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FREE AUTOMORPHISMS ON K3 SURFACES

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ABSTRACT. In the paper of Keiji Oguiso [15] is proved that there is a projective K3 surface admitting a fixed-point-free automorphism of positive entropy and that no smooth compact Kähler surface other than projective K3 surfaces and their blow-ups admit such an automorphism. Our goal in this paper is to construct exactly that surface and that automorphism.

1. PART I

Let M be a complex compact Kähler manifold. Let g be a biholomorphic automorphism of M.

Definition 1.1. The first dynamical degree of g, denoted $d_1(g)$, is the maximum of absolute values of eigenvalues of the $\mathbb{C}$-linear extension $g^*\colon H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{Z})$.

Definition 1.2. We say that g is of *positive entropy* if $d_1(g) > 1$. We say that g is of *null entropy* if $d_1(g) = 1$.

Remark 1.1. Last definition is the result of the research contained in the papers of Gromov-Yomdin [11], Dinh-Sibony [8] and Friedland [9].

(1) As g^* is an automorphism for $H^2_{DR}(M,\mathbb{Z})$, then $\det(g^*|_{H^2(X,\mathbb{Z})}) = \pm 1$. As

$$|\det g^*| = \left| \pm \prod_{i=1}^{b_2} \lambda_i \right|$$

then $1 = \prod_{i=1}^{b_2} |\lambda_i|$, then we have:

   (a) if there exists i such that $|\lambda_i| < 1$, then there exists j such that $|\lambda_j| > 1$ and g is of positive entropy;
   (b) if for all i, $|x_i| = 1$, then g is of null entropy.

(2) $(g^*)^k$ has eigenvalues λ_i^k, where λ_i are the eigenvalues for g^*. If $|\lambda_i| > 1$ and we have $|\lambda_i|^k = 1$, then $k = 0$ and so g has infinite order.

Definition 1.3. Let $M^g := \{x \in M \mid g(x) = x\}$. Map g is said to be *free* if $M^g = \emptyset$.

Remark 1.2. Non-trivial translation on a complex torus is free, but it is of null entropy.

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1.1. **Cantat's and Oguiso's results.** In the paper [4] by Cantat and in the paper [15] by Oguiso, next theoremes are shown :

Theorem 1.1. *Let X be a smooth complex compact Kähler surface admitting a free automorphism of positive entropy. Then X is birational to a projective K3 surface of Picard number greater than* 1*, and conversely there is a projective K3 surface of Picard number 2 with a free automorphism of positive entropy.*

In the paper [4] by Cantat it is shown the next theorem:

Theorem 1.2. *(Cantant's theorem) Let S be a smooth compact Kähler surface admitting an automorphism of positive entropy. Then S is bimeromorphic to either $\mathbb{P}^2$, to a 2-dimensional complex torus, an Enriques surface or a K3 surface.*

Now, if X is a smooth compact Kähler surface admitting an automorphism of positive entropy, we will show that that automorphism will be free just in case when S is a K3 surface.

Theorem 1.3. *Assume that X is birational to either $\mathbb{P}^2$ or to an Enriques surface. If $g \in Aut(X)$ is a biholomorphic morphism then $X^g \neq \emptyset$.*

So by this theorem we have that $\mathbb{P}^2$ and Enriques surfaces do not admit free automorphisms.

Theorem 1.4. *Assume that X is bimeromorphic to either a 2-dimensional torus or to a K3 surface. Let $\overline{X}$ be the minimal model of X and $\pi\colon X \to \overline{X}$ be the naturally induced morphism. Then g descends to an automorphism $\overline{g}$ of $\overline{X}$. Moreover:*

(1) *$\overline{g}$ is of positive entropy if and only if g is of positive entropy;*
(2) *$\overline{g}$ is free if and only if g is free.*

Theorem 1.5. *Let's suppose that X is bimeromorphic to a 2-dimensional complex torus. Let's suppose that X has a free automorphism g. Then g is of null entropy.*

Theorem 1.6. *Let X be bimeromorphic to a K3 surface $\overline{X}$. Let ω_X be a generator of $H^{2,0}(X)$. Assume furthermore that X has a free automorphism g. Then $g^*\omega_X = -\omega_X$ and X is projective. Moreover $\rho(\overline{X}) \geq 2$, where $\rho(\overline{X})$ is the Picard number of $\overline{X}$.*

The first part of our main theorem says if X is smooth complex compact Kähler surface admitting a free automorphism of positive entropy then X is birational to projective K3 surface of Picard number ≥ 2. This part is a consequence of Cantant theorem and the previous three theorems, which directly led us to the fact that X is birational to a projective K3 surface of Picard number ≥ 2.

And now I will state the most important theorem from Ougiso's paper , which was starting point for my result.

Theorem 1.7. *There exists a projective K3 surface X of Picard number $\rho(X) = 2$ such that $NS(X) = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2$, where*

$$((h_i \cdot h_j)) = \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$$

Any such K3 surface X admits a free automorphism g of positive entropy.

2. Part II

In this part I tried ,using result of Oguiso, to show explicitly how we can obtain surface and automorphism on that surface with properties described in previous theorem .
Let X be a K3 surface with $NS(X) = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2$ such that

$$(h_i \cdot h_j)_{i,j} = \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$$

We have that there are no divisors with self-intersection ± 2 and 0. We have that for any $ax + by \in NS(X)$ it holds:

$$(ax + by)^2 = 4(a^2 + ab - b^2)$$

Let us take D from $\Omega^+ \setminus \{0\}$ positive cone . Then $D^2 > 0$ and by Saint-Donat [16] we have an embedding $\varphi_D : X \hookrightarrow \mathbb{P}^N$.

Let D be of type $(1, 0)$ and we have $D^2 = 4$. A curve in $|D|$ is defined as $D_H = H \cap X$, where H is the hyperplane in $\mathbb{P}^3$, and will have class $(1, 0)$. This curve D_H is irreducible. Let us prove this. If we suppose that $D_H = C + C'$ and if C has class (a, b) and C' has class (c, d), thus we have that $a + c = 1$ and so $a = 0$ or $c = 0$. Hence, we get that one class is zero class and thus, D_H is irreducible.

Let D' be curve on X of type $(1, 1)$. Then we get: $(D')^2 = 4(1 + 1 - 1) = 4$. We want to see how is obtained curve D'. For sure we know that $D' \neq X \cap H$, since D' is of type $(1, 1)$ and $X \cap H$ is of type $(1, 0)$. What is the degree of D'?

$$D' \cdot H = D' \cdot D = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6$$

Is $D' \subseteq Q \cap X$, where Q is quadric and Q is of type $2H$, so with class $(2, 0)$? If we suppose that $D' \subseteq Q \cap X$ we have $Q \cap X = D' \cup D''$ and D' has class $(1, 1)$ and D'' has class $(1, -1)$. But as $(D'')^2 = -4 < 0$ we have that D'' is not from Ω^+ and by this we have that D' is not obtained as intersection of quadric Q and surface X.

Now we will investigate intersection of the cubic surface R and surface X. The cubic R has class $3H = (3, 0)$. We will suppose that $D' \subset R \cap X$. Thus we have $R \cap X = D' + D''$, and D' is of class $(1, 1)$ and so D'' will have class $(2, -1)$. As $(D'')^2 = 4$ then we have that $D'' \in \Omega^+$. What is the degree of D''?

$$D'' \cdot H = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6$$

Let us prove the existence of cubics $R \in |3H|$ such that $R = D' + D$". We have exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-X + 3H) \to \mathcal{O}_{\mathbb{P}^3}(3H) \to \mathcal{O}_X(3H) \to 0$$

and the first map is $f \mapsto f \cdot G$, where G is homogeneous polynomial defines X . We get a long exact sequence

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3H)) \to H^0(\mathcal{O}_X(3H)) \to H^1(\mathcal{O}_{\mathbb{P}^3}(3H - X)).$$

But $H^1(\mathcal{O}_{\mathbb{P}^3}(3H - X)) = H^1(\mathcal{O}_{\mathbb{P}^3}((3 - d)H) = 0$, where d is the degree of X. So we get

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3H)) \xrightarrow{\varphi} H^0(\mathcal{O}_X(3H))) \longrightarrow 0$$

and so φ is surjective. This means that for every $s \in H^0(\mathcal{O}_X(3H))$ there is polynomial function F of degree 3 (of course it is an element of $H^0(\mathcal{O}_{\mathbb{P}^3}(3H))$ such that $(s = 0) \subseteq X$

is given by $(F = 0) \cap X$. Since $D' + D''$ has class $(3, 0)$ we have that $D' + D'' \in |3H|$ on X and so there exists a cubic surface R such that $R \cap X = D' + D''$.

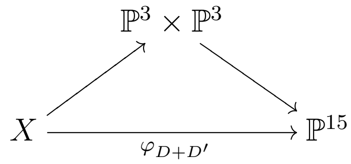
Let D and D' be divisors with corresponding class $(1, 0)$ and $(1, 1)$. Map $\varphi_D \times \varphi_{D'}$ will define an embedding of X into $\mathbb{P}^3 \times \mathbb{P}^3$:

$$\varphi_D \times \varphi_{D'} \colon X \hookrightarrow \mathbb{P}^3 \times \mathbb{P}^3$$

Now we will use Segre's embedding of $\mathbb{P}^3 \times \mathbb{P}^3$ into $\mathbb{P}^{4\cdot 4-1} = \mathbb{P}^{15}$ defined as

$$((x_0 \colon \dots \colon x_3), (y_0 \colon \cdot \colon y_3)) \mapsto (\dots \colon x_i y_i \colon \dots)$$

We denote $x_i y_j = z_{ij}$. So we define embedding of X into $\mathbb{P}^{15}$ using z_{ij}. We have a diagram



But the dimension of space of section which defines embedding $\varphi_{D+D'}$ is $h^0(D + D') = \frac{20}{2} + 2 = 12$ (by Riemann-Roch), so we have that the projective space where we can embed X is of dimension $12 - 1 = 11$. Hence , we can conclude that:$15 - 11 = 4$ linear equations of type $(1, 1)$ in $\mathbb{P}^{15}$ define X. In fact X is a complete intersection of 4 divisors of type $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$.

We will now show that the surface obtained as complete intersection of 4 divisors with classes (1,1) is K3 surface with properties which are desired in the begining of III part.

Let D be divisor on X. We will define cycle class of the divisor D as $[D] = c_1(D) \in H^2(X, \mathbb{Z}) \hookrightarrow H^2_{DR}(X)$.

Remark 2.1. Let $Z \subseteq X$ have codimension k in the n-dimensional complex manifold X, then

$$\int_Z (H_{DR}(X)^{2(n-k)})^{\text{dual}} \cong H^{2k}_{DR}(X).$$

The last isomorphism is provided by Poincare's theorem. So we can represent $\int_Z$ as $[v]$ such that for all $[\omega] \in H^{2(n-r)}_{DR}(X)$ we have $\int_Z \omega = \int_X v \wedge \omega$. If Y is complete intersection of two divisors D_1, D_2 then we have $[Y] = [D_1] \wedge [D_2]$.

If we return now to our case we have that X is intersection of 4 divisors of class $(1, 1)$. So we have that

$$[D_1 \cap D_2 \cap D_3 \cap D_4] = [D_1] \wedge [D_2] \wedge [D_3] \wedge [D_4]$$

and $D_i \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ and we know that

$$H^i_{DR}(\mathbb{P}^3) = \begin{cases} 0 & i \text{ odd} \\ \mathbb{R} & i \text{ even} \end{cases}$$

By Künneth formula we have that

$$\begin{aligned} H^2_{DR}(\mathbb{P}^3 \times \mathbb{P}^3) &= H^0(\mathbb{P}^3) \otimes H^2(\mathbb{P}^3) + H^1(\mathbb{P}^3) \otimes H^1(\mathbb{P}^3) + H^2(\mathbb{P}^3) \otimes H^0(\mathbb{P}^3) \\ &= \mathbb{R}[\omega_{FS,2}] \oplus \mathbb{R}[\omega_{FS,1}] \end{aligned}$$

where ω_{FS} is the Fubini- Study form.

If $Y \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ is a manifold of codimenison 1, then we have that $[Y] = ax + by$ where $x = \pi_1^*\omega_{FS}$ and $y = \pi_2^*\omega_{FS}$ since we know that $\mathrm{Pic}(\mathbb{P}^3 \times \mathbb{P}^3) = \mathrm{Pic}(\mathbb{P}^3) \oplus \mathrm{Pic}(\mathbb{P}^3)$ $(a, b \in \mathbb{Z})$. We say that Y has type (a, b).

Again in our case we have 4 divisors D_i of type $(1, 1)$ and then

$$\begin{aligned}[D_1] \wedge [D_2] \wedge [D_3] \wedge [D_4] &= (x + y) \wedge \ldots \wedge (x + y) \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

$x^4 = (\pi_1^*\omega_{FS})^4 = \pi_1^*(\omega_{FS}^4) \in H^8_{DR}(\mathbb{P}^3)$, since $\dim_{\mathbb{R}}(\mathbb{P}^3) = 6$ we have that $H^8_{DR}(\mathbb{P}^3) = 0$ and $x^4 = 0$. Same is for y^4. X is complete intersection of divisors D_1, D_2, D_3, D_4 and

$$\begin{aligned}[X] &= [D_1] \wedge [D_2] \wedge [D_3] \wedge [D_4] \\ &= 4x^3y + 6x^2y^2 + 4xy^3 \in H^8_{DR}(\mathbb{P}^3 \times \mathbb{P}^3).\end{aligned}$$

We have an embedding $i \colon X \hookrightarrow \mathbb{P}^3 \times \mathbb{P}^3$ and we naturally have $i^* \colon H^*_{DR}(\mathbb{P}^3 \times \mathbb{P}^3) \to H^*_{DR}(X)$. Let $x|_X$ and $y|_X$ be classes in $H^2_{DR}(X)$. Our aim is to find what are $(x|_X)^2$ and $(y|_X)^2$. We have that

$$(x|_X)^2 = (i^*x)^2 = i^*(x^2)$$

We claim that

$$i^*(x^2) = [x^2] \wedge [X] \in H^{12}_{DR}(\mathbb{P}^3 \times \mathbb{P}^3)$$

and

$$\begin{aligned}[x^2] \wedge [X] &= [x] \wedge [x] \wedge [D_1] \wedge [D_2] \wedge [D_3] \wedge [D_4] \\ &= x^2 \cdot (4x^3y + 6x^2y^2 + 4xy^3) = 4x^3y^3 = 4 \cdot 1 = 4\end{aligned}$$

Similarly $[y^2] \wedge [X] = 4$ and $[xy] \wedge [X] = 6$. So intersection matrix on X will be

$$\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$$

(since in basis (1,0), (0,1) intersection matrix will have this form) . And it is exactely surface with desired properties. So our surface is obtained as complete intersection of 4 divisors with classes $(1, 1)$.

How to find an automorphism g on the surface X, as before, such that automorphism is of positive entropy and free?

Motivation step:

Divisors $D' \sim (1, 1)$, $D'' \sim (2, 3)$, $D^* \sim (5, 8)$ are such that $(D')^2 = 4$, $(D'')^2 = 4$, and $(D^*)^2 = 4$. All of them define embeddings $\varphi_{|D'|}, \varphi_{|D''|}, \varphi_{|D^*|}$ into projective space. Divisor D^* with class $(5, 8)$ in $NS(X)$ correspond to the element η^6 in N because $\eta^6 = 8\eta + 5$. This element η^6, using the isomorphism φ between N and $NS(X)$, defines automorphism g with desired properties.

Let us try to write explicitly automorphism g. First of all let us consider divisor $D' \sim (1, 1)$ on X. That divisor, as it is shown before, is obtained by intersection X with some cubic surface R. So $R \cap X = D' + D_0$, where $D_0 \sim (2, -1)$. D_0 is curve on X and it is defined by global section t from $H^0(\mathcal{O}_X(D_0))$ such that $(t = 0) = D_0$. We will now construct $\varphi_{(1,1)}$.

We know that $\mathcal{O}_X(D') \xrightarrow{\cdot t} \mathcal{O}_X(D' + D_0) \cong \mathcal{O}_X(3)$, where $\mathcal{O}_X(3)$ represents all cubics. Let us now see what is space of all cubics containing D_0. We will denote that space with I_0. In fact, we have next situation: $R \in I_0$ is defined by $(F \equiv 0)$, so we have that

$X \cap (F \equiv 0) = D_0 + D'_F$, where $D'_F \in |D'|$, and it is valid for all $R \in I_0$. So $F|_X = s \cdot t$, where $(s = 0)$ defines D_0 and $(t = 0)$ defines D'_F.

And so we have map:

$$F|_X = s \cdot t \mapsto t \in H^0(D')$$

where as we said s is a section which is zero on D_0. This map is isomorphism between I_0 and $H^0(D')$, since:

- injectivity: if we take $R_F = D_0 + D'_F$ and $R_G = D_0 + D'_G$ such that $D'_F, D'_G \in |D'|$ and such that $D'_F \neq D'_G$ then
$$F|_X = s \cdot t_1 \mapsto t_1$$
$$G|_X = s \cdot t_2 \mapsto t_2$$
and so we have $t_1 \neq t_2$, where t_1 and t_2 belong to $H^0(D')$.
- surjectivity: $\forall t \in H^0(D')$ is such that $(t = 0) = \widetilde{D}$ and $\widetilde{D} \in |D'|$. Hence $\widetilde{D} \sim (1, 1)$ and $\widetilde{D} + D_0 \sim (3, 0)$ will be the cubic which contains D_0 .Thus our map is surjective.

Then we have that $I_0 \cong H^0(D')$. By Riemann-Roch we have

$$h^0(D') = 2 + \frac{(D')^2}{2} = 2 + \frac{4}{2} = 4,$$

so $I_0 \cong \mathbb{C}^4$. Space I_0 we can write down as $I_0 = \langle R_0, R_1, R_2, R_3 \rangle$. As we know that $\varphi_{|D'|}$ is an embedding into $\mathbb{P}^3$ of X we have that $\varphi_{|D'|} \colon X \hookrightarrow X_1 \subseteq \mathbb{P}^3$ defined by

$$x \mapsto (R_0(x) \colon \dots \colon R_3(x))$$

. If it happens that $R_i(x)$ is 0 for all i on D' then we just take some other divisor $D_1 \in |D'|$ and everything will fits. The embedding $\varphi_{|D'|} \colon X \hookrightarrow X_1$ we will call $\varphi_{(1,1)}$.

Let us now consider all cubics on X_1 cut out by divisor D'_1, i.e. all cubics of class $3 \cdot (1, 1) = (3, 3)$. We can write them as $(3, 3) = (2, 3) + (1, 0)$. So now we will consider all cubics on X_1 which contains $\varphi_{1,1}(D)$, where D is the hyperplane curve defined by $X \cap H$, $D \sim (1, 0)$. These cubics are making space which is isomorphic to $H^0(\mathcal{O}_X(D''))$,as we proved before, and $D'' \sim (2, 3)$. In this way we defined base of section on X_1 which provide embedding $\varphi_{|D''|} \colon X_1 \hookrightarrow X_2$. This embedding we will call $\varphi_{2,3}$.

Now we will consider cubics on X_2 cut out by divisor $D'' \sim (2, 3)$. So we have that $3 \cdot (2, 3) = (6, 9) = (1, 1) + (5, 8)$. These cubics all contains $\varphi_{23}(D')$. Divisor with class $(5, 8)$ we will denote as D^*. Space of cubics on X_2 contains $\varphi_{2,3}(D')$ is isomorphic to the space $H^0(\mathcal{O}_X(D^*))$. This space gives bases of space of sections which define an embedding $\varphi_{|D^*|} \colon X_2 \hookrightarrow X_3 \subseteq \mathbb{P}^3$, and as we get used to, we will denote it with $\varphi_{5,8}$. On the end we have $\varphi_{5,8} \colon X_2 \hookrightarrow X_3 \subseteq \mathbb{P}^3$. Explicitly we can write that g is:

$$g \colon X \xrightarrow{\varphi_{1,1}} X_1 \xrightarrow{\varphi_{2,3}} X_2 \xrightarrow{\varphi_{5,8}} X_3 \xrightarrow{M} X$$

M is a linear map and $MX_3 = X$. This linear map exists since $g^*(1, 0) = (5, 8)$ and explicitly if $\{t_0, \dots, t_3\}$ is a basis of $H^0(D^*)$ and $\{s_0, \dots, s_3\}$ is a basis for $H^0(D)$ then $g^* s_i = \sum a_{ij} t_j$ and $[a_{ij}]_{i,j}$ will represent matrix M and it will give our linear transformation M between isomorphic quartic surfaces in $\mathbb{P}^3$.

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