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DIRECT AND INVERSE SPECTRAL ASSIGNMENT FOR THE OPERATOR STURM-LIUOVILLE TYPE WITH LINEAR DELAY

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Summary. In this paper we constructed the solution of the spectral boundary problem on $[0,\pi]$ with zero initial function and separated boundary conditions. In this work, characteristic function is constructed and asymptotics of its large zeroes and igenvalues asymptotics of the operator are found. In the second part of this paper, it is assumed that two sets of eigenvalues are given when $h = 0$. Under certain conditions of the linear delay, the parameters of operators are found. It means that we solved the inverse spectral problem using a Fourier's series method.

1. INTRODUCTION

This work is dedicated to solve the direct and inverse Sturm-Liouville problem with linear delay $\tau(x) = \alpha x + \beta$, where $0 < \alpha < 1, \beta > 0$ and separated boundary conditions.

So we will study the boundary task which is given by

$$-y''(x) + q(x)y\big(\gamma(x)\big) = \lambda y(x) = z^2 y(x) \tag{1}$$

$$\gamma(x) = (1-\alpha)x - \beta, q \in \ L_2[0,\pi]$$

$$y\big(\gamma(x)\big) \equiv 0, x \in [0,\xi_1)\,,\ \ \xi_1 = \frac{\beta}{1-\alpha} \tag{2}$$

$$y'(0,z) - hy(0,z) = 0, h \in R \tag{3}$$

$$y'(\pi,z) + Hy(\pi,z) = 0, \qquad H \in R \tag{4}$$

Spectral problem is defined by (1,2,3,4)and we will use abbreviation $D^2 y = z^2 y$.The coefficients of $h, H,$; delay function $\tau(x) = \alpha x + \beta$ and potential q are parameters of the operator

$$D^2 = D^2(q, h, H, \alpha, \beta).$$

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2. DIRECT SPECTRAL PROBLEM

In this paper, under the direct spectral problem, we mean the construction of eigenvaluesof the operator D^2

2.1. Problem (1-3)

LEMMA 1. Problem (1-3) is equivalent to the Volterra integral equation

$$y(x,z) = cosxz + \frac{h}{z}sinxz + \frac{1}{z}\int_{\xi_1}^{x} q(t_1)sinz(x-t_1)y(\gamma(t_1),z)dt_1. \quad (5)$$

Actually , by the method of variation of constants in the equation (1) and then using the initial condition (2) and the boundary condition (3), equation (5) follows directly.

Since the function $\gamma(x) = (1-\alpha)x - \beta$ is strictly increasing due to the $\gamma'(x) = 1-\alpha > 0$. It also has inverse function given by $\gamma^{-1}(x) = \frac{x}{1-\alpha} + \frac{\beta}{1-\alpha}$. It follows $\gamma^{-1}(0) = \xi_1 = \frac{\beta}{1-\alpha}$.

Further

$$\gamma^{-2}(x) = (1-\alpha)^2x - (1-\alpha)\beta - \beta \text{ and } \gamma^{-2}(0) = \frac{\beta}{1-\alpha} + \frac{\beta}{(1-\alpha)^2}.$$

We use mathematical induction to prove

$$\xi_l = \gamma^{-l}(0) = \frac{\beta}{1-\alpha} + \frac{\beta}{(1-\alpha)^2} + \cdots + \frac{\beta}{(1-\alpha)^l} = \frac{\beta}{1-\alpha}\left(1 + \frac{1}{1-\alpha} + \cdots + \frac{1}{(1-\alpha)^{l-1}}\right)$$

Because $(1-\alpha) \in (0,1)$ is a geometric series

$$\sum_{k=0}^{\infty} \frac{1}{(1-\alpha)^k}$$

divergent, however there is $k_0 \in N$ such that $\xi_{k_0} < \pi < \xi_{k_0+1}$. Therefore

$$(\xi_1, \pi] = \bigcup_{k=2}^{k_0} (\xi_{k-1}, \xi_k] \cup (\xi_{k_0}, \pi]$$

The solution of integral equation (5) is obtained by the variable step method. So, for$x \in (0, \xi_1]$solution has the form

$$y(x,z) = cosxz + \frac{h}{z}sinxz \quad (6_0)$$

In the sequel, we use the functions

$$a_{sc}(x,z) = \int_{\xi_1}^{x} (t_1) sin(x - t_1) cosz\gamma(t_1) dt_1$$

$$a_{s^2}(x,z) = \int_{\xi_1}^{x} (t_1) sin(x - t_1) sinz\gamma(t_1) dt_1$$

The solution $y(x,z)$ on the interval$(\xi_1, \xi_2]$ is given with

$$y(x,z) = cosxz + \frac{h}{z} sinxz + \frac{1}{z} a_{sc}(x,z) + \frac{h}{z^2} a_{s^2}(x,z) \qquad (6_1)$$

Now, we introduce the recurrence relations

$$a_{s^k c}(x,z) = \int_{\xi_k}^{x} (t_1) a_{s^{k-1}c}(\gamma(t_1), z) dt_1 \quad k = 2,3, \dots, k_0$$

$$a_{s^{k+1}}(x,z) = \int_{\xi_k}^{x} (t_1) a_{s^K}(\gamma(t_1), z) dt_1$$

On the intervals $(\xi_l, \xi_{l+1}]$ $l = 1,2, \dots, k_0$ the solution has the form

$$y(x,z) = cosxz + \frac{h}{z} sinxz + \sum_{k=1}^{l} \frac{1}{z^k} \left[a_{s^k c}(x,z) + \frac{h}{z} a_{s^{k+1}}(x,z) \right] \qquad (6_l)$$

2.2. Construction of characteristic functions of operator D^2

In the following discussion, we will use the solution $y(x,z)$on the interval$\left(\xi_{k_0}, \pi\right]$, which is contained in $\left(\xi_{k_0}, \xi_{k_0+1}\right]$

$$y(x,z) = cosxz + \frac{h}{z} sinxz + \sum_{k=1}^{k_0} \frac{1}{z^k} \left[a_{s^k c}(x,z) + \frac{h}{z} a_{s^{k+1}}(x,z) \right] \qquad (6_{k_0})$$

After differentiation, it follows that

$$\frac{dy}{dx}(x,z) = -zsinxz + hcosxz + \sum_{k=1}^{k_0} \frac{1}{z^{k-1}} \left[a_{cs^{k-1}c}(x,z) + \frac{h}{z} a_{cs^k}(x,z) \right] \qquad (6'_{k_0})$$

Where

$$a_{cs^{k-1}c}(x,z) = \int_{\xi_k}^{x} (t_1) cosz(x-t_1) a_{s^{k-1}c}(\gamma(t_1),z)dt_1$$

$$a_{cs^k}(x,z) = \int_{\xi_k}^{x} (t_1) cosz(x-t_1) a_{s^k}(\gamma(t_1),z)dt_1 \qquad k=\overline{1,k_0}$$

We will use the abbreviations

$$a_{cs^{k-1}c}(\pi,z) = a_{cs^{k-1}c}(z)\,, \qquad a_{ck}(\pi,z) = a_{cs^k}(z)$$

$$a_{s^k c}(\pi,z) = a_{s^k c}(z),\;\; a_{s^{k+1}}(\pi,z) = \; a_{s^{k+1}}(z)\;\; k=\overline{1,k_0}$$

Using the boundary condition (4) we obtain the characteristic function F as

$$F(z) = \left(-z+\frac{hH}{z}\right) sin\pi z + (h+H)cos\pi z +$$

$$+\sum_{k=1}^{k_0}\left\{\frac{1}{z^{k-1}} a_{cs^{k-1}c}(z) + \frac{1}{z^k}\left[h a_{cs^k}(z) + H a_{s^k c}(z)\right] + \frac{hH}{z^{k+1}} a_{s^{k+1}}(z)\right\} \qquad (7_{k_0})$$

The function F is the whole exponential function with apparent singularity with $z=0$. Because $F(-z)=F(z)$ is a even function and from $F(z_n)=0$ it follows $F(-z_n)=0$. In the circle $|z-n|=r_n$, where the radius $r_n$is small enough, there is exactly one zero z_n function F. The result is proved for the ordinary case without delay in [3], and the constant delays proof is given in [9].

2.3. Partial transformation of functions F

If $k_0=1$ then we have:

$$F(z) = \left(-z+\frac{hH}{z}\right) sin\pi z + (h+H)cos\pi z + a_{c^2}(z) +$$

$$+\frac{1}{z}(h a_{cs}(z) + H a_{sc}(z)) + \frac{hH}{z^2} a_{s^2}(z) \qquad (7_1)$$

Further

$$a_{c^2}(z) = \frac{1}{2}\int_{\xi_1}^{\pi} q(t_1)\left\{cosz\big(\pi-(\alpha t_1+\beta)\big) + cosz[\pi-((2-\alpha)t_1+\beta)]\right\}dt_1 \qquad (8_1)$$

$$a_{sc}(z) = \frac{1}{2}\int_{\xi_1}^{\pi} q(t_1)\left\{sinz\big(\pi-(\alpha t_1+\beta)\big) + sinz\big[\pi-\big((2-\alpha)t_1+\beta\big)\big]\right\}dt_1 \qquad (8_2)$$

$$a_{s^2}(z) = \frac{1}{2}\int_{\xi_1}^{\pi} q(t_1)\{cosz(\pi - (\alpha t_1 + \beta)) - cosz[\pi - ((2-\alpha)t_1 + \beta)]\}dt_1 \qquad (8_3)$$

$$a_{cs}(z) = \frac{1}{2}\int_{\xi_1}^{\pi} q(t_1)\{sinz(\pi - (\alpha t_1 + \beta)) - sinz[\pi - ((2-\alpha)t_1 + \beta)]\}dt_1 \qquad (8_4)$$

In the following work,we will define so-called transitive function $\tilde{q}^{\pm}$

$$\tilde{q}^{\pm}(\theta) = \begin{cases} \frac{1}{\alpha}q\left(\frac{2\theta-\beta}{\alpha}\right) \pm \frac{1}{2-\alpha}q\left(\frac{2\theta+\beta}{2-\alpha}\right), & \theta \in \left[\frac{1}{2}\xi_1, \frac{1}{2}\tau(\pi)\right) \\ \pm\frac{1}{2-\alpha}q\left(\frac{2\theta+\beta}{2-\alpha}\right), & \theta \in \left(\frac{1}{2}\tau(\pi), \pi - \frac{1}{2}\tau(\pi)\right] \\ 0, & \theta \in \left[0, \frac{1}{2}\right) \cup \left(\pi - \frac{1}{2}\tau(\pi), \pi\right] \end{cases} \qquad (9)$$

By introducing a new variable $2\theta_1 = \alpha t_1 + \beta$ and $2\theta_2 = (2-\alpha)t_1 - \beta$ and using (9) from (8_i) $i = \overline{1,4}$., it follows that

$$a_{c^2}(z) = \int_0^{\pi} \tilde{q}^{+}(\theta)cosz(\pi - 2\theta)d\theta = \tilde{a}_c^{\ +}(z)$$

$$a_{sc}(z) = \int_0^{\pi} \tilde{q}^{+}(\theta)sinz(\pi - 2\theta)d\theta = \tilde{a}_s^{\ +}(z) \qquad (10)$$

$$a_{cs}(z) = \int_0^{\pi} \tilde{q}^{-}(\theta)sinz(\pi - 2\theta)d\theta = \tilde{a}_s^{\ -}(z)$$

$$a_{s^2}(z) = \int_0^{\pi} \tilde{q}^{-}(\theta)cosz(\pi - 2\theta)d\theta = -\tilde{a}_c^{\ -}(z)$$

Based on (10) function (7_1) is rewritten as follows

$$F(z) = \left(-z + \frac{hH}{z}\right)sin\pi z + (h+H)cos\pi z + \tilde{a}_c^{\ +}(z) + \frac{1}{z}$$
$$+ \frac{1}{z}\left(H\tilde{a}_s^{\ +}(z) + h\tilde{a}_s^{\ -}(z)\right) - \frac{hH}{z^2}\tilde{a}_c^{\ -}(z) \qquad (11)$$

Remark: For $k_0 \geq 2$ transformation of function F can be realized but this procedure will not be discussed now.

2.4. The asymptotic behaviour of large zeroes of the function F

Let us introduce the following number series

$$\tilde{a}^{\pm}{}_{2n} = \int_0^{\pi} \tilde{q}^{\pm}(\theta) cos2n\theta d\theta;\ \ \tilde{b}^{\pm}{}_{2n} = \int_0^{\pi} \tilde{q}^{\pm}(\theta) sin2n\theta d\theta;\ \ b_{2n}^{\pm *} = \int_0^{\pi} \theta \tilde{q}^{\pm}(\theta) sin2n\theta d\theta \qquad (12)$$

We know that

$$z_n = n + \frac{c_{1n}}{n} + \frac{c_{2n}}{n^2} + O\left(\frac{c_{1n}}{n^3}\right), \qquad (\ as\ n \to \infty) \qquad (13)$$

is valid.

Based on (13) and (12) we can write

$$sin\pi z_n = (-1)^n \left[\frac{\pi c_{1n}}{n} + \frac{\pi c_{2n}}{n^2} + O\left(\frac{c_{1n}}{n^3}\right)\right] \quad (as\ \ n \to \infty) \qquad (13_1)$$

$$cos\pi z_n = (-1)^n + O\left(\frac{1}{n^2}\right) \quad (as\ \ n \to \infty) \qquad (13_2)$$

$$\tilde{a}^{+}{}_c(z_n) = (-1)^n \tilde{a}^{+}{}_{2n} + \frac{1}{n}\left[(-1)^n \pi c_{1n} \tilde{b}^{+}{}_{2n} + (-1)^{n+1} 2c_{1n} b_{2n}^{\pm *}\right] + O\left(\frac{c_{1n}}{n^2}\right) \qquad (13_3)$$

$$\tilde{a}^{+}{}_s(z_n) = (-1)^{n+1} b_{2n}^{\pm} + O\left(\frac{c_{1n}\tilde{a}^{\pm}{}_{2n}}{n}\right) \qquad (13_4)$$

Based on the relation (13_k) $k = 0,1,2,3,4$ from (11), we obtain

$$F(z_n) = (-1)^{n+1}\left[\pi c_{1n} + \frac{\pi c_{2n}}{n}\right] + (-1)^{n+1}(h+H) + (-1)^n \tilde{a}^{+}{}_{2n} + \frac{1}{n}\left[(-1)^n \pi c_{1n} \tilde{b}^{+}{}_{2n} + \right.$$

$$\left.(-1)^{n+1} 2c_{1n} b_{2n}^{\pm *}\right] + \frac{1}{n}\left[(-1)^{n+1} H \tilde{b}^{+}{}_{2n} + (-1)^{n+1} h \tilde{b}^{-}{}_{2n}\right] + O\left(\frac{c_{1n}}{n^2}\right) = 0$$

Therefore,

$$c_{1n} = \frac{1}{\pi}(h+H) + \frac{1}{\pi}\tilde{a}^{+}{}_{2n}$$

$$c_{2n} = \frac{h}{\pi}\left(\tilde{b}^{+}{}_{2n} - \tilde{b}^{-}{}_{2n}\right) - \frac{2}{\pi^2}(h+H) b_{2n}^{\pm *} + O\left(\tilde{b}^{+}{}_{2n}\right) \qquad (14)$$

In this way, we have proved a solution and we give it in the following theorem:

Theorem 1*:* If $q \in L_2[0,\pi]$ and $\frac{\beta}{1-\alpha} < \pi$, and $\frac{\beta}{1-\alpha} + \frac{\beta}{(1-\alpha)^2} \geq \pi$ then zeroes z_n of function F have following asymptotics:

$$z_n = \pm\left\{n + \frac{c_{1n}}{n} + \frac{c_{2n}}{n^2} + O\left(\frac{c_{1n}}{n^3}\right)\right\} \qquad (as\ n \to \infty) \qquad (15)$$

where are c_{1n} and c_{2n} given by (14).

The consequence 1: Eigenvalues of operator D^2 have the following asymptotic decomposition

$$\lambda_n = n^2 + 2c_{1n} + \frac{2c_{2n}}{n} + O\left(\frac{c_{1n}}{n^2}\right), \qquad (as\ n \to \infty) \tag{15_1}$$

3. INVERSE SPECTRAL TASK

Determination of the parameters $q, h, H, \alpha\ and\ \beta$ represents the solution of inverse spectral problem. It is based on the given values of operators. We assume that two series $\lambda_{nj}\ j = 1{,}2\ n \in N_0$of own values, which are obtained by varying the boundary conditions on the right end of the segment, are given. In addition, an initial function is identically equal to zero at some distance from $[0, \nu]$. So, the series have its own values $\lambda_{nj}$and asymptotic decomposition forms:

$$\lambda_{nj} = n^2 + c_j + \tilde{a}^+{}_{2n} + O\left(\frac{\tilde{b}^+{}_{2n}}{n}\right), \quad (n \to \infty) \tag{16}$$

where $c_j$are numbers that do not depend on n, and $\tilde{a}^+{}_{2n}$ is the cosine Fourier coefficient of some of the functions $\tilde{q}^+$ which is identically equal to zero in the intervalas $\left[0, \frac{\nu}{2}\right]$ and $[\mu, \pi]$. We assume that the numbers $\frac{\nu}{2}$ and μ are known.

3.1. Basic identitiy

According to Hadamard's theorem, we can reconstruct the characteristic function F_j of operator D^2 in the form of

$$F_j(z) = \pi\lambda_{0j} \prod_{n=1}^{\infty} \frac{\lambda_{nj}}{n^2}\left(1 - \frac{z^2}{\lambda_{0j}}\right) \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_{nj}}\right) \tag{17}$$

Parameters $\alpha\ and\ \beta$ are obtained from the system of linear equations:

$$\beta + \nu\alpha = \nu$$
$$\beta + \alpha\pi = 2(\pi - \mu)$$

From this follows:

$$\alpha = \frac{2(\pi - \mu)}{\pi - \nu}\ ; \quad \beta = \nu\frac{2\mu - \pi}{\pi - \nu} \tag{18}$$

Let,

$$\frac{\beta}{1 - \alpha} < \pi \le \frac{\beta}{1 - \alpha} + \frac{\beta}{(1 - \alpha)^2} \tag{19}$$

The condition (19) means that the characteristic function of the operator $D^2(q,h,H,\alpha,\beta)$ has the form (11).Therefore, we start from the basic identities

$$F_j(z) = \left(-z + \frac{hH_j}{z}\right) sin\pi z + \left(h + H_j\right) cos\pi z + \tilde{a}_c{}^+(z) +$$

$$+\frac{1}{z}\left(H_j \tilde{a}_s{}^+(z) + h\tilde{a}_s{}^-(z)\right) - \frac{hH_j}{z^2}\tilde{a}_c{}^-(z) \qquad (20)$$

where a F_j $j = 1,2$ are given by (17).

3.2. Determination of coefficients h, H_j $j = 1,2$.

From (16) it follows directly:

$$c_j = \lim_{n\to\infty}\left(\lambda_{nj} - n^2\right)$$

Based on the necessary conditions, from(14) and (15_1) we get thefollowing equation:

$$\frac{2}{\pi}\left(h + H_j\right) = c_j \quad j = 1,2. \qquad (21_1)$$

From (20) when $z_n = \left(2n + \frac{1}{2}\right)$ $n = 1,2,...$we have

$$hH_j = \lim_{n\to\infty}\left(2n + \frac{1}{2}\right)\left[F_j\left(2n + \frac{1}{2}\right) + \left(2n + \frac{1}{2}\right)\right] = d_j \;\; j = 1,2. \qquad (21_2)$$

From (21_2) we have:

$$H_2 = \frac{d_2}{d_1}H_1 \qquad (21_3)$$

From (21_1) follows $H_2 - H_1 = \frac{\pi}{2}\quad(c_2 - c_1)$ (21_4)

From (21_3) and (21_4) we get

$$H_1 = \frac{\pi(c_2 - c_1)}{2(d_2 - d_1)}d_1 \qquad (21_5)$$

and then

$$H_2 = \frac{\pi(c_2 - c_1)}{2(d_2 - d_1)}d_2 \qquad (21_6)$$

$$h = \frac{2}{\pi}\cdot\frac{d_2 - d_1}{c_2 - c_1} \qquad (21_7)$$

Conclusion: Two given series of its own values of the operator D^2clearly define the coefficients h, H_1, H_2

3.3. The formation of integral equation according to the transition function

First, we will define the function:

$$A(z) = \frac{H_2 F_1(z) - H_1 F_2(z)}{H_2 - H_1} + z sin\pi z - h cos\pi z$$

$$B(z) = z\frac{F_2(z) - F_1(z)}{H_2 - H_1} - h sin\pi z - z cos\pi z \qquad z \in C \tag{22}$$

From the system (20), we obtain:

$$\tilde{a}_c{}^{+}(z) + h\frac{\tilde{a}_s{}^{-}(z)}{z} = A(z)$$

$$\tilde{a}_s{}^{+}(z) - h\frac{\tilde{a}_c{}^{-}(z)}{z} = B(z) \tag{23}$$

We should point out the following facts:

$$\lim_{z\to 0} \tilde{a}_s{}^{+}(z) = \lim_{z\to 0}\frac{\tilde{a}_c{}^{-}(z)}{z} = \lim_{z\to 0} B(z) = 0 \tag{24}$$

By applying partial integration, we obtain:

$$\frac{\tilde{a}_c{}^{-}(z)}{z} = \int_{\frac{\xi_1}{2}}^{\pi-\frac{\tau(\pi)}{2}} \tilde{q}^{-}(\theta)d\theta \frac{cosz\tau(\pi)}{z} - 2\int_{\frac{\xi_1}{2}}^{\pi-\frac{\tau(\pi)}{2}}\int_{\frac{\xi_1}{2}}^{\theta} \tilde{q}^{-}(\theta_1)d\theta_1 sinz\,(\pi - 2\theta)d\theta$$

Based on (24) it follows that the integral is equal to zero:

$$\int_{\frac{\xi_1}{2}}^{\pi-\frac{\tau(\pi)}{2}} \tilde{q}^{-}(\theta)d\theta = 0 \tag{25}$$

It follows:

$$\frac{\tilde{a}_c{}^{-}(z)}{z} = -2\int_{\frac{\xi_1}{2}}^{\pi-\frac{\tau(\pi)}{2}}\int_{\frac{\xi_1}{2}}^{\theta} \tilde{q}^{-}(\theta_1)d\theta_1 sinz\,(\pi - 2\theta)d\theta \tag{26}$$

We get:

$$\frac{\tilde{a}_s^{\,-}(z)}{z} = 2\int_{\frac{\xi_1}{2}}^{\pi-\frac{\tau(\pi)}{2}}\int_{\frac{\xi_1}{2}}^{\theta}\tilde{q}^{-}(\theta_1)d\theta_1 cosz\,(\pi-2\theta)d\theta \tag{27}$$

By using (26) and (27), the relation (23) becomes a system (28)

$$\begin{gathered}\tilde{a}_c^{\,+}(z) + 2h\int_{\frac{\xi_1}{2}}^{\pi-\frac{\tau(\pi)}{2}}\int_{\frac{\xi_1}{2}}^{\theta}\tilde{q}^{-}(\theta_1)d\theta_1 cosz\,(\pi-2\theta)d\theta = A(z)\\ \tilde{a}_s^{\,+}(z) + 2h\int_{\frac{\xi_1}{2}}^{\pi-\frac{\tau(\pi)}{2}}\int_{\frac{\xi_1}{2}}^{\theta}\tilde{q}^{-}(\theta_1)d\theta_1 sinz\,(\pi-2\theta)d\theta = B(z)\end{gathered} \tag{28}$$

Identities (28) are equivalent to infinite systems of equations which areobtained in:

$$z_n = m, \quad m \in N_0.$$

$$\frac{2}{\pi}\int_0^{\pi}\tilde{q}^{+}(\theta)cos2m\theta d\theta + \frac{2h}{\pi}\int_0^{\pi}\left[\int_0^{\theta}\tilde{q}^{-}(\theta_1)d\theta_1 cos2m\theta d\theta\right] = (-1)^m\frac{2}{\pi}A(m)$$

$$\frac{2}{\pi}\int_0^{\pi}\tilde{q}^{+}(\theta)sin2m\theta d\theta + \frac{4h}{\pi}\int_0^{\pi}\left[\int_0^{\theta}\tilde{q}^{-}(\theta_1)d\theta_1 sin2m\theta d\theta\right] = (-1)^{m+1}\frac{2}{\pi}B(m)$$

Theorem 2: The following relations are valid

$$\lim_{m\to\infty}A(m) = \lim_{m\to\infty}B(m) = 0$$

Proof of theorem 2 follows directly from the asymptotic series λ_{nj} , the identities (17) and the relation (22).

Consequence 2: Sequences

$$a_{2n}^{f} = \frac{(-1)^m}{2\pi}A(m) \tag{29_1}$$

$$a_{2n}^{f} = \frac{(-1)^{m+1}}{2\pi}B(m) \tag{29_2}$$

represent respectively cosine and sine Fourier's coefficients of a function $f \in L_2[0,\pi]$.

Multiplying (29_1) with $cos2mx$, and (29_2) with $sin2mx$ for $m \in N$ and then $m = 0$ from (29_1), we get the relation

$$\frac{1}{\pi}\int_0^{\pi}\tilde{q}^{+}(\theta)d\theta+\frac{2h}{\pi}\int_0^{\pi}\int_0^{\theta}(\tilde{q}^{-}(\theta_1)d\theta_1)d\theta=a_0^f$$

Summarizing for m we get

$$\tilde{q}^{+}(x)+2h\int_0^{x}\tilde{q}^{-}(\theta)d\theta=f(x)$$

Respectively:

$$\tilde{q}^{+}(x)-f(x)=-2h\int_0^{x}\tilde{q}^{-}(\theta)d\theta,\quad x\in[0,\pi] \tag{30}$$

Remark: The integral equation (30) is the Volterra integral equation. It is fulfilled with the transition function $\tilde{q}^{\pm}(x)$ in the whole interval $[0,\pi]$. Since $\tilde{q}^{\pm}(x)$is identically equal to zero in$\left[0,\frac{1}{2}\xi_1\right)\cup\left(\pi-\frac{1}{2}\tau(\pi),\pi\right]$, the actual significance of the equation (30) is reduced to the interval $\left[\frac{1}{2}\xi_1,\pi-\frac{1}{2}\tau(\pi)\right]$.

3.4. Solving the integral equation (30)

We assume $h=0$

If $d_1=d_2=0$ according to $H_1\neq H_2$ must be $h=0$, then must be $H_1=\frac{\pi}{2}c_1$ and $H_2=\frac{\pi}{2}c_2$.

In this case, equation (30) is simplified and becomes

$$\tilde{q}^{+}(x)=f(x),\quad x\in\left(\frac{1}{2}\xi_1,\pi-\frac{\tau(\pi)}{2}\right] \tag{31}$$

Based on the definition of the function $\tilde{q}^{+}(x)$which is given by (9),at a distance $\left[\frac{\tau(\pi)}{2},\pi-\frac{1}{2}\tau(\pi)\right]$, the relation (31) becomes

$$q(x)=(2-\alpha)f\left(\left(1-\frac{\alpha}{2}\right)x-\frac{\beta}{2}\right),\qquad x\in\left[\frac{\alpha\pi+2\beta}{2-\alpha},\pi\right] \tag{32}$$

Using (32), the potential q at a distance $x\in\left[\frac{\alpha\pi+2\beta}{2-\alpha},\pi\right]$ when$h=0$ is determined.

At a distance $\left[\xi_1,\frac{\alpha\pi+2\beta}{2-\alpha}\right]$, the potentital q will be successively determined using (9).

Theorem 3 : There are single partitions of segments

$$[\xi_1,\pi]=\bigcup_{k=0}^{\infty}[t_{k+1},t_k];\ \left[\frac{1}{2}\xi_1,\frac{1}{2}\tau(\pi)\right]=\bigcup_{k=0}^{\infty}\left(\theta_{k+1}^{(1)},\theta_k^{(1)}\right)$$

$$\left[\frac{1}{2}\xi_1, \pi - \frac{1}{2}\tau(\pi)\right] = \bigcup_{k=0}^{\infty}\left[\theta_{k+1}^{2}, \theta_k^{(2)}\right]$$

where is

$$\theta^{(1)} = \frac{\alpha_X + \beta}{2};\ \theta^{(2)} = \frac{(2-\alpha)X - \beta}{2}, \qquad t_0 = \pi, \quad t_\infty = \xi_1$$

$$\theta_\infty^{(1)} = \frac{1}{2}\xi_1 \qquad \theta_\infty^{(2)} = \frac{1}{2}\xi_1$$

$$\theta_0^{(1)} = \frac{1}{2}\tau(\pi) \qquad \theta_0^{(2)} = \pi - \frac{1}{2}\,\tau(\pi)$$

So that is valid $\theta^{(1)}(t_k) = \theta_k^{(1)} = \theta^{(2)}(t_{k+1}) = \theta_{k+1}^{(2)}$ for every $k \in N_0$(33).

Proof : Let's $k = 0$. Then $\theta^{(1)}(t_0) = \theta^{(1)}(\pi) = \theta_0^{(1)} = \theta^{(2)}(t_1) = \theta^{(2)}$ is

$$t_1 = \left(\theta^{(2)}\right)^{-1}\left(\theta_0^{(1)}\right) = \frac{2 \cdot \frac{1}{2}\tau(\pi+\beta)}{2-\alpha} = \frac{\alpha\pi + 2\beta}{2-\alpha}$$

In the following, let's $k = 1$. Then from (33), it follows

$$\theta^{(1)}(t_1) = \theta_1^{(1)} = \frac{\alpha^2\pi}{2-\alpha} + \frac{2\alpha\beta}{2-\alpha} + \beta = \theta^{(2)}(t_2) = \theta_2^{(2)} = \frac{(2-\alpha)t_2 - \beta}{2}$$

Then we get

$$t_2 = \frac{\alpha^2\pi}{(2-\alpha)^2} + \frac{2\alpha\beta}{(2-\alpha)^2} + \frac{2\beta}{2-\alpha}$$

For $k = 2$ in (33) we obtain:

$$\theta^{(1)}(t_2) = \theta_2^{(1)} = \frac{1}{2}\left[\frac{\alpha^3\pi}{(2-\alpha)^2} + \frac{2\alpha^2\pi}{(2-\alpha)^2} + \frac{2\beta}{2-\alpha} + \beta\right] = \frac{(2-\alpha)t_3 - \beta}{2} = \theta_3^{(2)}$$

From this follows:

$$t_3 = \frac{\alpha^3\pi}{(2-\alpha)^3} + \frac{2\alpha^2\pi}{(2-\alpha)^3} + \frac{2\alpha\beta}{(2-\alpha)^3} + \frac{2\beta}{2-\alpha}$$

From the cases $k = 0,1,2,3$, we anticipate formula

$$t_l = \left(\frac{\alpha}{2-\alpha}\right)^l \pi + \frac{2\beta}{2-\alpha}\left[\left(\frac{\alpha}{2-\alpha}\right)^{l-1} + \left(\frac{\alpha}{2-\alpha}\right)^{l-2} + \cdots + \frac{\alpha}{2-\alpha} + 1\right] \tag{34}$$

Let us prove by using the mathematical induction that (34) is valid for every $l \in N_0$. Since the evidence was implemented for $l = 0,1,2,3$, we assume that (34) is valid for any l. Prove that it is also true for $l + 1$.

From (33), we have

$$\theta^{(1)}(t_k) = \theta_l^{(1)} = \frac{1}{2}[\alpha t_l + \beta] = \frac{1}{2}\left\{\frac{\alpha^{l+1}}{(2-\alpha)^l}\pi + \frac{2\alpha\beta}{2-\alpha}\left[\left(\frac{\alpha}{2-\alpha}\right)^{l-1} + \cdots + 1\right] + \beta\right\}$$

According to (33) it should be equalto$\theta_{l+1}^{(2)} = \theta_{l+1}^{(2)} = \frac{1}{2}\left((2-\alpha)t_{l+1} - \beta\right)$

Therefore,

$$t_{l+1} = \left(\frac{\alpha}{2-\alpha}\right)^{l+1}\pi + \frac{2\beta}{2-\alpha}\left[\left(\frac{\alpha}{2-\alpha}\right)^{l} + \left(\frac{\alpha}{2-\alpha}\right)^{l-1} + \cdots + \frac{\alpha}{2-\alpha} + 1\right]$$

(34) is proved for any $l \in N_0$.

In addition, we have

$$\lim_{l\to\infty} t_l = \lim_{l\to\infty}\left\{\left(\frac{\alpha}{2-\alpha}\right)^{l}\pi + \frac{2\beta}{2-\alpha}\sum_{K=0}^{l-1}\left(\frac{\alpha}{2-\alpha}\right)^{k}\right\} = \frac{2\beta}{2-\alpha}\lim_{l\to\infty}\frac{1-\left(\frac{\alpha}{2-\alpha}\right)^{l}}{1-\frac{\alpha}{2-\alpha}} = \frac{\beta}{1-\alpha} = \xi_1$$

Similarly, we can prove that

$$\lim_{l\to\infty}\theta_l^{(1)} = \frac{1}{2}\xi_1 = \lim_{l\to\infty}\theta_l^{(2)}$$

is valid.

This theorem 3 is proved.

The result of the theorem 3 allows us to constructe the potential q and to use the relation (9) at the distance $[t_{k+1}, t_k], k \in N$, which shows us the connection between q̃ and q.

For $x \in [t_2, t_1]$, we get

$$q(x) = -\frac{\alpha}{2-\alpha}q\left(\frac{\alpha+2\beta}{2-\alpha}\right) + \alpha f\left(\frac{\alpha x+\beta}{2}\right)$$

Further, for $x \in [t_3, t_2]$

$$q(x) = -\frac{\alpha}{2-\alpha}q\left(\frac{\alpha x+2\beta}{2-\alpha}\right) + \alpha f\left(\frac{\alpha x+\beta}{2}\right)$$

Following this procedure, we obtain the potential q at a distance $[t_{k+1}, t_k]$, $k \in N$ using already specified potential q at a distance$[t_{k+1}, t_k]$, $k \in N$. In this way we have proved our fundamental statement which follows.

Theorem: Let two series $\lambda_{nj}$of eigenvalues of operators $D^2\left(q, 0, H_j, \alpha, \beta\right)$. $j = 1{,}2$are given and let $0 < \frac{\beta}{1-\alpha} < \pi$and$\frac{\beta}{1-\alpha} + \frac{\beta}{(1-\alpha)^2} \geq \pi$ where the asymptotic relations hold

$$\lambda nj = n^2 + C_j + \tilde{a}^{+}{}_{2n} + O\left(\frac{\tilde{b}^{+}{}_{2n}}{n}\right) \qquad (n \to \infty)$$

When the series $\tilde{a}^{+}{}_{2n}$is the series of cosine coefficients Fourier's expansion of a function $\tilde{q} \in L_2[0,\pi]$, and besides such that there exist $\nu < \mu$ such that $\tilde{q}(x) \equiv 0$ $x \in \left[0, \frac{1}{2}\nu\right] \cup [\mu, \pi]$ then the numbers α, β, H_1, H_2 and potential q are uniquely defined.

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