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CERTAIN CONGRUENCES FOR HARMONIC NUMBERS

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Summary. For given positive integers *n* and *m*, the harmonic numbers of order *m* are those rational numbers $H_{n,m}$ defined as

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}.$$

If m=1, then $H_n := H_{n,1} = \sum_{k=1}^{n} 1/k$ is the *n*th harmonic number. In [12] Z.W. Sun obtained basic congruences modulo a prime $p > 3$ for several sums involving harmonic numbers. Further generalizations and extensions of these congruences have been obtained by R. Tauraso in [16], by Z.W. Sun and L.L. Zhao in [14] and by R. Me_strovi_c in [6] and [7]. In this paper we prove that for each prime $p > 3$ and all integers $m = 0, 1, \ldots, p-2$ there holds

$$\sum_{k=m}^{p-1} \binom{k}{m} H_k \equiv \frac{(-1)^m}{m+1}\left(1 - pH_{m+1} + \frac{p^2}{2}\left(H_{m+1}^2 - H_{m+1,2}\right)\right) \qquad \left(\mathrm{mod}\ p^3\right)$$

As an application, we determine the mod p^3 congruences for the sums $\sum_{k=1}^{p-1} k^r H_k$ with r = 0, 1, 2, 3 and a prime $p > 3$.

1 INTRODUCTION

Given positive integers *n* and *m*, the *harmonic numbers of order m* are those rational numbers $H_{n,m}$ defined as

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}.$$

For simplicity, we will denote by

$$H_n := H_{n,1} = \sum_{k=1}^{n} \frac{1}{k}$$

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the nth *harmonic number* (in addition, we define $H_0 = 0$).

Harmonic numbers play important roles in mathematics. Throughout this paper, for a prime p and two reduced rational numbers a/b and c/d such that b and d are not divisible by p, we write $a/b \equiv c/d \pmod{p^s}$ (with $s \in \mathbb{N}$) to mean that $ad - bc$ is divisible by p^s.

In 2012 Z.W. Sun [12] investigated their arithmetic properties and obtained various basic congruences modulo a prime $p > 3$ for several sums involving harmonic numbers. In particular, Sun established the congruences $\sum_{k=1}^{p-1}(H_k)^r \pmod{p^{4-r}}$ for $r = 1, 2, 3$. Further generalizations and extensions of these congruences have been obtained by R. Tauraso in [16], by Z.W. Sun and L.L. Zhao [14] and by R. Meštrović in [6] and [7]. Furthermore, Z.W. Sun [13] initiated and studied congruences involving both harmonic and Lucas sequences (especially, including Fibonacci numbers or Lucas numbers). Moreover, some congruences involving multiple harmonic sums were established in [9], [18] and [19].

Recall that Bernoulli numbers $B_0, B_1, B_2, \ldots$ are recursively given by

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \ (n = 1, 2, 3, \ldots).$$

It is easy to find the values $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, and $B_n = 0$ for odd $n \geq 3$. Furthermore, $(-1)^{n-1}B_{2n} > 0$ for all $n \geq 1$. These and many other properties can be found, for instance, in [3]. Recently, the first author of this paper in [6, Theorem 1.1] established the following six congruences involving harmonic numbers contained in the following result.

Theorem 1.1 ([6, Theorem 1.1]). *Let $p > 5$ be a prime, and let $q_p(2) = (2^{p-1} - 1)/p$ be the Fermat quotient of p to base 2. Then*

$$\sum_{k=1}^{p-1} \frac{2^k H_k}{k} \equiv -q_p(2)^2 + \frac{2}{3}pq_p(2)^3 + \frac{p}{12}B_{p-3} \pmod{p^2}, \tag{1}$$

$$\sum_{k=1}^{p-1} \frac{2^k H_k}{k^2} \equiv -\frac{1}{3}q_p(2)^3 + \frac{23}{24}B_{p-3} \pmod{p}, \tag{2}$$

$$\sum_{k=1}^{p-1} \frac{H_k}{k^2 \cdot 2^k} \equiv \frac{5}{8}B_{p-3} \pmod{p}, \tag{3}$$

$$\sum_{k=1}^{p-1} \frac{2^k H_k^2}{k} \equiv -\frac{1}{3}q_p(2)^3 + \frac{11}{24}B_{p-3} \pmod{p}, \tag{4}$$

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k \cdot 2^k} \equiv \frac{7}{8}B_{p-3} \pmod{p} \tag{5}$$

and

$$\sum_{k=1}^{p-1} \frac{2^k H_{k,2}}{k} \equiv -\frac{1}{3}q_p(2)^3 - \frac{25}{24}B_{p-3} \pmod{p}. \tag{6}$$

In this paper we prove the following result.

Theorem 1.2. *Let $p>3$ be a prime. Then for each $m=0,1,\ldots,p-2$ there holds*

$$\sum_{k=m}^{p-1}\binom{k}{m}H_k\equiv\frac{(-1)^m}{m+1}\left(1-pH_{m+1}+\frac{p^2}{2}(H_{m+1}^2-H_{m+1,2})\right)\pmod{p^3}. \tag{7}$$

The particular cases of Theorem 1.2 yield the following result.

Corollary 1.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}H_k\equiv 1-p\pmod{p^3}, \tag{8}$$

$$\sum_{k=1}^{p-1}kH_k\equiv -\frac{p^2-3p+2}{4}\pmod{p^3}, \tag{9}$$

$$\sum_{k=1}^{p-1}k^2H_k\equiv \frac{15p^2-17p+6}{36}\pmod{p^3}, \tag{10}$$

and

$$\sum_{k=1}^{p-1}k^3H_k\equiv -\frac{21p^2-10p}{48}\pmod{p^3}, \tag{11}$$

Reducing the modulus in congruences (8), (9), (10) and (11) of Corollary 1.3, immediately gives the following two corollaries.

Corollary 1.4. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}H_k\equiv 1-p\pmod{p^2}, \tag{12}$$

$$\sum_{k=1}^{p-1}kH_k\equiv \frac{3p-2}{4},\pmod{p^2}, \tag{13}$$

$$\sum_{k=1}^{p-1}k^2H_k\equiv -\frac{17p-6}{36}\pmod{p^2}, \tag{14}$$

and

$$\sum_{k=1}^{p-1}k^3H_k\equiv \frac{5p}{24}\pmod{p^2}. \tag{15}$$

Corollary 1.5. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}H_k\equiv 1\pmod{p}, \tag{16}$$

$$\sum_{k=1}^{p-1} kH_k \equiv -\frac{1}{2} \pmod{p}, \tag{17}$$

$$\sum_{k=1}^{p-1} k^2H_k \equiv \frac{1}{6} \pmod{p}, \tag{18}$$

and

$$\sum_{k=1}^{p-1} k^3H_k \equiv 0 \pmod{p}. \tag{19}$$

Remark 1.5. Notice that the congruences (8) and (9) are proved by Z.W. Sun in [12, p. 419 and p. 417].

2 PROOF OF THEOREM 1.2 AND COROLLARY 1.3

For the proof of Theorem 1.2 we will need the following three auxiliary results.

Lemma 2.1 (see the identity (6.70) in [1]; also [11, p. 2]). *If m and n are nonnegative integers such that $m \le n$, then*

$$\sum_{k=m}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right). \tag{20}$$

The following result is well known as Wolstenholme's theorem established in 1862 by J. Wolstenholme [17].

Lemma 2.2 (see [17]; also [2], [5] and [10]). *If $p > 3$ is a prime, then*

$$H_{p-1} \equiv 0 \pmod{p^2}. \tag{21}$$

The following result is well known and elementary.

Lemma 2.3 (see, e.g., [12, Lemma 2.1 (2.2)]). *If $p \ge 3$ is a prime, then*

$$\binom{p-1}{k} \equiv (-1)^k \left(1 - pH_k + \frac{p^2}{2}(H_k^2 - H_{k,2})\right) \pmod{p^3}, \tag{22}$$

for each $k = 0, 1, \dots, p-1$.

Proof of Theorem 1.2. Taking $n = p-1$ into the identity (20) of Lemma 2.1 and using the identities $\binom{p}{m+1} = \frac{p}{m+1}\binom{p-1}{m}$ and $\binom{p}{m+1} - \binom{p-1}{m} = \binom{p-1}{m+1}$, we find that

$$\begin{aligned}\sum_{k=m}^{p-1}\binom{k}{m}H_k &= \binom{p}{m+1}\left(H_p - \frac{1}{m+1}\right)\\ &= \binom{p}{m+1}\left(H_{p-1} + \frac{1}{p} - \frac{1}{m+1}\right)\\ &= \binom{p}{m+1}H_{p-1} + \frac{1}{p}\binom{p}{m+1} - \frac{1}{m+1}\binom{p}{m+1}\\ &= \binom{p}{m+1}H_{p-1} + \frac{1}{m+1}\binom{p-1}{m} - \frac{1}{m+1}\binom{p}{m+1}\\ &= \binom{p}{m+1}H_{p-1} - \frac{1}{m+1}\left(\binom{p}{m+1} - \binom{p-1}{m}\right)\\ &= \frac{p}{m+1}\binom{p-1}{m}H_{p-1} - \frac{1}{m+1}\binom{p-1}{m+1}.\end{aligned} \tag{23}$$

Using the congruence (21) of Lemma 2.2 and the assumption $0 \le m \le p-2$, we obtain

$$\frac{p}{m+1}\binom{p-1}{m}H_{p-1} \equiv 0 \pmod{p^3}. \tag{24}$$

Furthermore, by the congruence (22) of Lemma 2.3, we have

$$-\binom{p-1}{m+1} \equiv (-1)^m\left(1 - pH_{m+1} + \frac{p^2}{2}(H_{m+1}^2 - H_{m+1,2})\right) \pmod{p^3}. \tag{25}$$

Applying the congruences (24) and (25) to the right hand side of the identity (23), we immediately get

$$\sum_{k=m}^{p-1}\binom{k}{m}H_k \equiv \frac{(-1)^m}{m+1}\left(1 - pH_{m+1} + \frac{p^2}{2}(H_{m+1}^2 - H_{m+1,2})\right) \pmod{p^3}. \tag{26}$$

The congruence (26) is actually the congruence (7) of Theorem 1.2. This completes the proof. □

Proof of Corollary 1.3. Taking $m = 0$ and $m = 1$ into the congruence (7) of Theorem 1.2, we immediately give the congruences (8) and (9), respectively.

Taking $m = 2$ into the congruence (7), we find that

$$\sum_{k=2}^{p-1}\binom{k}{2}H_k \equiv \frac{1}{3}\left(1 - \frac{11p}{6} + p^2\right) \pmod{p^3},$$

which can be written as

$$\sum_{k=2}^{p-1}\frac{k^2H_k}{2} - \sum_{k=2}^{p-1}\frac{kH_k}{2} \equiv \frac{1}{3}\left(1 - \frac{11p}{6} + p^2\right) \pmod{p^3}. \tag{27}$$

By using the congruences (27) and (9), we obtain

$$\begin{aligned}\sum_{k=1}^{p-1} k^2 H_k &\equiv \sum_{k=1}^{p-1} kH_k + \frac{2}{3}\left(1 - \frac{11p}{6} + p^2\right) \pmod{p^3}\\ &\equiv -\frac{p^2-3p+2}{4} + \frac{2}{3}\left(1 - \frac{11p}{6} + p^2\right) \pmod{p^3}\\ &= \frac{15p^2-17p+6}{36} \pmod{p^3}.\end{aligned}$$

The above congruence is in fact the congruence (10) of Corollary 1.3.

Finally, in order to prove the congruence (11), we put $m = 3$ into the congruence (7). This immediately yields

$$\sum_{k=3}^{p-1} \binom{k}{3} H_k \equiv -\frac{1}{4}\left(1 - \frac{25p}{12} + \frac{35p^2}{24}\right) \pmod{p^3}.$$

By substituting $\binom{k}{3} = \frac{k^3-3k^2+2k}{6}$ into above congruence, it can be written as

$$\sum_{k=3}^{p-1} \frac{k^3 H_k}{6} - \sum_{k=3}^{p-1} \frac{k^2 H_k}{2} + \sum_{k=3}^{p-1} \frac{kH_k}{3} \equiv -\frac{1}{4}\left(1 - \frac{25p}{12} + \frac{35p^2}{24}\right) \pmod{p^3}. \tag{28}$$

By using the congruences (28), (9) and (10), we have

$$\begin{aligned}\sum_{k=1}^{p-1} k^3 H_k &\equiv 3\sum_{k=1}^{p-1} k^2 H_k - 2\sum_{k=1}^{p-1} kH_k - \frac{3}{2}\left(1 - \frac{25p}{12} + \frac{35p^2}{24}\right) \pmod{p^3}\\ &\equiv \frac{15p^2-17p+6}{12} + \frac{p^2-3p+2}{2} - \frac{35p^2-50p+24}{16} \pmod{p^3}\\ &= -\frac{21p^2-10p}{48} \pmod{p^3}.\end{aligned}$$

The above congruence coincides with the congruence (11) of Corollary 1.3, and thus, the proof is completed. □

Remark 2.4. Of course, by applying the recursive method used in proof of Corollary 1.3, it is possible to determine the expression for $\sum_{k=1}^{p-1} k^m H_k \pmod{p^3}$ for each positive integer m, where $p > 3$ is a prime. Furthermore, it is obvious that each of these congruences can be written in the form

$$\sum_{k=1}^{p-1} k^m H_k \equiv a_m p^2 + b_m p + c_m \pmod{p^3},$$

where a_m, b_m and c_m are rational numbers depending on m whose denominators are not divisible by p.

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