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ON A SHARP ESTIMATE FOR A DISTANCE FUNCTION IN BERGMAN TYPE ANALYTIC SPACES IN SIEGEL DOMAINS OF SECOND TYPE.

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Summary. We provide a new sharp result for distance function in analytic Bergman spaces in Siegel domains and in products of Siegel domains of second type. These are the first type results for such type product domains.

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1 INTRODUCTION

The most classical example of the most typical bounded Siegel domain a bounded strictly pseudoconvex domain Ω with smooth boundary in C^n is a unit ball $B = \{z : |z| < 1\}$. The basic facts of the theory of functions analytic by each variable in products of unit balls $B \times ... \times B$ were developed recently in [9,10,18]. It is natural to pose various problems even in more general situations namely to consider various problems in products of more general Siegel domains in C^n.

It is well-known fact that products of strictly pseudoconvex bounded Ω domains in C^n is again bounded and pseudoconvex and the paper [20] was probably the first one where interpolation properties of analytic functions on such product $\Omega \times ... \times \Omega$ domains were studied. (see also recent paper [14] and references there). Later various results in products of such pseudoconvex domains appeared in literature (see, for example [21] and references there). The natural question is to consider products of even more general unbounded Siegel and bounded Siegel domains in C^n simultaneously.(in very particular case it is a simple polydisk).

And the most natural general examples here are general Siegel domains of the second type (direct generalizations of bounded strictly pseudoconvex and unbounded tubular domains over symmetric cones simultaneously) $\Omega \times ... \times \Omega \subset C^{mn}$. Note for $m = 1$ case these general Siegel domains of the second type in C^n were studied before (see, for example [1,2,19,16], and [6] and various references there). The main goal of this paper to try to find complete analogues of our previous sharp results on distances on analytic function spaces in unit ball (and products of unit balls) $B \times ... \times B$ in C^n in more general case of Siegel domains of second type.(and even in products of such domains).

First we plan in this note to extend a sharp result on distance function from [6] where it is given without proof for $p = 1$ case to some values of $p > 1$. Moreover our first result then will be extended even to more general situation to Bergman-type analytic spaces on products of Siegel

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domains of second type (see also for related results on distances [6,7,8,11,12,17] and references there).

The base of all our proofs are properties of Bergman projections in Siegel domains of the second type given in [1,2] and in [16,19]. The estimates of Bergman kernel and Bergman representation formula from [1,2] and [16,19] are also playing an important role in our proof below. Note in addition arguments we used in this paper are close to arguments which were used before in [6,8] and [7,11,12] in less general domains. Let us mention [13,14,17] where recently some new sharp results on products of the most typical bounded and unbounded Siegel domains of second type(bounded strictly pseudoconvex and tubular domains over symmetric cones) were also obtained. We denote various constants as usual by C or c with indexes.

2 NOTATIONS, DEFINITIONS AND PRELIMINARIES

We first recall some basic facts on Siegel domains of second type and establish basic notations for our main theorems in Siegel domains of second type and in products of such Siegel domains. All facts which we indicate below can be seen in [1,2,3,16] and [19]. Recall first the explicit formula for the Bergman kernel function is known for very few domains. The explicit forms and zeros of the Bergman kernel function for Hartogs domains and Hartogs type domains (Cartan-Hartogs domains) were found only recently. On the other hand in strictly pseudoconvex domains the principle part of the Bergman kernel can be expressed explicitly by kernels closely related to so-called Henkin-Ramirez kernel.(see, for example [20,21,6,17] and references there). The Bergman kernel

$$b\left((\tau_1, \tau_2), (\tau_3, \tau_4)\right)$$

for the Siegel domain of the second type was computed explicitly. It is an integral via V^* a convex homogeneous open irreducible cone of rank l in R^n, a conjugate cone of V cone and which also contains no straight line and in that integral the fixed Hermitian form from definition of D Siegel domain(see below for definition) participates.(see for details of this [1,2] and also an important paper [3]). This fact was heavily used in [1,2] in solutions of several classical problems in Siegel domains of the second type. We will need now some short, but more concrete review of certain results from [1,2] to make this exposition more complete. To be more precise the authors in [1,2] showed that on homogeneous Siegel domain of type 2 under certain conditions on parameters the subspace of a weighted L^p space for all positive p consisting of holomorphic functions is reproduced by a concrete weighted Bergman kernel which we just mentioned. They also obtain some L^p estimates for weighted Bergman projections in this case. The proof relies on direct generalization of the Plancherel-Gindikin formula for the Bergman space A^2 (see [1,2]). We remind the reader that the Siegel domain of type 2 associated with the open convex homogeneous irreducible cone V of rank l which contains no straight line, $V \in R^n$, and a V-Hermitian homogeneous form F which act from product of two C^m into C^n is a set of points (w, τ) from C^{m+n} so that the difference D of $\Im w$ and the value of F on (τ, τ) is in V cone. This domain is affine homogeneous and we now should recall the following expression for the Bergman kernel of

$$D = D(V, F).$$

Let D be an affine-homogeneous Siegel domain of type 2. Let $dv(z)$ or $d\nu(z)$ denote the Lebesgue measure on D domain and let $H(D)$ denote the space of all holomorphic functions on D. The Bergman kernel is given by the following formula (see [1]) for $(\tau_1, \tau_2) \in D$ and $(\tau_3, \tau_4) \in D$

$$b\left((\tau_1, \tau_2), (\tau_3, \tau_4)\right) = (\frac{\tau_1 - \overline{\tau}_3}{2i} - (F(\tau_2, \tau_4))^{2d-q},$$

where two vectors $q = (q_i)$ and $d = (d_i)$ and in addition $n = (n_i)$ here the i index is running from 1 to l are specified via $n_{i,k}$, where these $n_{i,k}$ numbers are dimensions of certain $(R_{i,k})$ and $(C_{i,j})$ subspaces of the certain canonical decomposition of C^{m+n} and R^n via the V cone from definition of our D domain(see for some additional details about this [1,2,16]). We will call this family of triples parameters of a Siegel domain D of second type. They will constantly appear in all our main theorems. As usual $H(D)$ is endowed with the topology of uniform convergence on compact subsets of D.

The Bergman projection P of D is as usual the orthogonal projection of Hilbert space $L^2(D, d\nu)$ onto its subspace $A^2(D)$ consisting of holomorphic functions. Moreover it is known P is the integral operator defined on Hilbert space $L^2(D, d\nu)$ by the Bergman kernel $b(z, \zeta)$ which for our D domains was computed for example in [3].

Let r be a real number, for example. We fix it. Since D is homogeneous the function $\zeta \to B(\zeta, \zeta)$ does not vanish on D, we can set weighted L^p spaces as follows.

$$L^{p,r}(D) = L^p(D, b^{-r}(\zeta, \zeta) d\nu(\zeta)), 0 < p < \infty.$$

Let p be an arbitrary positive number. The weighted Bergman space will be denoted as usual by $A^{p,r}(D)$ it is the analytic part of $L^{p,r}(D)$, with usual modification for $p = \infty$ case [1,2]. We put also $A^{p,0} = A^p(D)$. The so-called weighted Bergman projection P_ε is the orthogonal projection of Hilbert space $L^{2,\varepsilon}(D)$ onto $A^{2,\varepsilon}(D)$. These facts can be found in [1,2]. It is proved in [1,2] that there exist a real number $\varepsilon_D < 0$ such that $A^{2,\varepsilon}(D) = \{0\}$ if $\varepsilon \leq \varepsilon_D$; and that for $\varepsilon > \varepsilon_D$, P_ε is the integral operator defined on $L^{2,\varepsilon}(D)$ by the weighted Bergman kernel $c_\varepsilon b^{1+\varepsilon}(\zeta, z)$. In all our work we shall assume that $\varepsilon > \varepsilon_D$.

The norm $\| \|_{p,r}$ of $A^{p,r}(D)$ with $r > \varepsilon_D$ is defined by

$$\|f\|_{p,r} = \left(\int_D |f(z)|^p b^{-r}(z, z) d\nu(z) \right)^{\frac{1}{p}}, f \in A^{p,r}(D).$$

with usual modification for $p = \infty$ case.

We need some assertions (see [1,2,16]), some basic facts on Bergman kernel and Bergman projection in Siegel domains of second type. They will be partially used by us below in proofs of our theorems. Some proofs of these preliminaries are rather intricate [1,2,16]. We indicate for readers in advance that some assertions below involving integrals can be easily extended to m- products of Siegel domains of second type by simple application of "one variable" result m -times by each variable separately. This procedure is well-known in simpler case of polydisk. (see for example [4,5]).

Lemma A. Let $h \in L^{\infty}(D)$. Take $\rho > \rho_0$, for large fixed ρ_0. Then the function

$$z \to G(z) = \int_D b^{1+\rho}(z,\zeta)h(\zeta)d\nu(\zeta)$$

satisfies the estimate $\sup\limits_{z\in D} |G(z)|b^{-\rho}(z,z) \leq c\|h\|_{\infty}$ and $G \in H(D)$.

The following lemma is a complete analogue of so-called Forelly-Rudin type estimate for our Siegel domains of second type.

Lemma B. Let α and ε be in R^l, $(\zeta, v) \in D$. Then we have

$$\int_D |b^{1+\alpha}((\zeta,v),(z,u)|b^{-\varepsilon}((z,u),(z,u))dv(z,u) < \infty$$

if and only if $\varepsilon_i > \left(\frac{n_i+2}{2(2d-q)i}\right)$ and $(\alpha_i - \varepsilon_i) > \frac{n_i}{(-2)(2d-q)_i}$, $i = 1,...,l$.

The following lemma is another complete analogue of so-called Forelly-Rudin type estimates for our Siegel domains of second type.

Lemma C. Let α and ε be in R^l, $(\zeta, v) \in D$. Then for $\varepsilon_i > \left(\frac{n_i+2}{2(2d-q)i}\right)$ and $(\alpha_i - \varepsilon_i) > \frac{n_i}{(-2)(2d-q)_i}$, $i = 1,...,l$

$$\int_D |b^{1+\alpha}((\zeta,v),(z,u)|b^{-\varepsilon}((z,u),(z,u))dv(z,u) = c_{\alpha,\varepsilon}b^{\alpha-\varepsilon}((\zeta,v),(\zeta,v))$$

Lemma D. Let r be a vector of R^l such that $r_i > (\frac{n_i+2}{2(2d-q)_i})$ for all $i = 1,...,l$ and a p is a real number such that $1 \leq p < \min\{\frac{n_i-2(2d-q)_i(1+r_i)}{n_i}\}$. Then for all $\varepsilon \in R^l$ such that $(\varepsilon_i) > \frac{n_i+2}{2(2d-q)_i}(\frac{p-1}{p}) + (\frac{r_i}{p})$, $i = 1...l : P_{\varepsilon}f = f$, $f \in A^{p,r}$.

We list in lemma E other properties of Bergman kernel. The last estimate in assertion below is an embedding theorem which connect so-called growth spaces with Bergman spaces This allows to pose a distance problem immediately (see also the complete analogue of this result in other simpler domains in [11,12] and in [4,5]).

Lemma E. Let $\alpha \in R^l$; $\alpha_j > 0$ or $\alpha_j = 0$, $j = 1,...,l$. Then $|b^{\alpha}((\zeta,v),(z,u))| \leq c_{\alpha}b^{\alpha}((\zeta,v),(\zeta,v))$ and $|b^{\alpha}((\zeta,v) + (\zeta',v');(z,u) + (z',u'))| \leq c_{\alpha}b^{\alpha}((\zeta,v),(\zeta,v))$ for all $(\zeta,v),(\zeta',v'),(z,u),(z',u)$ in D. For all $f \in A^{p,r}(D)$, $p > 0$

$$|f(z,u)|^p \leq cb^{1+r}((z,u),(z,u)\|f\|^p_{p,r},$$

for all (z,u) points taken from D.

The following result is crucial for the proof of our theorem 3. It concerns the boundedness of Bergman type projection with positive Bergman kernel in weighted Bergman spaces. Note this fact is classical in simpler domains and it has also many applications in analytic function theory (see [4,5]).

Proposition A. Let k and r be in R^l such that $k_i > \frac{1}{(2d-q)_i}$ and $r_i > \frac{n_i+2}{2(2d-q)_i}$, $i = 1, ..., l$. Then P_k is bounded from $L^{p,r}(D)$ into $A^{p,r}(D)$ if

$$\max_{i=1,...,l}\{1, \frac{2n_i + 2 - 2(2d-q)_i r_i}{n_i + 2 - 2(2d-q)_i k_i}\} < p < \min_{i=1,...,l} \frac{2n_i + 2 - 2(2d-q)_i r_i}{n_i}$$

We denote below everywhere by p_1 and p_0 the right and the left end of the interval for p parameter in proposition A.

The following assertion is a base of proof of theorem 1. It provides integral representation for a certain so called analytic "growth space" on Siegel domains of the second type.

Proposition B. Let r and ε be two vectors of R^l such that $\varepsilon_i > \frac{n_i}{(-2)(2d-q)_i}$; $r_i > \frac{n_i+2}{2(2d-q)_i} + \varepsilon_i$, $i = 1, ..., l$. Let G be in $H(D)$ such that

$$\sup_{z \in D}\{|G(z)| b^{-\varepsilon}(z, z)\} < \infty.$$

Then $P_r G = G$.

The following result explains the structure of functions from Bergman spaces on Siegel domains of second type. It is an extension of a classical theorem on atomic decomposition of Bergman spaces in the unit disk on a complex plane.(see, for example, [19] and references there).

Proposition C. Let $D \subset C^n$ be a symmetric Siegel domain of type II, $p \in (\frac{2n}{2n+1}, 1)$, $r \in R^l$; $r_j > \frac{n_i+2}{2(2d-q)_i}$. Then there are two constants $c = c(p, r)$ and $c_1 = c_1(p, r)$ such that for every $f \in A^{p,r}(D)$ there exists an l^p sequence $\{\lambda_i\}$ such that

$$f(z) = \sum_{i=0}^{\infty} \lambda_i b^{\alpha/p}(z, z_i) b^{\frac{1+r-\alpha}{p}}(z_j, z_j)$$

where $\{z_i\}$ is a lattice in D and the following estimate holds $c\|f\|_{p,r}^p \leq \sum_{i=1}^{\infty} |\lambda_i|^p \leq c_1 \|f\|_{p,r}^p$ where α is a special fixed vector (see [19]).

3 SHARP THEOREMS ON DISTANCES IN ANALYTIC SPACES ON SIEGEL DOMAINS AND ON PRODUCTS OF SIEGEL DOMAINS OF SECOND TYPE

The goal of this main section to extend the main result from [11] on distances given there for $p = 1$ in two directions to some values of $1 < p < \infty$ and to products of Siegel domains. We start with some discussion however related with these issues. First we define and fix some properties of Bergman-type spaces on products of Siegel domains of the second type $A^{p_1,...,p_m,\varepsilon_1,...,\varepsilon_m}(D^m)$, $m > 1$. These spaces are closely related with Trace operator and multifunctional analytic function spaces in higher dimension (see, for example [9,10,18] and various references there). More precisely we consider spaces of all analytic functions (analytic by each variable separately) $f(z_1, ..., z_m) \in H(D \times ... \times D)$. Then we define for $p_i \in (0, \infty)$, $\varepsilon_i \in R$, $i = 1, ..., m$ the following spaces on product domains. We define first a subset of Locally integrable functions on D^m.

$$(L^{p_1,...,p_m,\varepsilon_1,...,\varepsilon_m})(D^m) = \{locallyintegrable : \|f\|_{\vec{p},\vec{\varepsilon}} =$$

$$= \left(\int_D \left(\int_D \cdots \left(\int_D |f(z_1,...,z_m)|^{p_1} b^{-\varepsilon_1}(z_1,z_1)dv(z_1)\right)^{p_2/p_1} \times b^{-\varepsilon_2}(z_2,z_2)dv(z_2)\right)\right)^{1/p_m} < \infty\}$$

and $\tilde{A}^{\vec{p},\vec{\varepsilon}} = H(D^m) \cap L^{\vec{p},\vec{\varepsilon}}(D^m)$. Many general issues related with various functional spaces on product domains can be seen in [15]. Later this topic was developed by many authors (see,for example, for analytic spaces on products domains, [18] and references there). We defined Bergman-type mixed norm spaces in products of Siegel domains of second type. For $\min_i(p_i) > 1$,$j = 1,...,m$ these are Banach spaces and complete metric for other values. Note for very particular case when D is C_+ (upper half space) or when D is unit disk on a complex plane C and all $p_j = p$ for each j from 1 to m these are well-known analytic Bergman type spaces in polydisk or polyhalspace (see[4,5] and [18]).

A natural question is to try to understand the structure of these new interesting spaces, in particular to extend results obtained for $m = 1$ in [1,2] for this general $m > 1$ case. We in this paper consider only $p_j = p$ case, where $j = 1,...,m$. Our last theorem is a sharp distance theorem for this spaces for this particular case. However the proof of one side estimate for distances($l_1 < cl_2$ implication see below) for general case also follows from the proof of particular case immediately. In addition many questions concerning various embedding between such Bergman type spaces with vector p also arise naturally. We note the study of these type classes on product domains in most typical Siegel domains (bounded or unbounded) bounded strictly pseudoconvex domains or tubular domains over symmetric cones or polyball was started in particular recently in papers of first author and coauthors (see [13,14,22]). We mention also [20] and [21] for some results in this area. Some new results about these spaces in particular case of polyball can be seen also in [18]. We will heavily need a simple observation. In some situations we easily note that just using $m-$times a "one dimensional result" by each variable separately we get a result for $D \times ... \times D$ product domains in a ready form. This observation can be applied for various assertions we had above.

As an example we note easily that, for example, based on last part of Lemma E we have.

$$(\sup_{z_1,...,z_m})|f(z_1,...,z_m)|^p \times b^{-(1+r_1)}(z_1,z_1)...b^{-(r_m+1)}(z_m,z_m) \le C_k(f) \le$$

$$\le c\int_D \cdots \int_D |f(z_1,...,z_m)|^p \prod_{j=1}^m b(z_j,z_j)^{-r_j} dv(z_1)...dv(z_m)$$

where

$$C_k(f) = \sup_{z_1,...,z_k} \int_D \cdots \int_D |f(z_1,...,z_k,z_{k+1},...,z_m)|^p \times \left(\prod_{j=k+1}^m b(z_j,z_j)^{-r_j}\right) dv(z_{k+1})...dv(z_m) \times$$

$$\times \left(\prod_{j=1}^k b(z_j,z_j)^{-(1+r_j)}\right)$$

Since $C_k(f) \le \tilde{C}_k(f) \le c\|f\|_{A^{\vec{p},\vec{r}}}$ where

$$\tilde{C}_k(f) = \sup_{z_1,\dots,z_{k-1}} \int_D \dots \int_D \left[\sup_{z_k} |f(\vec{z})|^p b^{-(1+r_k)}(z_k, z_k)\right] \prod_{j=k+1}^{m} b(z_j, z_j)^{-r_j} dv(z_{k+1})\dots dv(z_m)\times$$

$$\times \prod_{j=1}^{k-1} b^{-(1+r_j)}(z_j, z_j).$$

Defining $L^{\vec{p},\vec{\varepsilon}}$ and $A^{\vec{p},\vec{\varepsilon}}$ it is natural to consider $p_i = \infty$ cases also. For $m = 2$ case these "norms" will look like

$$\sup_{z_2} \left(\int_D |f(z_1, z_2)|^{p_1} (b^{-\varepsilon}(z_1, z_1))dv(z_1)\right)^{p_2/p_1} (b^{-\varepsilon}(z_2, z_2))$$

or

$$\int_D \left(\sup |f(z_1, z_2)|(b^{-\varepsilon}(z_1, z_1))\right)^{p_2} (b^{-\varepsilon}(z_2, z_2))dv(z_2)$$

Next we can easily note that almost all lemmas above can be extended also to product $D\times\dots\times D$ case. For example, using Bergman reproducing formula we mentioned in lemmas above namely $P_\varepsilon f = f$ equation by each variable separately we will set the reproducing formula for $A^{p,r}(D^m)$, if as Bergman kernel we put $m-$ products of one dimensional kernels (see the same idea in [4,5] for simpler cases of unit disk and polydisk). This simple observation is crucial for the proof of last theorem of this note a sharp theorem on distances in products of Siegel domains of second type.

As one more application of productive idea given above we add an embedding for analytic Bergman type spaces on products of Siegel domains of the second type below. The study of these new analytic spaces from various points is an interesting and separate problem.

Lemma 1. We have the following estimate

$$\left(\int_D \dots \int_D |f(z_1, \dots, z_m)|(\prod_{i=1}^{m} b^{-\rho_i}(z_i, z_i))dv(z_1)\dots dv(z_m)\right)^p \le$$

$$\int_D \dots \int_D |f(z_1, \dots, z_m)|^p(\prod_{i=1}^{m} b^{-\alpha_i}(z_i, z_i))dv(z_1)\dots dv(z_m); wherep \in (0, 1); \rho_i = \frac{1+\alpha_i}{p} - 1,$$

$i = 1, \dots, m$.

The proof follows from estimate above and the equality $|f| = |f|^{1-p}|f|^p$, $p \in (0, 1)$ and estimate of lemma E $|f(z)| \le cb^{\frac{1+\alpha}{p}}(z, z)\|f\|_{p,\alpha}$ for $z \in D$, $f \in H(D)$. Note for m=1 case this long estimate can be seen in [2]. The following sharp theorem on distances was formulated without proof in [6]. Here we provide a complete proof of it.This is the first sharp theorem on distances in analytic spaces in Siegel domains of second type.

Theorem 1. Let

$$r^0 = \max\left(\frac{n_j + 2}{2(2d - q)_j}, -1 - \frac{n_j}{2(2d - q)_j}\right), j = 1, \dots, l$$

Let

$$N_{\tilde{\varepsilon},r}(f) = \{z \in D : |f(z)|b^{1+r}(z,z) \geq \tilde{\varepsilon}$$

where $\tilde{\varepsilon}$ is a positive number. Then the following two quantities are equivalent $l_1 \asymp l_2$, where

$$l_1 = dist_{A^{\infty}_{1+r}}(f, A^{1,r}).$$

$$l_2 = \inf\{\tilde{\varepsilon} > 0 : \int_D \left(\int_{N_{r,\tilde{\varepsilon}}} b^{-k+1+r}(\tau,\tau)|b(\tau,z)|^{k+1} dv(\tau) \right) b^{-r}(z,z)dv(z) < \infty\}$$

for all r and k so that, $r = (r_1, \dots, r_l)$, $k = (k_1, \dots, k_l)$, $r_j \in (r^0, \infty)$ and $k_j \in (k_0^j, \infty)$, $j = 1, \dots, l$. And for certain fixed vector $k_0 = (k_0^1, \dots, k_0^l)$ depending on r_j and on parameters of the Siegel D domain (d_i) and (q_i) and (n_i).

One of the goals of this paper to give a complete proof of this theorem. Complete analogues of all theorems of this paper in spaces of harmonic or analytic functions of several variables in unit ball of Euclidean space R^n or unit ball or polydisk or in bounded pseudoconvex domains with smooth boundary in C^n can be seen for example in [7,11,12,17] (see also various references there)

The proof of theorem 1.

First we show that $l_1 \leq cl_2$. Let $r \in R^l$,$r = (r_1, \dots, r_l)$. Then by Bergman representation formula we mentioned in previous section we have.

$$f(z) = c_k \int_D f(u)b^{1+k}(z,u)b^{-k}(u)du, \ f \in A^{\infty}_{1+r}, \ z \in D, \ k_j > m_j$$

for some large enough m_j vector elements depending on parameters of Siegel domain. Hence we have putting $f = f_1 + f_2$

$$|f_1(z)| \leq c \int_{D \setminus N_{\tilde{\varepsilon},r}} |f(u)||b^{1+k}(z,u)|(b^{-k}(u,u))du \leq$$

$$\leq c\varepsilon \int_D |b^{1+k}(z,u)|(b^{-k+1+r}(u,u))dv(u) \leq c\varepsilon b(z,z)^{1+r};$$

$z \in D$. And we also have that

$$\int_D |f_2(z)|(b^{-r}(z,z))dv(z) \leq \int_D \left(\int_{N_{\tilde{\varepsilon},r}} b^{-k+1+r}(\tau,\tau)|b(\tau,z)|^{k+1} dv(\tau) \right) (b^{-r}(z,z))dv(z) \leq c$$

Note that we used also that $k_j - 1 - r_j > \frac{n_j+2}{2(2d-q)_i}$

$$k_j - 1 - r_j - \frac{n_j}{2(2d-q)_i} < k_j; \ j = 1, \dots, l$$

Since we have according to lemmas above

$$\int_D |b^{1+k}(z,u)|b^{-k+1+r}(u,u)dv(u) \leq cb(z,z)^{1+r};\ z \in D.$$

So we have finally $dist_{A^\infty_{1+r}}(f, A^{1,r}) \leq c\|f - f_2\|_{A^\infty_{1+r}} = c\|f_1\|_{A^\infty_{1+r}} \leq c\varepsilon$.

We show the reverse implication now following the unit disk case arguments (see, for example [11,12] and references there). Let us assume that $l_1 < l_2$. Then we can find two numbers ε, ε_1 such that $\varepsilon > \varepsilon_1 > 0$, and a function $f_{\varepsilon_1} \in A^{1,r}$, $\|f - f_{\varepsilon_1}\|_{A^{1,r}} \leq \varepsilon_1$ and hence

$$(\varepsilon - \varepsilon_1)[\chi_{N_\varepsilon(f)}(z)]b(z,z)^{(1+r)} \leq c|f_{\varepsilon_1}(z)|$$

and hence

$$\int_D \left(\int_D (\chi_{N_\varepsilon(f)}(z))b(z,z)^{1+r-k}|b(\tau,z)|^{k+1}dv(z) \right) (b^{-r}(\tau,\tau))dv(\tau) \leq$$

$$\leq c\int_D \left(\int_D |f_{\varepsilon_1}(z)|(b(\tau,z)^{k+1})(b(z,z)^{-k})dv(z) \right) b^{-r}(\tau,\tau)dv(\tau) \leq c\int_D |f_{\varepsilon_1}(z)|(b(z,z)^{-r})dv(z).$$

We used first Fubini's theorem and then lemma B namely the estimate,

$$\int_D |b(\tau,z)|^{k+1}(b^{-r}(\tau,\tau))dv(\tau) \leq c(b(z,z))^{k-r},\ z \in D,$$

where

$$r_j > \frac{n_i+2}{2(2d-q)_i};\ k_i > r_i - \frac{n_i}{2(2d-q)_i};\ i = 1,\dots,l.$$

These estimates provide some estimate from below for $k = (k^1,\dots,k^l)$ fixed in formulation of theorem 1. Theorem 1 is proved.

The following theorem was proved in [1]. As a direct corollary of this result and arguments of the proof of theorem 1 we will get immediately theorem 3 (a direct generalization of theorem 1 to some values of p,$p > 1$). Details of proof of theorem 3 will be hence omitted by us.

Theorem 2. Let p_0 and p_1 be fixed numbers as above. Let also r and k be two vectors from R^l. Let $k_i > \frac{1}{(2d-q)_i}$ and $r_i > \frac{n_i+2}{2(2d-q)_i}$, $i = 1,\cdots,l$. Let $p \in (1,\infty)$ and $p \in (p_0,p_1)$. Then we have P_k^* is a bounded operator on $L^{p,r}(D)$ where

$$P_k^* f(\xi) = C_k \int_D |b^{1+k}(\xi,z)|(b^{-k}(z,z))|f(z)|dv(z),$$

and ξ is any point from D. It is a bounded Bergman projection with positive Bergman kernel, for some positive constant C_k.

Let us note this theorem can be easily extended (for $p = 1$ case) to the product of Siegel domains of second type following standard procedure of adding variables and remarks we did above.

Theorem 3. Let $p > 1$ and $p \in (p_0, p_1)$. Let $t = \frac{1+r}{p}$. Let also

$$N^p_{\tilde{\varepsilon},r}(f) = \{z \in D : |f(z)|b^t(z, z) \geq \tilde{\varepsilon},$$

where $\tilde{\varepsilon}$ is a positive number. Then the following two quantities are equivalent $l_1 \asymp l_2$, where

$$l_1 = dist_{A^{\infty}_t}(f, A^{p,r}).$$

$$l_2 = \inf\{\tilde{\varepsilon} > 0 : \int_D \left(\int_{N^p_{r,\tilde{\varepsilon}}} b^{-k+t}(\tau, \tau)|b(\tau, z)|^{k+1} dv(\tau) \right)^p b^{-r}(z, z) dv(z) < \infty\}$$

for all r and k so that, $r = (r_1, \ldots, r_l)$, $k = (k_1, \ldots, k_l)$, $r_j \in (r_0, \infty)$ and $k_j \in (k^j_0, \infty)$, $j = 1, \ldots, l$ and for certain fixed vector $k_0 = (k^1_0, \ldots, k^l_0)$ depending on r_j and on parameters of the Siegel D domain (d_i) and (q_i) and (n_i), and number r_0 depending on p and on parameters of the Siegel D domain (d_i) and (q_i) and (n_i).

Finally we formulate a new general sharp theorem for distances in products of Siegel domains of second type,but only for $p = 1$ case. Note by $\tilde{c}(z, w) = c_m(z, w)$ we denote below m-products of Bergman kernels in Siegel domains of second type, z and w belong to D^m, and G, $G(z) = G(z_1, ..., z_m)$, $m > 1$ is an analytic function by each variable separately in m-products of Siegel domains. Adding below wave˜on objects we already used above we mean same objects defined on products of Siegel domains of second type. This concerns Lebeques measure on D^m and analytic spaces on them.

Theorem 4. Let

$$r^0 = \max\left(\frac{n_j + 2}{2(2d - q)_j}, -1 - \frac{n_j}{2(2d - q)_j} \right), j = 1, \ldots, l.$$

Let

$$N^1_{\tilde{\varepsilon},r}(G) = \{z \in D^m : |G(z)|c^{1+r}(z, z) \geq \tilde{\varepsilon}$$

where $\tilde{\varepsilon}$ is a positive number. Then the following two quantities are equivalent $l_1 \asymp l_2$, where

$$l_1 = dist_{\tilde{A}^{\infty}_{1+r}}(f, \tilde{A}^{1,r}).$$

and

$$l_2 = \inf\{\tilde{\varepsilon} > 0 : \int_{D^m} \left(\int_{N^1_{r,\tilde{\varepsilon}}} \tilde{c}^{-k+1+r}(\tau, \tau)|\tilde{c}(\tau, z)|^{k+1} d\tilde{v}(\tau) \right) \tilde{c}^{-r}(z, z) d\tilde{v}(z) < \infty\}$$

for all r and k vectors so that, $r = (r_1, \ldots, r_l)$, $k = (k_1, \ldots, k_l)$, $r_j \in (r^0, \infty)$ and $k_j \in (k^j_0, \infty)$, $j = 1, \ldots, l$ for certain fixed vector $k_0 = (k^1_0, \ldots, k^l_0)$ depending on r_j and on parameters of the Siegel D domain (d_i) and (q_i) and (n_i).

For the proof of this general sharp theorem for products of Siegel domains of second type we only note that all what is needed can be seen in proof of theorem 1. Indeed we have to use a simple induction and transfer each lemma above which is needed in proof to product domains. As we indicated already the base of the proof is the fact that as a Bergman kernel on product domains we can take products of "onedimensional" Bergman kernels, then apply estimates in our proof for them by each variable m times separately if needed. Note this simple procedure of passing to product domains using products of Bergman kernels was used also in much simpler case of unit disk and polydisk (see [4,5]). This observation and arguments of proof of theorem 1 will lead to the completion of proof of theorem 4 and we leave the proof of theorem 4 to interested readers. The last theorem probably is valid also for all $p > 1$ this however will not be discussed in this paper. Next our last theorem is a sharp distance theorem for product spaces we defined above when $p_j = p$ for all j. The proof of(not sharp)but more general estimate -one side estimate for distances(implication $l_1 < cl_2$) for general case of mixed norm spaces however follows from arguments of proof of particular case immediately.

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