ON SOME METRIC TOPOLOGIES ON PRIVALOV SPACES ON THE UNIT DISK

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Summary. Let N^p $(1 < p < \infty)$ be the Privalov class N^p of holomorphic functions on the open unit disk $\mathbb{D}$ in the complex plane. In 1977 M. Stoll proved that the class N^p equipped with the topology given by the metric λ_p defined by

$$\lambda_p(f,g) = \left(\int_0^{2\pi} \left(\log(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|) \right)^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad f,g \in N^p,$$

becomes an F-algebra. In the recent overview paper by Meštrović and Pavićević (2017) a survey of some known results on the topological structures of the Privalov spaces N^p $(1 < p < \infty)$ and their Fréchet envelopes F^p are presented.

In this article we continue a survey of results concerning the topological structures of the spaces N^p $(1(p < \infty)$. In particular, for each $p > 1$, we consider the class N^p as the space M^p equipped with the topology induced by the metric ρ_p defined as

$$\rho_p(f,g) = \left(\int_0^{2\pi} \log^p(1+M(f-g)(\theta)) \frac{d\theta}{2\pi} \right)^{1/p}, \quad f,g \in M^p, \text{ where } Mf(\theta) = \sup_{0\leqslant r<1} \left| f\left(re^{i\theta}\right) \right|.$$

On the other hand, we consider the class N^p with the metric topology introduced by Meštrović, Pavićević and Labudović (1999) which generalizes the Gamelin-Lumer's metric which is generally defined on a measure space (Ω, Σ, μ) with a positive finite measure μ. The space N^p with the associated modular in the sense of Musielak and Orlicz becomes the Hardy-Orlicz class. It is noticed that the all considered metrics induce the same topology on the space N^p.

2010 Mathematics Subject Classification: 30H50, 46E10, 46E30, 46J15, 30H15.
Keywords and Phrases: Privalov space N^p, the class M^p, metric topology, equivalent topologies, Orlicz space, Hardy-Orlicz class, modular space.

1 INTRODUCTION AND PRELIMINARY RESULTS

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let $\mathbb{T}$ denote the boundary of $\mathbb{D}$. Let $L^q(\mathbb{T})$ $(0 < q \leq \infty)$ be the familiar Lebesgue spaces on the unit circle $\mathbb{T}$. The *Privalov class* N^p $(1 < p < \infty)$ consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\sup_{0\leq r<1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p \frac{d\theta}{2\pi} < +\infty, \tag{1}$$

where for $z \in \mathbb{C}$, $\log^+ |z| = \max(\log |z|, 0)$ if $z \neq 0$ and $\log^+ 0 = 0$. These classes were firstly considered by I.I. Privalov in [40, p. 93], where N^p is denoted as A_q.

Notice that for $p = 1$ the condition (1) defines the *Nevanlinna class* N of holomorphic functions on $\mathbb{D}$. Recall that the *Smirnov class* N^+ is the set of all functions $f \in N$ such that

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi} < +\infty,$$

where f^* is the boundary function of f on $\mathbb{T}$, i.e.,

$$f^*(e^{i\theta}) = \lim_{r\to 1-} f(re^{i\theta})$$

is the *radial limit* of f which exists for almost every $e^{i\theta} \in \mathbb{T}$. Recall that the classical *Hardy space* H^q $(0 < q \leq \infty)$ consists of all functions f holomorphic on $\mathbb{D}$ such that

$$\sup_{0\leq r<1} \int_0^{2\pi} \left|f\left(r\,\mathrm{e}^{i\theta}\right)\right|^q \frac{d\theta}{2\pi} < \infty$$

if $0 < q < \infty$, and which are bounded when $q = \infty$:

$$\sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

It is known that (see [36] and [25])

$$N^r \subset N^p \ (r > p), \quad \bigcup_{q>0} H^q \subset \bigcap_{p>1} N^p, \quad \text{and} \quad \bigcup_{p>1} N^p \subset M \subset N^+ \subset N,$$

where the above containment relations are proper.

It is well known (see, e.g., [4, p. 26]) that a function $f \in N^+$ has a unique factorization of the form

$$f(z) = B(z)S_\mu(z)F(z), \quad z \in \mathbb{D},$$

where B is the *Blaschke product* with respect to zeros $\{z_k\} \subset \mathbb{D}$ of f, S_μ is a *singular inner function* and F is an *outer function* in N^+, i.e.,

$$B(z) = z^m \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \cdot \frac{z_k - z}{1 - \bar{z}_k z}, \quad z \in \mathbb{D},$$

with $\sum_{k=1}^{\infty}\big(1-|z_k|\big)<\infty$, m a nonnegative integer,

$$S_\mu(z)=\exp\Big(-\int\limits_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\Big)$$

with positive singular measure $d\mu$ and

$$F(z)=\lambda\exp\Big(\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\big|F^*\big(e^{it}\big)\big|\,dt\Big), \tag{2}$$

where $\lambda\in\mathbb{C}$ with $|\lambda|=1$ and $\log|F^*|\in L^1(\mathbb{T})$.

Recall that a function I of the form

$$I(z)=B(z)S_\mu(z),\quad z\in\mathbb{D},$$

is called an *inner function*. Furthermore, it is well known that $|I^*(e^{it})|=1$ for almost every $e^{it}\in\mathbb{T}$ and hence, $|f^*(e^{it})|=|F^*(e^{it})|$ for almost every $e^{it}\in\mathbb{T}$.

I.I. Privalov [40, p. 98] (alo see [25, Theorem 5.3]) proved that a function f holomorphic on $\mathbb{D}$ belongs to the class N^p if and only if $f=IF$, where I is an inner function on $\mathbb{D}$ and F is an outer function given by (2) such that $\log^+|f^*|\in L^p(\mathbb{T})$ (or equivalently, $\log^+|F^*|\in L^p(\mathbb{T})$).

M. Stoll [44, Theorem 4.2] showed that the space N^p (with the notation $(\log^+ H)^\alpha$ in [44]) equipped with the topology given by the metric λ_p defined by

$$\lambda_p(f,g)=\left(\int\limits_0^{2\pi}\big(\log(1+|f^*(e^{i\theta})-g^*(e^{i\theta})|)\big)^p\,\frac{d\theta}{2\pi}\right)^{1/p},\quad f,g\in N^p, \tag{3}$$

becomes an F-algebra. Recall that the function $\lambda_1=\lambda$ defined on the Smirnov class N^+ by (3) with $p=1$ induces the metric topology on N^+. N. Yanagihara [45] proved that under this topology, N^+ is an F-space.

Furthermore, in connection with the spaces N^p $(1<p<\infty)$, Stoll [44] (also see [5] and [29, Section 3]) also studied the spaces F^q $(0<q<\infty)$ (with the notation $F_{1/q}$ in [44]), consisting of those functions f holomorphic on $\mathbb{D}$ for which

$$\lim_{r\to 1}(1-r)^{1/q}\log^+ M_\infty(r,f)=0,$$

where

$$M_\infty(r,f)=\max_{|z|\le r}|f(z)|.$$

Stoll [44, Theorem 3.2] also proved that the space F^q with the topology given by the family of seminorms $\{|||\cdot|||_{q,c}\}_{c>0}$ defined for $f\in F^q$ as

$$|||f|||_{q,c}=\sum_{n=0}^{\infty}|\hat{f}(n)|e^{-cn^{1/(q+1)}}<\infty$$

for each $c>0$, where $\hat{f}(n)$ is the n-th Taylor coefficient of f, is a countably normed *Fréchet algebra*. By a result of C.M. Eoff [5, Theorem 4.2], F^p is the *Fréchet envelope* of N^p and hence, F^p and N^p have the same topological duals.

Following H.O. Kim ([13] and [14]), the class M consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\int_0^{2\pi} \log^+ Mf(\theta) \frac{d\theta}{2\pi} < \infty,$$

where

$$Mf(\theta) = \sup_{0 \leqslant r < 1} \left| f\left(re^{i\theta}\right) \right|$$

is the *maximal radial function* of f.

The study on the class M on the disk $\mathbb{D}$ has been exstensively investigated by H.O. Kim in [13] and [14], V.I. Gavrilov and V.S. Zaharyan [9] and M. Nawrocky [39]. Kim [14, Theorems 3.1 and 6.1] showed that the space M with the topology given by the metric ρ defined by

$$\rho(f, g) = \int_0^{2\pi} \log(1 + M(f-g)(\theta)) \frac{d\theta}{2\pi}, \quad f, g \in M, \tag{4}$$

becomes an F-algebra. Furthermore, Kim [14, Theorems 5.2 and 5.3] gave an incomplete characterization of multipliers of M into H^∞. Consequently, the topological dual of M is not exactly determined in [14], but as an application, it was proved in [14, Theorem 5.4] (also cf. [39, Corollary 4]) that M is not locally convex space. Furthermore, the space M is not locally bounded ([14, Theorem 4.5] and [39, Corollary 5]).

Nevertheless that as noticed above, the class M is essentially smaller than the class N^+, M. Nawrocky [39] showed that the class M and the Smirnov class N^+ have the same corresponding locally convex structure which was already established by N. Yanagihara for the Smirnov class in [45] and [46]. More precisely, it was proved in [39, Theorems 1] that the Fréchet envelope of the class M can be identified with the space F^+ of holomorphic functions on the open unit disk $\mathbb{D}$ such that

$$|||f|||_c := \sum_{n=0}^{\infty} |\hat{f}(n)| e^{-c\sqrt{n}} < \infty$$

for each $c > 0$, where $\hat{f}(n)$ is the n-th Taylor coefficient of f. Notice that F^+ coincides with the space F^1 defined above. It was shown in [46] (also see [45]) that F^+ is actually the containing Fréchet space for N^+ (also see [43]). Moreover, Nawrocky [39, Theorem 1] characterized the set of all continuous linear functionals on M which by a result of Yanagihara [45] coincides with those on the Smirnov class N^+.

Motivated by the mentioned investigations of the classes M and N^+, and the fact that the classes N^p $(1 < p < \infty)$ are generalizations of the Smirnov class N^+, in [20, Chapter 6] and [22] the first author of this paper investigated the classes M^p $(1 < p < \infty)$ as generalizations of the class M. Accordingly, the *class* M^p $(1 < p < \infty)$ consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\int_0^{2\pi} \left(\log^+ Mf(\theta)\right)^p \frac{d\theta}{2\pi} < \infty.$$

Obviously,

$$\bigcup_{p>1} M^p \subset M.$$

By analogy with the topology defined on the space M ([13] and [14]), the space M^p can be equipped with the topology induced by the metric ρ_p defined as

$$\rho_p(f,g) = \Big(\int_0^{2\pi} \log^p (1 + M(f-g)(\theta)) \, \frac{d\theta}{2\pi} \Big)^{1/p},$$

with $f, g \in M^p$.

After Privalov, the study of the spaces N^p $(1 < p < \infty)$ was continued in 1977 by M. Stoll [44] (with the notation $(\log^+ H)^\alpha$ instead of N^p in [44]). Further, the linear topological and functional properties of these spaces were extensively investigated by C.M. Eoff in [5] and [6], N. Mochizuki [36], Y. Iida and N. Mochizuki [12], Y. Matsugu [17], J.S. Choa [2], J.S. Choa and H.O. Kim [3], A.K. Sharma and S.-I. Ueki [42] and in works [19]-[35] of authors of this paper; typically, the notation varied and Privalov was mentioned in [17], [21]-[24], [29]-[32], [34], [35] and [42]. In particular, it was proved in [21, Corollary] that N^p is not locally convex space and in [30, Theorem 1.1] that N^p is not locally bounded space. We refer the recent monograph [10, Chapters 2, 3 and 9] by V.I. Gavrilov, A.V. Subbotin and D.A. Efimov for a good reference on the spaces N^p $(1 < p < \infty)$.

Let us recall that in our recent overview paper [32] it was given a survey of some known results on different topologies on the Privalov classes N^p $(1 < p < \infty)$ and their Fréchet envelopes F^p $(1 < p < \infty)$ on the open unit disk. Here we give a survey on related extended results involving some other metrics and the induced topologies on the classes N^p.

The remainder of this overview paper is organized in three sections. For any fixed $p > 1$, in Section 2 we present some results concerning the topological and functional structures on the classes M^p $(1 < p < \infty)$. Section 3 is devoted to the consideration of the Privalov class N^p as a closed subspace of some Orlicz space. In this setting N^p with the associated modular in the sense of Musielak and Orlicz becomes the Hardy-Orlicz class whose topology coincides with both metric topologies λ_p and ρ_p. Concluding remarks are presented in the last section.

2 THE ρ_p-METRIC TOPOLOGY ON PRIVALOV SPACE N^p

Here we focus our attention to certain results from [20, Chapter 6] and [22] concerning the classes M^p $(1 < p < +\infty)$. In [22] it is proved the following basic result.

Theorem 1 ([22, Theorem 2]). *The function ρ_p defined on M^p as*

$$\rho_p(f,g) = \Big(\int_0^{2\pi} \log^p (1 + M(f-g)(\theta)) \, \frac{d\theta}{2\pi} \Big)^{1/p}, \quad f, g \in M^p, \tag{5}$$

is a translation invariant metric on M^p. Further, the space M^p is a complete metric space with respect to the metric ρ_p.

Remark 1. Notice that the expression (5) with $p = 1$ defines the metric $\rho_1 = \rho$ on the class M (given by (4)) introduced by H.O. Kim in [13] and [14]. As noticed above, it was proved in [14] that the metric ρ induces the topology on M under which M is also an F-algebra.

Moreover, the following two statements are also proved in [20, Chapter 6].

Theorem 2 ([22, Theorem 11]). *$M^p = N^p$ for each $p > 1$; that is, the spaces M^p and N^p coincide.*

Theorem 3 ([22, Theorem 15]). *M^p with the topology given by the metric ρ_p defined by* (5) *becomes an F-space.*

Using Theorem 3 and the open mapping theorem (see, e.g., [41, Corollary 2.12 (b)]), the following result was also proved in [22].

Theorem 4 ([22, Theorem 16]). *For each $p > 1$ the classes M^p and N^p coincide, and the metric spaces (M^p, ρ_p) and (N^p, λ_p) have the same topological structure, where the metrics ρ_p and λ_p are given on M^p and N^p by* (5) *and* (3)*, respectively.*

As an immediate consequence of Theorem 4 and [22, Lemma 8], we obtain the following assertion.

Proposition 1. *The convergence with respect to the metric ρ_p given by* (5) *on the space M^p is stronger than the metric of uniform convergence on compact subsets of the disk $\mathbb{D}$.*

Remark 2. For an outer function h let $H^2(|h^*|^2)$ denote the closure of the (analytic) polynomials in the space $L^2(|h^*|^2 d\,\theta)$. By using the famous Beurling's theorem for the Hardy space H^2 ([1]; also see [11, Ch. 7, p. 99]), it was proved in [6] (also see [27, Section 1]) that the class N^p can be represented as a union of certain weighted Hardy classes. Using this representation, the following two topologies are defined on N^p in [27]: the usual locally convex inductive limit topology, which we shall call the *Helson topology* and denote by $\mathcal{H}_p$, in which a neighborhood base for 0 is given by those balanced convex sets whose intersection with each $H^2\,(|h^*|^2)$ is a neighborhood of zero in $H^2\,(|h^*|^2)$, and a not locally convex topology, denoted by I_p, in which a neighborhood base for zero is given by all sets whose intersection with each space $H^2\,(|h^*|^2)$ is a neighborhood of zero. It was proved in [27, Theorem E] (cf. [20, Chapter 3]) that the topology $\mathcal{H}_p$ coincides with the metric topology induced on N^p by the Stoll's metric topology λ_p given by (3). Moreover, it was proved in [6] that the topology I_p coincides with the metric topology λ_p and by Theorem 4, I_p also coincides with the metric topology ρ_p, which are not locally convex. Hence, I_p is strictly stronger than $\mathcal{H}_p$. The analogous results for the space N^+ are proved by J.E. McCarthy in [18].

3 THE SPACE N^p AS THE HARDY-ORLICZ CLASS

In this section we we give a short survey about Privalov classes N^p $(1 < p < +\infty)$ as the Hardy-Orlicz classes. Related results are mainly obtained in [33].

Let (Ω, Σ, μ) be a measure space, i.e., Ω is a nonempty set, Σ is a σ-algebra of subsets of Ω and μ is a nonnegative finite complete measure not vanishing identically. Denote by $L^p(\mu) = L^p$ $(0 < p \leq \infty)$ the familiar Lebesgue spaces on Ω. For each real number $p > 0$ in [33] it was considered the class $L_p^+(\mu) = L_p^+$ of all (equivalence classes of) Σ-measurable complex-valued functions f defined on Ω such that the function $\log^+|f|$ belongs to the space L^p, i.e.,

$$\int_\Omega \left(\log^+|f(x)|\right)^p \, d\mu < +\infty,$$

where $\log^+|a| = \max(\log|a|, 0)$. Clearly, $L_q^+ \subset L_p^+$ for $q > p$ and $\bigcup_{p>0} L^p \subset \bigcap_{p>0} L_p^+$ [33, Section 2]. For each $p > 0$ the space L_p^+ is an algebra with respect to the pointwise addition and multiplication. For each $p > 0$ we define the metric d_p on L_p^+ by

$$\begin{aligned} d_p(f,g) &= \inf_{t>0} \left[t + \mu\left(\{x \in \Omega : |f(x) - g(x)| \geq t\}\right)\right] \\ &\quad + \int_\Omega \left| \left(\log^+|f(x)|\right)^p - \left(\log^+|g(x)|\right)^p \right| \, d\mu. \end{aligned} \tag{6}$$

Recall that the space L_1^+ was introduced by T. Gamelin and G. Lumer in [8, p. 122] (also see [7, p. 122], where L_1^+ is denoted as $L(\mu)$). Note that the metric d_p given by (6) with $p = 1$ coincides with the Gamelin-Lumer's metric d defined on L_1^+. It was proved in [8, Theorem 1.3, p. 122] (also see [7, Theorem 2.3, p. 122]) that the space L_1^+ with the topology given by the metric d_1 becomes a topological algebra. The following result is a generalization of the corresponding result for the case $p = 1$ given in [8, p. 122] (also see [7, p. 122]).

Theorem 5 ([33, Theorem 2.1]). *The space L_p^+ with the metric d_p given by* (6) *is a topological algebra, i.e., a topological vector space with a complete metric in which multiplication is continuous.*

By the inequality

$$\left(\log(1 + |z|)\right)^p \leq 2^{\max(p-1,0)} \left((\log 2)^p + (\log^+|z|)^p\right), \quad z \in \mathbb{C},$$

it follows that a function f belongs to the space L_p^+ if and only if

$$\|f\|_p := \left(\int_\Omega (\log(1 + |f(x)|))^p \, d\mu \right)^{1/p} < \infty. \tag{7}$$

Furthermore [33, Section 2], the function σ_p defined as

$$\sigma_p(f,g) = \left(\|f - g\|_p\right)^{\min(1,p)}, \quad f, g \in L_p^+, 0 < p \leq 1, \tag{8}$$

is a translation invariant metric on L_p^+ for all $p > 0$. Notice that in the case of Privalov space N^p $(1 < p < \infty)$, the metric σ_p given by (8) coincides with Stoll's metric λ_p defined by (3).

Recall that two metrics (or norms) defined on the same space will be called equivalent if they induce the same topology on this space.

Theorem 6 ([33, Theorem 2.3]). *The metric d_p given by* (6) *defines the topology for L_p^+ which is equivalent to the topology defined by the metric σ_p given by* (8).

Remark 4. It was pointed out in [33, Remark, Section 2] that using the same argument applied in the proof of Theorem 2.3 of [33], it is easy to show that the metrics σ_p and d_p are equivalent with the metric δ_p given on L_p^+ by

$$\begin{aligned}\delta_p(f,g) &= \inf_{t>0}\left[t+\mu\left(\{x\in\Omega:\ |f(x)-g(x)|\geq t\}\right)\right]\\ &+\left(\int_\Omega \left|\log^+|f(x)|-\log^+|g(x)|\right|^p\,d\mu\right)^{1/\max(p,1)},\ f,g\in L_p^+. \qquad (9)\end{aligned}$$

Remark 5. In [45, Remark 5, p. 460] M. Hasumi pointed out that the Yanagihara's metric $\lambda=\lambda_1$ on the Smirnov class (given by (3) with $p=1$) defines the topology on the space $L_1^+ = L(\mu)$ which is equivalent to the metric topology $d_1 = d$ (given by (6) with $p = 1$).

As a consquence of Theorems 5 and 6, it can be obtained the following result.

Theorem 7 ([33, Corollary 2.4]). *For each $p > 0$ the space L_p^+ with the topology given by the metric σ_p is an F-algebra, i.e., a topological algebra with a complete translation invariant metric σ_p.*

Remark 6. In view of Theorem 7, note that L_p^+ may be considered as the generalized Orlicz space L_p^w with the constant function $w(t)\equiv 1$ on $[0, 2\pi)$ defined in [33, Section 6].

The real-valued function $\psi : [0,\infty) \mapsto [0,\infty)$ defined as $\psi(t) = \big(\log(1+t)\big)^p$, is continuous and nondecreasing in $[0,\infty)$, equals zero only at 0, and hence it is a φ-*function* (see, e.g., [37, p. 4, Examples 1.9]). Moreover, ψ is a log-*convex* function since it can be represented in the form $\psi(x) = \Psi(\log x)$ for $x > 0$, where $\Psi(u) := \max(u^p, 0)$ $(u \in [0,\infty))$ is a *convex* function on the whole real axis, satisfying the condition $\lim_{u\to+\infty}\frac{\Psi(u)}{u} = +\infty$. Notice that convex φ-functions are a particular case of log-convex φ-functions.

Further, observe that [33, Section 4] the space $L_p^+(dt/(2\pi) = L_p^+$ $(p > 0)$, consisting of all complex-valued functions f, defined and measurable on $[0, 2\pi)$, for which

$$\|f\|_p := \left(\int_0^{2\pi} (\log(1+|f(t)|))^p\,\frac{dt}{2\pi}\right)^{1/p} < +\infty \qquad (10)$$

is the *Orlicz class* (see [37, p. 5]; cf. [33, Section 4]), whose generalization was given in [33, Section 6]. It follows by the dominated convergence theorem that the class L_p^+

coincides with the associated *Orlicz space* (see [37, Definition 1.4, p. 2]) consisting of those functions $f \in L_p^+$ such that

$$\int_0^{2\pi} (\log(1+c|f(t)|))^p \, \frac{dt}{2\pi} \to 0 \quad \text{as} \quad c \to 0+ \,.$$

Since $\sigma_p(f,g) = \big(\|f-g\|_p\big)^{\min(p,1)} (f,g \in L_p^+)$ is an invariant metric on L_p^+, the function $\|\cdot\|_p$ given by (10) is a *modular* in the sense of Definition 1.1 in [37, p. 1], where σ_p is the metric defined by (8). For any function $f \in L_p^+$, by the monotone convergence theorem, it follows that $\lim_{c\to 0}\|cf\| = 0$ and thus (L_p^+, σ_p) is a *modular space* in the sense of Definition 1.4 in [37, p. 2]. In other words, the function $\|\cdot\|_p$ is an F-norm. It is known (see [37, Theorem 1.5, p. 2 and Theorem 7.7, p. 35]) that the functional $|\cdot|_p$ defined as

$$|f|_p = \inf\left\{\varepsilon > 0 : \int_0^{2\pi} \left(\log\left(1+\frac{|f(t)|}{\varepsilon}\right)\right)^p \frac{dt}{2\pi} \le \varepsilon\right\}, \quad f \in L_p^+, \tag{11}$$

is a complete F-*norm* on L_p^+. Furthermore (see [16, p. 54]), if we denote by L_p^0 the class of all functions f such that $\alpha f \in L_p^+$ for every $\alpha > 0$, then L_p^0 is the closure of the space of all continuous functions on $[0, 2\pi)$ in the space $(L_p^+, |\cdot|_p)$.

We also give the following two results.

Theorem 8 ([33, Theorem 4.1]). *The F-norms $\|\cdot\|_p$ and $|\cdot|_p$ (given by* (10) *and* (11)*, respectively), induce the same topology on the space L_p^+. In other words, the norm and modular convergences are equivalent.*

Proposition 2 ([33, Corollary 4.2]). *There does not exist a nontrivial continuous linear functional on the space $(L_p^+, \|\cdot\|_p)$.*

Note that [33, Section 5] the algebra N^p may be considered as the *Hardy-Orlicz space* with the Orlicz function $\psi : [0,\infty) \mapsto [0,\infty)$ defined as $\psi(t) = \big(\log(1+t)\big)^p$. These spaces were firstly studied in 1971 by R. Leśniewicz [15]. For more information on the Hardy-Orlicz spaces, see [37, Ch. IV, Sec. 20]. Identifying a function $f \in N$ with its *boundary function* f^*, by [16, 3.4, p. 57], the space N^p is identical with the closure of the space of all functions holomorphic on the open unit disk $\mathbb{D}$ and continuous on $\overline{\mathbb{D}} : |z| \le 1$ in the space $(L_p^+(dt/2\pi) \cap N, |\cdot|_p)$, where $dt/2\pi$ is the usual normalized Lebesgue measure on the unit circle $\mathbb{T}$. Using this fact, Theorem 4 and Theorem 6, the main surveyed results of this paper can be summarized as follows.

Theorem 9 ([33, Theorem]). *For each $p > 1$ Privalov class is the Hardy-Orlicz space with the Orlicz function $\psi(t) = \big(\log(1+t)\big)^p$ $(t \in [0, 2\pi))$. Moreover, the metrics λ_p, ρ_p, d_p, δ_p and the functional $|\cdot|_p$ (defined respectively by* (3), (5), (6), (9) *and* (11)*) induce the same topology on N^p under which N^p becomes an F-algebra.*

4 CONSLUSION

This paper continues an overview of topologies on the Privalov spaces N^p $(1 < p < +\infty)$ induced by different metrics. Notice that the class N^p equipped with the topology given by the metric λ_p introduced by M. Stoll becomes an F-algebra. The same statement is also true for the class M^p with respect to the ρ_p-metric topology. These facts are used in [20] to prove that for each $p > 1$ the classes M^p and N^p coincide and the metric spaces (M^p, ρ_p) and (N^p, λ_p) have the same topological structure In Section 3 we give a short survey about Privalov spaces N^p $(1 < p < +\infty)$ whose topology is induced by the generalized Gamelin-Lumer's metric d_p defined on the space $L_p^+(dt/(2\pi)$. Notice that the space $L_p^+(dt/(2\pi)$ coincides with the Orlicz class associated to the log-convex φ-function $\psi(t) = \big(\log(1+t)\big)^p$ $(t \in [0, +\infty)$. Accordingly, it follows that for each $p > 1$ Privalov space is the Hardy-Orlicz space with the Orlicz function $\psi(t) = \big(\log(1+t)\big)^p$ $(t \in [0, 2\pi))$. Moreover, the metrics λ_p, ρ_p and d_p induce the same topology on N^p under which N^p becomes an F-algebra. We believe that presented results would be useful for future research on related topics, as well as for some applications in Functional and Complex Analysis.

Acknowledgements. This work was supported by the MEPhI Academic Excellence Project (agreement with the Ministry of Education and Science of the Russian Federation of August 27, 2013, project no. 02.a03.21.0005).

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Received February 23, 2017