such that $A\Gamma B \subseteq P$. By Theorem 1.3(2), we have $(A\Gamma B] \cap (B\Gamma A] \subseteq (A\Gamma B] \subseteq (P] \subseteq P$. By hypothesis, we have $A \subseteq P$ or $B \subseteq P$. Therefore, P is a weakly prime Γ-hyperideal of S.

In the following, we define right weakly prime Γ-hyperideals in ordered Γ-semihypergroups and investigate some of their related results.

Definition 2.6 A right Γ-hyperideal I of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is said to be a right weakly prime Γ-hyperideal of S if $(A\Gamma B] \cap (B\Gamma A] \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ for all right Γ-hyperideals A,B of S.

Theorem 2.7 Let $(S,\Gamma,\leq)$ be anordered Γ-semihypergroup. If M is a maximal right Γ-hyperideal of S such that $I \cap I^2 \neq \phi$, where $I=S \backslash M$, then M is a right weakly prime Γ-hyperideal of S.

Proof. Suppose that M is a maximal right Γ-hyperideal of S such that $I \cap I^2 \neq \phi$, where $I=S \backslash M$. If M is not a right weakly prime Γ-hyperideal of S, then there exist right Γ-hyperideals A,B of S such that $(A\Gamma B] \cap (B\Gamma A] \subseteq M$, $A \not\subset M$ and $B \not\subset M$. Since $A \not\subset M$, it follows that $M \subseteq A \cup M$. By Lemma 2.4, $A \cup M$ is a right Γ-hyperideal of S. So, we get $A \cup M=S$. Similarly, we have $B \cup M=S$. Hence,

$$I=S \backslash M=(A \cup M) \backslash M=A \backslash M \subseteq A \text{ and } I=S \backslash M=(B \cup M) \backslash M=B \backslash M \subseteq B,$$

which imply that

$$I^2 \subseteq A\Gamma B \cap B\Gamma A \subseteq (A\Gamma B] \cap (B\Gamma A] \subseteq M.$$

Since $I \cap I^2 \neq \phi$, it follows that $I \cap M \neq \phi$, which is a contradiction. Therefore, M is a right weakly prime Γ-hyperideal of S.

An element a of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is called an idempotent of S if $a\in a\gamma a$ for every $\gamma\in\Gamma$. In view of Theorem 2.7, we have the following corollaries.

Corollary 2.8 Let $(S,\Gamma,\leq)$ be an ordered Γ-semihypergroup. If M is a maximal right Γ-hyperideal of S such that $S\backslash M$ contains an idempotent of S, then M is a right weakly prime Γ-hyperideal of S.

Proof. Suppose that M is a maximal right Γ-hyperideal of S such that $I=S\backslash M$ contains an idempotent a of S. Set $I=S\backslash M$. Since $a\in a\gamma a\subseteq I\Gamma I=I^2$, it follows that $I\cap I^2\neq\phi$. By Theorem 2.7, M is a right weakly prime Γ-hyperideal of S.

Corollary 2.9 Let $(S,\Gamma,\leq)$ be an ordered Γ-semihypergroup. If $S=(a\gamma S]$ for some $a\in S$ and $\gamma\in\Gamma$, then every maximal right Γ-hyperideal of S is a right weakly prime Γ-hyperideal of S.

Proof. Suppose that $S=(a\gamma S]$ for some $a\in S$ and $\gamma\in\Gamma$. Let M be a maximal right Γ-hyperideal of S and $I=S\backslash M$. If a is not in I, then $a\in M$. It follows that

$$S=(a\gamma S]\subseteq(M\Gamma S]\subseteq(M]=M,$$

which is a contradiction. This leads to $a\in I$. So, we have $a\gamma a\subseteq I\Gamma I=I^2$. Now, let $a\gamma a\cap I=\phi$. Then $a\gamma a\subseteq M$. This implies that

$$S=(a\gamma S]=(a\gamma(a\gamma S]]\subseteq((a]\gamma(a\gamma S]]=(a\gamma a\gamma S]\subseteq(M\Gamma S]\subseteq(M]=M.$$

This is a contradiction. Hence, $a\gamma a\cap I\neq\phi$. Thus there exists $x\in a\gamma a\subseteq I^2$ such that $x\in I$. Therefore, $I\cap I^2\neq\phi$. Now, by Theorem 2.7, M is a right weakly prime Γ-hyperideal of S.

REFERENCES

[1] F. Marty, Sur une generalization de la notion de groupe, *Eight Congress Math. Scandinaves*, Stockholm, 45-49 (1934).

[2] P. Corsini, *Prolegomena of Hypergroup Theory*, Second edition, Aviani Editore, Italy, (1993).

[3] B. Davvaz, *Semihypergroup Theory, Elsevier*, (2016).

[4] D. Heidari and B. Davvaz, “On ordered hyperstructures”, *U.P.B. Sci. Bull. Series A.*, **73(2)**, 85-96 (2011).

[5] B. Davvaz, P. Corsini and T. Changphas, “Relationship between ordered semihypergroups and ordered semigroups by using pseudoorder”, *European J. Combinatorics*, **44**, 208-217 (2015).

[6] Z. Gu and X. Tang, “Ordered regular equivalence relations on ordered semihypergroups”, *J. Algebra*, **450**, 384-397 (2016).

[7] S. Omidi and B. Davvaz, “A short note on the relation *N* in ordered semihypergroups”, *GU J. Sci.*, **29(3)**, 659-662 (2016).

[8] S. Omidi, B. Davvaz and C. Abdioglu, “Some properties of quasi-Γ-hyperideals and hyperfilters in ordered Γ-semihypergroups”, *Southeast Asian Bull. Math.*, **42**, 223-242 (2018).

[9] J. Tang, B. Davvaz and X. Y. Xie, “A study on (fuzzy) quasi-Γ-hyperideals in ordered Γ-semihypergroups”, *Journal of Intelligent & Fuzzy Systems* **32,** 3821-3838 (2017).

[10] B. Davvaz and S. Omidi, "Basic notions and properties of ordered semihyperrings", *Categ. General Alg. Structures Appl.* **4(1),** 43-62 (2016).
[11] S. Omidi and B. Davvaz, "Contribution to study special kinds of hyperideals in ordered Semihyperrings", *J. Taibah Univ. Sci.* **11**, 1083-1094 (2017).
[12] S. Omidi and B. Davvaz, "Ordered Krasner hyperrings", *Iranian Journal of Mathematical Sciences and Informatics*, **12(2)**, 35-49 (2017).
[13] Y. I. Kwon and S. K. Lee, "On weakly prime ideals of ordered Γ-semigroups", *Comm. Korean Math. Soc.,* **13(2)**, 251-256 (1998).
[14] T. Changphas, "Right prime ideals and maximal right ideals in ordered semigroups", *Far East Journal of Mathematical Sciences*, **73(1),** 65-72 (2013).
[15] M.Y. Abbasi and A. Basar, "Weakly prime ideals in involution po-Γ-semigroups", *Kyungpook Math. J.*, **54,** 629-638 (2014).
[16] S. M. Anvariyeh, S. Mirvakili and B. Davvaz, "On Γ-hyperideals in Γ-semihypergroups", *Carpathian J. Math*., **26(1)**, 11-23 (2010).
[17] J. Tang, B. Davvaz, X. Xie and N. Yaqoob, "On fuzzy interior Γ -hyperideals in ordered Γ –semihypergroups", *Journal of Intelligent & Fuzzy Systems* **32**, 2447-2460 (2017).

Received April 15, 2018