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ON THE INCREMENT OF SOME CLASSES ANALYTICAL FUNCTIONS

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Abstract. In this paper estimates of the growth for the functions from the well known classes D^p_α and A^p_α, when z tends to the unit circle are obtained. First the known estimate $|f(z)| \le M(1-|z|)^{-\frac{\alpha+2}{p}}$ for the functions from the class A^p_α is sharpened for the functions belonging to the particular class $A^2_\alpha(-1<\alpha<\infty)$, then it is generalized for the whole space A^p_α.

Finally it is proved that this estimate is not possible to improve, i. e. the exponent $\frac{1}{p}$ is exact.

Lets $\mathcal{D}$ be the unit disk in the complex plane $\mathcal{C}$, and $Hol(\mathcal{D})$ be a set of holomorphic functions in $\mathcal{D}$. We say that the function $f(z)$ from $Hol(\mathcal{D})$ belongs to the class A^p_α, $0<p<+\infty,\ \alpha>-1$, if

$$\int_0^1\int_{-\pi}^{\pi}(1-r)^\alpha \left|f(re^{i\theta})\right|^p r dr d\theta < +\infty,\ z=re^{i\theta}.$$

The research of the behavior of functions of these classes are of interest of many authors. In M. M. Jrbashyan's work [1] these classes were denoted by $H_p(\alpha)$. Some authors call them the classes of Bergman.

In [1] and in several other works (see for instance [2]) it is proved that if $f \in A^p_\alpha$, $0<p<+\infty$, $-1<\alpha<+\infty$ then

$$|f(z)| \le \frac{M}{(1-|z|)^{\frac{\alpha+2}{p}}},\ \ z\in\mathcal{D}, \tag{1}$$

where M is a constant.

If the sequence $\mathbf{a}=\{a_j\}$ has

$$\sum(1-|a_j|)^2<\infty$$

A condition which is certainly met by all A^p_α, $0<p<+\infty$, $-1<\alpha<+\infty$ space zero sequences, C. Horowitz [3] introduced a product

$$H_{\mathbf{a}}(z)=\prod_j b(z,a_j)(2-b(z,a_j)),\ z\in\mathcal{D},$$

where

$$b(z,\beta)=\frac{\overline{\beta}}{|\beta|}\frac{\beta-z}{1-\overline{\beta}z},\ z\in\mathcal{D},\ \beta\in\mathcal{D},$$

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denotes a single Blashke factor. The Horowitz product is that it may grow wildly towards the boundary, and, in general, it itself will not belong to the A^p_α space. It is known [2], that if $f \in A^p_\alpha$, and $\mathbf{a}$ is the sequence of zeroes of the function f, then $\frac{f(z)}{H_{\mathbf{a}}(z)} \in A^p_\alpha$.

We say that the function f from $Hol(\mathcal{D})$ belongs to the class D^p_α, where $0 < p < +\infty$, $-1 < \alpha < +\infty$, if

$$\int_0^1 \int_{-\pi}^{\pi} (1-r)^\alpha \left| f'(re^{i\theta}) \right|^p r dr d\theta < +\infty, \; z = re^{i\theta}.$$

The class of functions D^2_0 coincides with the usual class of analytic in $\mathcal{D}$ functions with finite Dirichlet integral. If $\alpha + 1 \le p$, D^p_α is called the class of functions with bounded Dirichlet type integral.

Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in Hol(\mathcal{D})$. Denote

$$M(r,f) = \max_{|z|=r} |f(z)|; \; \overline{M}(r,f) = \sum_{n=0}^{\infty} |a_n| r^n.$$

In [4] V. Gowling proved that if $f(z) \in D^2_0$, then

$$\lim_{r \to 1-} \left(\log \frac{1}{1-r} \right)^{-\frac{1}{2}} \cdot \overline{M}(r,f) = 0, \tag{2}$$

In [5] S. Yamashita proved that $-\frac{1}{2}$ in equation (2) is the best possible value, i.e. for any constant $q, 0 < q < \frac{1}{2}$ there exists $f \in D^2_0$ such that

$$\lim_{r \to 1-} \inf \left(\log \frac{1}{1-r} \right)^{-q} \cdot \overline{M}(r,f) \ge 1 \tag{3}$$

V. Zakaryan in [6] proved that if $f \in D^2_\alpha$, $\alpha > 0$ then

$$\lim_{r \to 1} (1-r)^{\frac{a}{2}} \cdot \overline{M}(r,f) = 0, \tag{4}$$

Moreover, for any constant q, $0 < q < \frac{1}{2}$ there exists a function $f(z) \in D^2_\alpha$ such that

$$\lim_{r \to 1} \inf ((1-r)^{q\alpha} \cdot M(r,f) \ge 1 \tag{5}$$

The results (2)-(5) as well as some similarities of the classes A^p_α and D^p_α suggest that the result (1) can be strengthened. In this work we get similar to (4), (5) for the functions from classes A^p_α.

Theorem 1. Let $-1 < \alpha < +\infty$, $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in A^2_\alpha$ and $z = re^{i\theta\varphi}$, then

$$\lim_{r \to 1-} (1-r)^{1+\frac{a}{2}} \cdot \overline{M}(r,f) = 0 \tag{6}$$

Proof. As $f(z) \in A^2_\alpha$, then (see [7])

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n^{\alpha+1}} < +\infty \tag{7}$$

Let's write $\overline{M}(r,f)$ in the following way:

$$\overline{M}(r,f) = \sum_{n=0}^{k} |a_n| r^n + \sum_{n=k+1}^{\infty} [\gamma(\alpha+1,n)]^{1/2} |a_n| \cdot [\gamma(\alpha+1,n)]^{-1/2} \cdot r^n,$$

where

$$\gamma(\alpha+1,n) = \frac{\Gamma(\alpha+1) \cdot \Gamma(1+n)}{\Gamma(\alpha+2+n)} = \int_0^1 (1-r^2)^\alpha \cdot r^{2n+1} dr \tag{8}$$

It is known that (see [8], page 885)

$$\gamma(\alpha+1,n)=O\left(\frac{1}{n^{\alpha+1}}\right) \tag{9}$$

Applying the Cauchy inequality, we get

$$\overline{M}(r,f)\le\sum_{n=0}^{k}|a_n|r^n+\left(\sum_{n=k+1}^{\infty}|a_n|^2\gamma(\alpha+1,n)\right)^{1/2}\cdot\left(\sum_{n=k+1}^{\infty}\frac{r^{2n}}{\gamma(\alpha+1,n)}\right)^{1/2}$$

From the last inequality, using (8) and (9), we receive

$$\overline{M}(r,f)\le\sum_{n=0}^{k}|a_n|r^n+C_1\left(\sum_{n=k+1}^{\infty}\frac{|a_n|^2}{n^{\alpha+1}}\right)^{\frac{1}{2}}\cdot\left(\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+1)\Gamma(1+n)}r^n\right)^{\frac{1}{2}}=$$

$$=\sum_{n=0}^{k}|a_n|r^n+C_1\left(\sum_{n=k+1}^{\infty}\frac{|a_n|^2}{n^{\alpha+1}}\right)^{\frac{1}{2}}(1-r)^{-(1+\frac{\alpha}{2})}$$

where C_1, is a constant. Hence, we have

$$(1-r)^{1+\frac{\alpha}{2}}\overline{M}(r,f)\le(1-r)^{1+\frac{\alpha}{2}}\sum_{n=0}^{k}|a_n|r^n+C_1\left(\sum_{n=k+1}^{\infty}\frac{|a_n|^2}{n^{\alpha+1}}\right)^{\frac{1}{2}}$$

Using this inequality, it is easily seen that

$$\lim_{r\to1-}\sup(1-r)^{1+\frac{\alpha}{2}}\cdot\overline{M}(r,f)\le C_2\left(\sum_{n=k+1}^{\infty}\frac{|a_n|^2}{n^{\alpha+1}}\right)^{\frac{1}{2}}$$

Now applying (7) and noting that the left hand side of the inequality doesn't depend on k we get (6).
From this theorem the following statement holds true:

Theorem 2. Let $0<p<\infty, -1<\alpha<+\infty,\ f(z)=\sum\limits_{n=0}^{\infty}a_nz^n\in A^p_\alpha$, and $\mathbf{a}=\{a_j\}$ is the sequence of zeroes of the function f. Then the following equality takes place

$$\lim_{\substack{z\to e^{i\varphi}\\ z\in\mathcal{D}}}\frac{(1-|z|)^{\frac{2+\alpha}{p}}}{|H_{\mathbf{a}}(z)|}|f(z)|=0 \tag{10}$$

where $\varphi\in[0,2\pi], H_{\mathbf{a}}(z)$ - Horowits products.

Proof. When $p=2$ noting that $\overline{M}\left(r,\frac{f}{H_{\mathbf{a}}}\right)\ge\left|\frac{f(z)}{H_{\mathbf{a}}(z)}\right|$ for any $z\in\mathcal{D}$, from theorem 1 we have

$$\lim_{\substack{z\to e^{i\varphi}\\ z\in\mathcal{D}}}\frac{(1-|z|)^{1+\frac{\alpha}{2}}}{|H_{\mathbf{a}}(z)|}|f(z)|=0 \tag{11}$$

Now note that if $f(z)\in A^p_\alpha$ then $\left(\frac{f(z)}{H_{\mathbf{a}}(z)}\right)^{\frac{p}{2}}\in A^2_\alpha$. It means that for the function $f(z)\in A^p_\alpha$ then $\left(\frac{f(z)}{H_{\mathbf{a}}(z)}\right)^{\frac{p}{2}}$ statement (11) holds true, i.e.

$$\lim_{\substack{z\to e^{i\varphi}\\ z\in\mathcal{D}}}(1-|z|)^{\frac{2+\alpha}{2}}\left|\frac{f(z)}{H_{\mathbf{a}}(z)}\right|^{\frac{p}{2}}=0.$$

This completes the proof of the theorem.

Theorem 3. Let $0<p<+\infty,\ -1<\alpha<+\infty$. For any constant $q,\ 0<q<\frac{1}{p}$ there exists a function $g(z)\in A^p_\alpha$ such that

$$\lim_{z\to1-}\inf\,(1-|z|)^{q(2+\alpha)}M\left(|z|,g\right)\ge1 \tag{12}$$

Proof. Let $z = re^{i\theta}$ and

$$g(z) = (1-z)^{-q(2+\alpha)}, \quad |z| < 1.$$

We show that $g(z) \in A^p_\alpha$. For this purpose we evaluate above the following integral:

$$\int_0^1\int_0^{2\pi} (1-r)^\alpha \left|g\left(re^{i\theta}\right)\right|^p r dr d\theta = \int_0^1\int_0^{2\pi} (1-r)^\alpha \cdot \frac{r dr d\theta}{|1-re^{i\theta}|^{qp(2+\alpha)}} \leq$$

$$\leq \int_0^1 (1-r)^\alpha \left(\int_0^{2\pi} \frac{d\theta}{\left[(1-r)^2 + 4r\sin^2\frac{\theta}{2}\right]^{\frac{qp(2+\alpha)}{2}}} \right) dr.$$

Hence, since for $0 < \theta \leq \pi$

$$\left|1-re^{i\theta}\right| = \left[(1-r)^2 + 4r\sin^2\frac{\theta}{2}\right]^{\frac{1}{2}} \geq C_1(1-r+\theta)$$

where C_1 is a constant, we get

$$\int_0^1\int_0^{2\pi} (1-r)^\alpha \cdot \left|g\left(re^{i\theta}\right)\right|^p r dr d\theta \leq C_2 \int_0^1 (1-r)^\alpha \int_0^\pi \frac{d\theta}{(1-r+\theta)^{qp(2+\alpha)}} \leq$$

$$\leq C_3 - C_2 \int_0^1 \frac{dr}{(1-r)^{qp(2+\alpha)-1-\alpha}},$$

where C_2, C_3 are constants. As $q < \frac{1}{p}$, it follows that $g(z) \in A^p_\alpha$. The inequality (12) now holds, as

$$M(r,g) \geq g(r) = (1-r)^{-q(2+\alpha)}, \; 0 < r < 1.$$

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