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# WEAK INSERTION OF A CONTRA-CONTINUOUS FUNCTION BETWEEN TWO COMPARABLE CONTRA-PRECONTINUOUS (CONTRA-SEMI-CONTINUOUS) FUNCTIONS

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**Summary**. A sufficient condition in terms of lower cut sets are given for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.

## 1 INTRODUCTION

The concept of preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [4], under the name locally dense. Subset A of a topological space $(X, \tau)$ is called preopen or locally dense or nearly open if $A \subseteq Int(Cl(A))$. Set A is called preclosed if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term ,preopen, was used for the first time by A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb [20].

The concept of semi-open set in a topological space was introduced by

N. Levine in 1963 [17]. Subset A of a topological space $(X, \tau)$ is called semi-open [10] if $A \subseteq Cl(Int(A))$. Set A is called semi-closed if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V$ −sets. Complements of $V$ −sets, i.e., sets that are intersection of open sets are called $\Lambda$−sets [19].

Recall that a real-valued function f defined on a topological space X is called A−continuous [23] if the preimage of every open subset of R belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of A−continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contra-continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 12, 13, 22].

Hence, a real-valued function f defined on a topological space X is called contra-continuous (resp. contra-semi−continuous , contra-precontinuous) if the preimage of every open subset of R is closed (resp. semi−closed , pre-closed) in X[6].

Results of Katˇetov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between

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two comparable real-valued functions on such topological spaces that Λ−sets or kernel of sets are open [19].

If g and f are real-valued functions defined on a space X, we write $g \le f$ in case $g(x) \le f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is cc−property provided that any constant function has property P and provided that the sum of a function with property P and any contra-continuous function also has property P.

If P1 and P2 are cc−properties, the following terminology is used: A space X has the weak cc−insertion property for (P1,P2) if and only if for any functions g and f on X such that $g \le f$, g has property P1 and f has property P2, then there exists a contra-continuous function h such that $g \le h \le f$.

In this paper, for a topological space whose Λ−sets or kernel of sets are open, is given a sufficient condition for the weak cc−insertion property. Several insertion theorems are obtained as corollaries of these results.

## 2 THE MAIN RESULT

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.

**Definition 2.1**. Let A be a subset of a topological space (X, τ ). We define the subsets $A^{\Lambda}$ and $A^{V}$ as follows:

$$A^{\Lambda} = \cap\{O : O \supseteq A, O \in \tau\} \text{ and } A^{V} = \cup\{F : F \subseteq A, F^{c} \in \tau\}.$$

In [7, 18, 21], $A^{\Lambda}$ is called the kernel of A.

The family of all preopen, preclosed, semi−open and semi−closed will be denoted by pO(X, τ), pC(X, τ), sO(X, τ ) and sC(X, τ), respectively.

We define the subsets $p(A^{\Lambda})$,$p(A^{V})$,$s(A^{\Lambda})$ and $s(A^{V})$ as follows:

$$p(A^{V})= \cup\{F : F \subseteq A, F \in pC(X, \tau)\},$$

$$p(A^{\Lambda})= \cap\{O : O \supseteq A, O \in pO(X, \tau)\},$$

$$s(A^{\Lambda})= \cap\{O : O \supseteq A, O \in sO(X, \tau)\} \text{ and}$$

$$s(A^{V})= \cup\{F : F \subseteq A, F \in sC(X, \tau)\}.$$

$p(A^{\Lambda})$ (resp. $s(A^{\Lambda})$) is called the prekernel (resp. semi − kernel) of A.

The following two definitions are modifications of conditions considered in [14, 15].

**Definition 2.2**. If ρ is a binary relation in set S then $\bar{\rho}$ is defined as follows:

$x \bar{\rho} y$ if and only if yρv implies xρv and uρx implies uρy for any u and v in S.

**Definition 2.3**. A binary relation ρ in the power set P (X) of a topological space X is called a strong binary relation in P (X) in case ρ satisfies each of the following conditions:

1) If $A_i \rho B_j$ for any $i \in\{1,...,m\}$ and for any $j \in\{1,...,n\}$, then there exists a set C in P (X)

such that Ai $\rho$C and C$\rho$Bj for any $i \in \{1,...,m\}$ and any $j \in \{1,...,n\}$.

2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.

3) If A$\rho$B, then $A^{\wedge} \subseteq B$ and $A \subseteq B^{\vee}$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 2.4**. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < l\} \subseteq A(f, l) \subseteq \{x \in X : f(x) \leq l\}$ for a real number l, then A(f, l) is called a lower indefinite cut set in the domain of f at the level I.

We now give the following main result:

**Theorem 2.1**. Let g and f be real-valued functions on a topological space X, in which kernel sets are open, with $g \leq f$. If there exists a strong binary relation $\rho$ on the power set of X and if there exist lower indefinite cut sets A(f, t) and A(g, t) in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1)\ \rho A(g, t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$.

**Proof**. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation $\rho$ on the power set of X and there exist lower indefinite cut sets A(f, t) and A(g, t) in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1)\ \rho A(g, t_2)$.

Define functions F and G mapping the rational numbers Qinto the power set of X by F (t)= A(f, t) and G(t)= A(g, t). If $t_1$ and $t_2$ are any elements of Q with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2)$, $G(t_1)\ \rho G(t_2)$, and $F(t_1)\ \rho G(t_2)$. By Lemmas 1 and 2 of [15] it follows that there exists a function H mapping Q into the power set of X such that if $t_1$ and $t_2$ are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho H(t_2)$, $H(t_1)\ \rho H(t_2)$ and $H(t1)\ \rho G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in Q : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in H(t) then x is in $G(t^{/})$ for any $t^{/} > t$; since x is in $G(t^{/}) = A(g, t^{/})$ implies that $g(x) \leq t^{/}$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in H(t), then x is not in $F(t^{/})$ for any $t^{/} < t$; since x is not in $F(t^{/}) = A(f, t^{/})$ implies that $f(x) > t^{/}$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers $t_1$ and $t_2$ with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = H(t_2)^{\vee} \setminus H(t_1)^{\wedge}$. Hence $h^{-1}(t_1,t_2)$ is closed in X, i.e., h is a contra-continuous function on X. •

The above proof used the technique of Theorem 1 in [14].

## 3 APPLICATIONS

The abbreviations cpc and csc are used for contra-precontinuous and contra-semi−continuous, respectively.

Before stating the consequences of Theorems 2.1, we suppose that X is a topological space whose kernel sets are open.

**Corollary 3.1**. If for each pair of disjoint preopen (resp. semi−open) sets $G_1,G_2$ of X , there exist closed sets $F_1$ and $F_2$ of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then X has the weak cc−insertion property for (cpc, cpc) (resp. (csc, csc)).

**Proof**. Let g and f be real-valued functions defined on X, such that f and g are cpc (resp. csc), and $g \le f$. If a binary relation $\rho$ is defined by $A\rho B$ in case $p(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $s(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis $\rho$ is a strong binary relation in the power set of X. If $t_1$ and $t_2$ are any elements of Q with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \le t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \le t_1\}$ is preopen (resp. semi−open) set and since $\{x \in X : g(x) < t_2\}$ is preclosed (resp. semi−closed) set, it follows that $p(A(f, t_1)^{\Lambda}) \subseteq p(A(g, t_2)^{V})$ (resp. $s(A(f, t_1)^{\Lambda}) \subseteq s(A(g, t_2)^{V})$). Hence $t_1 < t_2$ implies that $A(f, t_1)\, \rho A(g, t_2)$. The proof follows from Theorem 2.1. •

**Corollary 3.2**. If for each pair of disjoint preopen (resp. semi−open) sets $G_1, G_2$, there exist closed sets $F_1$ and $F_2$ such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then every contra-precontinuous (resp. contra-semi−continuous) function is contra-continuous.

**Proof**. Let f be a real-valued contra-precontinuous (resp. contra-semi−continuous) function defined on X. Set g = f, then by Corollary 3.1, there exists a contra-continuous function h such that g = h = f.•

**Corollary 3.3**. If for each pair of disjoint subsets $G_1, G_2$ of X , such that $G_1$ is preopen and $G_2$ is semi−open, there exist closed subsets $F_1$ and $F_2$ of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then X have the weak cc−insertion property for (cpc, csc) and (csc, cpc).

**Proof**. Let g and f be real-valued functions defined on X, such that g is cpc (resp. csc) and f is csc (resp. cpc), with $g \le f$.If a binary relation $\rho$ is defined by $A\rho B$ in case $s(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $p(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis $\rho$ is a strong binary relation in the power set of X. If $t_1$ and $t_2$ are any elements of Q with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \le t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \le t_1\}$ is semi−open (resp. preopen) set and since $\{x \in X : g(x) < t_2\}$ is preclosed (resp. semi−closed) set, it follows that $s(A(f, t_1)^{\Lambda}) \subseteq p(A(g, t_2)^{V})$ (resp. $p(A(f, t_1)^{\Lambda}) \subseteq s(A(g, t_2)^{V})$). Hence t1 $< t_2$ implies that $A(f, t_1)\, \rho A(g, t_2)$. The proof follows from Theorem 2.1. •

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## REFERENCES

[1] A. Al-Omari and M.S. Md Noorani, “Some properties of contra-bcontinuous and almost contra-b-continuous functions”, *European J. Pure. Appl. Math.*, **2(2)**, 213-230 (2009).
[2] F. Brooks, “Indefinite cut sets for real functions”, *Amer. Math. Monthly*, **78**, 1007-1010 (1971).
[3] M. Caldas and S. Jafari, “Some properties of contra-β−continuous functions”, *Mem. Fac. Sci. Kochi. Univ.*, **22**, 19-28 (2001).

[4] H.H. Corson and E. Michael, “Metrizability of certain countable unions”, *Illinois J. Math.*, **8**, 351-360 (1964).
[5] J. Dontchev, “The characterization of some peculiar topological space via α− and β−sets”, *Acta Math. Hungar.*, **69(1-2)**, 67-71 (1995).
[6] J. Dontchev, “Contra-continuous functions and strongly S-closed space”, *Intrnat. J. Math. Math. Sci.*, **19(2)**, 303-310 (1996).
[7] J. Dontchev, and H. Maki, “On sg-closed sets and semi−λ−closed sets”, *Questions Answers Gen. Topology*, **15(2)**, 259-266 (1997).
[8] E. Ekici, “On contra-continuity”, *Annales Univ. Sci. Bodapest*, **47**, 127-137 (2004).
[9] E. Ekici, “New forms of contra-continuity”, *Carpathian J. Math.*, **24(1)**, 37-45 (2008).
[10] A.I. El-Magbrabi, Some properties of contra-continuous mappings, *Int. J. General Topol.*, **3(1-2)**, 55-64 (2010).
[11] M. Ganster and I. Reilly, “A decomposition of continuity”, *Acta Math. Hungar.*, **56(3-4)**, 299-301 (1990).
[12] S. Jafari and T. Noiri, “Contra-continuous function between topological spaces”, *Iranian Int. J. Sci.*, **2**, 153-167 (2001).
[13] S. Jafari and T. Noiri, “On contra-precontinuous functions”, *Bull. Malaysian Math. Sc. Soc.*, **25**, 115-128 (2002).
[14] M. Katˇetov, ”On real-valued functions in topological spaces”, *Fund. Math.*, **38**, 85-91 (1951).
[15] M. Katˇetov, “Correction to On real-valued functions in topological spaces”, *Fund. Math.*, **40**, 203-205 (1953).
[16] E. Lane, “Insertion of a continuous function”, *Pacific J. Math.*, **66**, 181-190 (1976).
[17] N. Levine, “Semi-open sets and semi-continuity in topological space”, *Amer. Math. Monthly*, **70**, 36-41 (1963).
[18] S. N. Maheshwari and R. Prasad, “On ROs−spaces”, *Portugal. Math.*, **34**, 213-217 (1975).
[19] H. Maki, “Generalized Λ−sets and the associated closure operator”, *The special Issue in commemoration of Prof. Kazuada IKEDA’s Retirement*, 139-146 (1986).
[20] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, “On pre-continuous and weak pre-continuous mappings”, *Proc. Math. Phys. Soc. Egypt*, **53**, 47-53 (1982).
[21] M. Mrsevic, “On pairwise R and pairwise R1 bitopological spaces”, *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, **30**, 141-145 (1986).
[22] A.A. Nasef, “Some properties of contra-continuous functions”, *Chaos Solitons Fractals*, **24**, 471-477 (2005).
[23] M. Przemski, “A decomposition of continuity and α−continuity”, *Acta Math. Hungar.*, **61(1-2)**, 93-98 (1993).

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