

EMBEDDING THEOREMS FOR HARMONIC MULTIFUNCTION SPACES ON \mathbb{R}_+^{n+1}

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ABSTRACT. We introduce and study properties of certain new multi functional harmonic spaces in the upper half space. We prove several sharp embedding theorems for such multi functional spaces, these results are new even in the case of a single function.

1. INTRODUCTION AND AUXILIARY RESULTS

The theory of harmonic function spaces in the single function case is well developed by various authors during last decades, see [16], [17], [9]. The main goal of this paper is to define, for the first time in the literature, multi functional harmonic spaces and to establish some properties of these spaces. The proofs we found and provided below are new even in the case of one single function. We believe these new interesting objects can serve as a base for further generalizations and investigations.

Partially analytic analogues of theorems on multi functional spaces we obtained below were proved in recent papers of the first author [11], [12]. Here we intend to expand these results to more general new harmonic function spaces based on several functions in higher dimension. Let us note that these topics are new even in the classical case of harmonic function spaces on the unit disk.

We set $\mathbb{H} = \{(x, t) : x \in \mathbb{R}^n, t > 0\} \subset \mathbb{R}^{n+1}$. For $z = (x, t) \in \mathbb{H}$ we set $\bar{z} = (x, -t)$. We denote the points in \mathbb{H} usually by $z = (x, t)$ or $w = (y, s)$. The Lebesgue measure is denoted by $dm(z) = dz = dxdt$ or $dm(w) = dw = dyds$, the Lebesgue measure of a measurable set $E \subset \mathbb{H}$ is often denoted by $|E|$. We also use measures $dm_\lambda(z) = t^\lambda dxdt$, $\lambda \in \mathbb{R}$.

The space of all harmonic functions in a domain Ω is denoted by $h(\Omega)$. Weighted harmonic Bergman spaces on \mathbb{H} are defined, for $0 < p < \infty$ and $\lambda > -1$, by

$$A(p, \lambda) = A(p, \lambda)(\mathbb{H}) = \left\{ f \in h(\mathbb{H}) : \|f\|_{A(p, \lambda)} = \left(\int_{\mathbb{H}} |f(z)|^p dm_\lambda(z) \right)^{1/p} < \infty \right\}.$$

These spaces are complete metric spaces, for $1 \leq p < \infty$ they are Banach spaces.

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For $f \in h(\mathbb{H})$ and $t > 0$ we set $M_p(f, t) = \|f(\cdot, t)\|_{L^p(\mathbb{R}^n)}$, $0 < p < \infty$ with the usual convention in the case $p = \infty$. We use harmonic mixed norm spaces

$$(1) \quad B_\alpha^{p,q} = \left\{ f \in h(\mathbb{H}) : \|f\|_{B_\alpha^{p,q}}^q = \int_0^\infty M_p^q(f, t) t^{\alpha q - 1} dt < \infty \right\},$$

where $0 \leq p < \infty$, $0 < q < \infty$ and harmonic Triebel-Lizorkin spaces

$$(2) \quad F_\alpha^{p,q} = \left\{ f \in h(\mathbb{H}) : \|f\|_{F_\alpha^{p,q}}^p = \int_{\mathbb{R}^n} \left(\int_0^\infty |f(x, t)|^q t^{\alpha q - 1} dt \right)^{p/q} dx < \infty \right\},$$

where again $0 < p \leq \infty$ and $0 < q \leq \infty$, the case $q = \infty$ is covered by the usual convention. These spaces are complete metric spaces and for $\min(p, q) \geq 1$ they are Banach spaces. For details on $A(p, \lambda)$ spaces and more general $B_\alpha^{p,q}$ spaces see [7]; Triebel-Lizorkin spaces were studied, in the case of analytic functions, by many authors, see for example [15].

By X - L^p Carleson measure (or simply X Carleson measure when p is clear from the context) of a (quasi)-normed subspace X of $h(\Omega)$ we understand those positive Borel measures μ on Ω such that

$$(3) \quad \left(\int_\Omega |f|^p d\mu \right)^{1/p} \leq C \|f\|_X, \quad f \in X.$$

Typical cases are $\Omega = \mathbb{B} = \{x \in \mathbb{R}^n : |x| < 1\}$ and $\Omega = \mathbb{H}$. In the case Ω is the unit disc and X is the analytic Hardy space H^p we have a classical notion of Carleson measures on the unit disc. The cases of harmonic spaces can be found, for example, in [4] and much earlier in [14].

One of the goals of this paper is to find complete descriptions of Carleson measures for certain new harmonic function spaces in higher dimension and in the multi function case, in the latter case some restrictions on the parameters involved usually appear. We note that recently several new papers appeared where embedding theorems for analytic spaces in the unit disk were obtained and where the classical expression $\int_B |f|^p d\mu$ was modified, generalized or changed to certain expressions of $G(f, \mu, p)$, see for example [8], [18]; see also an earlier paper [5]. In these papers descriptions of measures for which $G(f, \mu, p) \leq C \|f\|_X$ were presented. Our sharp embedding theorems 2, 3, 4 and 8 we present below are of this character, and the spaces we deal with are defined using the above mentioned idea. On the other hand, our theorems 1, 5 and 6 give results modeled after (3), but in multi functional case.

We use common convention regarding constants: letter C denotes a constant which can change its value from one occurrence to the next one. Given two positive quantities A and B , we write $A \asymp B$ if there are two constants $c, C > 0$ such that $cA \leq B \leq CA$.

Many of the results of this paper rely on the following three key auxiliary results. The first one is a concrete Whitney decomposition into cubes of the upper half space, the second one is essentially subharmonic behavior of $|f|^p$ for harmonic f and for $0 < p < \infty$ and the third one describes geometric properties of Whitney cubes.

Lemma 1 ([16]). *There exists a collection $\{\Delta_k\}_{k=1}^\infty$ of closed cubes in \mathbb{H} with sides parallel to coordinate axes such that*

$$1^\circ. \cup_{k=1}^\infty \Delta_k = \mathbb{H} \text{ and } \text{diam} \Delta_k \asymp \text{dist}(\Delta_k, \partial \mathbb{H}).$$

2^o. *The interiors of the cubes Δ_k are pairwise disjoint.*

3°. If Δ_k^* is a cube with the same center as Δ_k , but enlarged $5/4$ times, then the collection $\{\Delta_k^*\}_{k=1}^\infty$ forms a finitely overlapping covering of R_+^{n+1} , i.e. there is a constant $C = C_n$ such that $\sum_k \chi_{\Delta_k^*} \leq C$.

Lemma 2 ([6]). Let Δ_k and Δ_k^* be the cubes from the previous lemma and let (ξ_k, η_k) be the center of Δ_k . Then, for $0 < p < \infty$ and $\alpha > 0$, we have

$$(4) \quad \eta_k^{\alpha p - 1} \max_{z \in \Delta_k} |f(z)|^p \leq \frac{C}{|\Delta_k^*|} \int_{\Delta_k^*} t^{\alpha p - 1} |f(x, t)|^p dx dt, \quad f \in h(\mathbb{H}), \quad k \geq 1.$$

Lemma 3 ([16]). Let Δ_k and Δ_k^* are as in the previous lemma, let $\zeta_k = (\xi_k, \eta_k)$ be the center of the cube Δ_k . Then we have:

$$(5) \quad m_\lambda(\Delta_k) \asymp \eta_k^{n+1+\lambda} \asymp m_\lambda(\Delta_k^*), \quad \lambda \in \mathbb{R},$$

$$(6) \quad |\bar{w} - z| \asymp |\bar{\zeta}_k - z|, \quad w \in \Delta_k^*, \quad z \in \mathbb{H},$$

$$(7) \quad t \asymp \eta_k, \quad (x, t) \in \Delta_k^*.$$

Since the cubes from the above Lemmata appear quite often in this paper, we fix the following notation: Δ_k and Δ_k^* are the cubes from Lemma 1, centered at $\zeta_k = (\xi_k, \eta_k)$, $k \geq 1$.

We also need the following integral estimate.

Lemma 4 ([10]). If $\alpha > -1$ and $n + \alpha < 2\gamma - 1$, then

$$(8) \quad \int_{\mathbb{H}} \frac{t^\alpha dz}{|z - \bar{w}|^{2\gamma}} \leq C s^{\alpha + n + 1 - 2\gamma}, \quad w = (y, s) \in \mathbb{H}.$$

The following definition introduces certain multi functional spaces that appeared in the setting of analytic functions in the unit ball in \mathbb{C}^n in [13].

Definition 1. Let $s > -1$ and $\vec{p} = (p_1, \dots, p_m)$ where $0 < p_i < \infty$. We denote by $A(\vec{p}, s, m)$ the set of all m -tuples (f_1, \dots, f_m) of functions harmonic in \mathbb{H} such that

$$(9) \quad (f_1, \dots, f_m)_{A(\vec{p}, s, m)} = \int_{\mathbb{H}} \prod_{i=1}^m |f_i(z)|^{p_i} t^s dm(z) < \infty.$$

If $p_i = p$, $i = 1, \dots, m$, we write simply $A(p, s, m)$.

For the first part of the following proposition see [2], for the second one see [1]. Note that the first part gives a simple estimate of the $A(\vec{p}, \lambda, m)$ norm, while the second one gives an estimate of trace in $A(p, \lambda)$ norm.

Proposition 1. 1°. Let $0 < p_i < \infty$, $-1 < s_i < \infty$, $f_i \in A(p_i, s_i)(\mathbb{H})$ for $i = 1, \dots, m$ and set $\lambda = (m - 1)(n + 1) + \sum_{i=1}^m s_i$. Then

$$(10) \quad (f_1, \dots, f_m)_{A(\vec{p}, \lambda, m)} \leq C \prod_{i=1}^m \|f_i\|_{A(p_i, s_i)}^{p_i}.$$

2°. Let $0 < p < \infty$ and $s_1, \dots, s_m > -1$. Set $\lambda = (m - 1)(n + 1) + \sum_{j=1}^m s_j$. Then there is a constant $C > 0$ such that for all $f \in h(\mathbb{H}^m)$ we have

$$(11) \quad \int_{\mathbb{H}} |f(z, \dots, z)|^p dm_\lambda(z) \leq C \int_{\mathbb{H}} \cdots \int_{\mathbb{H}} |f(z_1, \dots, z_m)|^p dm_{s_1}(z_1) \cdots dm_{s_m}(z_m).$$

2. EMBEDDING THEOREMS FOR MULTI FUNCTIONAL SPACES OF HARMONIC FUNCTIONS

In this section we give several sharp embedding theorems for multi functional spaces of harmonic functions, necessary and sufficient conditions turn out to be Carleson-type conditions.

The following theorem is an analogue for harmonic functions spaces of Theorem 3 from [11].

Theorem 1. *Let μ a positive Boreal measure on \mathbb{H} . Assume $0 < p_i, q_i < \infty$, $i = 1, \dots, m$, satisfy $\sum_{i=1}^m \frac{1}{q_i} = 1$ and let $\alpha > -1$. Then the following two conditions on the measure μ are equivalent.*

1°. *If f_i , $i = 1, \dots, m$ are functions harmonic in \mathbb{H} , then we have*

$$(12) \quad \int_{\mathbb{H}} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \leq C \prod_{i=1}^m \left[\sum_{k=1}^{\infty} \left(\int_{\Delta_k^*} |f_i(z)|^{p_i} t^\alpha dz \right)^{q_i} \right]^{1/q_i}.$$

2°. *The measure μ satisfies a Carleson type condition:*

$$(13) \quad \mu(\Delta_k) \leq C |\Delta_k|^{m(1+\frac{\alpha}{n+1})}, \quad k \geq 1.$$

Proof. Let us assume μ satisfies condition (13). Then we have, using Lemma 1 and Lemma 2:

$$\begin{aligned} \int_{\mathbb{H}} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) &= \sum_{k=1}^{\infty} \int_{\Delta_k} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \leq \sum_{k=1}^{\infty} \mu(\Delta_k) \prod_{i=1}^m \sup_{\Delta_k} |f_i|^{p_i} \\ &\leq C \sum_{k=1}^{\infty} \mu(\Delta_k) \eta_k^{-m(n+1+\alpha)} \prod_{i=1}^m \int_{\Delta_k^*} |f_i(w)|^{p_i} s^\alpha dw \\ &\leq C \sum_{k=1}^{\infty} \prod_{i=1}^m \int_{\Delta_k^*} |f_i(w)|^{p_i} s^\alpha dw. \end{aligned}$$

Now one arrives at estimate (12) by applying Holder's inequality for sums with exponents q_1, \dots, q_m .

Now we assume (12) holds. We fix $k \in \mathbb{N}$ and choose $f_i(z) = f(z) = |z - \bar{\zeta}_k|^{1-n}$. Clearly

$$\prod_{i=1}^m |f_i(z)|^{p_i} = |z - \bar{\zeta}_k|^{-(n-1) \sum_{i=1}^m p_i}.$$

Therefore (12), combined with Lemma 3, gives

$$\begin{aligned} \frac{\mu(\Delta_k)}{\eta_k^{(n-1) \sum_{i=1}^m p_i}} &\leq C \int_{\Delta_k} |z - \bar{\zeta}_k|^{-(n-1) \sum_{i=1}^m p_i} d\mu(z) \leq C \int_{\mathbb{H}} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \\ &\leq C \prod_{i=1}^m \int_{\Delta_k^*} \frac{t^\alpha dz}{|z - \bar{\zeta}_k|^{p_i(n-1)}} \leq C \prod_{i=1}^m |\Delta_k| \eta_k^{\alpha - p_i(n-1)}, \end{aligned}$$

and (13) easily follows. \square

The spaces defined below were considered, in the case of analytic functions on the unit ball in \mathbb{C}^n , in [12].

Definition 2. Let $0 < p, q < \infty$ and let μ be a positive Borel measure on \mathbb{H} . The space $A(p, q, m, d\mu)$ is the space of all (f_1, \dots, f_m) where $f_i \in h(\mathbb{H})$ for $i = 1, \dots, m$ such that

$$(14) \quad \|(f_1, \dots, f_m)\|_{A(p,q,m,d\mu)}^q = \sum_{k=1}^{\infty} \left(\int_{\Delta_k} \prod_{i=1}^m |f_i(z)|^p d\mu(z) \right)^{q/p} < \infty.$$

If $d\mu = dm_s$, then we write $A(p, q, m, s)$ for the corresponding space, if $m = 1$ we write $A(p, q, d\mu)$ or $A(p, q, s)$ if $d\mu = dm_s$.

Lemma 5. Let $0 < s < \infty$ and $\beta > -1$. Then we have

$$(15) \quad \int_{\mathbb{H}} |f(z)|^{st^\beta} dxdt \asymp \sum_{k=1}^{\infty} \eta_k^{n+1+\beta} \sup_{\Delta_k} |f|^s, \quad f \in h(\mathbb{H}).$$

We omit an easy proof based on Lemma 1 and Lemma 3.

Theorem 3.1 from [12] served as a model for the following theorem.

Theorem 2. Let $0 < p, q < \infty$, $0 < s \leq q$, $\beta_i > -1$ for $i = 1, \dots, m$ and let μ be a positive Borel measure on \mathbb{H} . Then the following conditions are equivalent.

1°. If $f_i \in A(s, \beta_i)$, $i = 1, \dots, m$, then

$$(16) \quad \|(f_1, \dots, f_m)\|_{A(p,q,m,d\mu)} \leq C \prod_{i=1}^m \|f_i\|_{A(s,\beta_i)}.$$

2°. The measure μ satisfies a Carleson type condition:

$$\mu(\Delta_k) \leq C \eta_k^{\sum_{i=1}^m \frac{p(n+1+\beta_i)}{s}}, \quad k \geq 1.$$

Proof. Assume 2° holds and choose $f_i \in A(s, \beta_i)$, $i = 1, \dots, m$. Since $0 < s/q \leq 1$ we have, using Lemma 5

$$\begin{aligned} \|(f_1, \dots, f_m)\|_{A(p,q,m,d\mu)}^s &= \left(\sum_{k=1}^{\infty} \left(\int_{\Delta_k} \prod_{i=1}^m |f_i(z)|^p d\mu(z) \right)^{q/p} \right)^{s/q} \\ &\leq \left(\sum_{k=1}^{\infty} \mu(\Delta_k)^{q/p} \max_{\Delta_k} \prod_{i=1}^m |f_i|^q \right)^{s/q} \\ &\leq C \sum_{k=1}^{\infty} \prod_{i=1}^m \eta_k^{n+1+\beta_i} \max_{\Delta_k} |f_i|^s \\ &\leq C \prod_{i=1}^m \left(\sum_{k=1}^{\infty} \eta_k^{n+1+\beta_i} \max_{\Delta_k} |f_i|^s \right) \leq C \prod_{i=1}^m \|f_i\|_{A(s,\beta_i)}^s, \end{aligned}$$

and we proved implication 2° \Rightarrow 1°.

Conversely, assume 1° holds and choose $l \in \mathbb{N}_0$ such that $s(n+l-1) > n+\beta+1$. We use, as test functions, functions $f_{\zeta_k, l}$, $k \in \mathbb{N}$ where

$$(17) \quad f_{w,l}(z) = \frac{\partial^l}{\partial t^l} \frac{1}{|z - \bar{w}|^{n-1}}, \quad z \in \mathbb{H},$$

see [1] for norm and pointwise estimates related to these functions. In particular we have

$$(18) \quad \|f_{\zeta_k, l}\|_{A(s,\beta_i)} \leq C \eta_k^{-\frac{(n-1+l)+\beta_i+1}{s}}, \quad k \geq 1, \quad 1 \leq i \leq m$$

and

$$(19) \quad \begin{aligned} \|(f_{\zeta_k, l}, \dots, f_{\zeta_k, l})\|_{A(p, q, m, d\mu)} &\geq \left(\int_{\Delta_k} |f_{\zeta_k, l}(z)|^{mp} d\mu(z) \right)^{1/p} \\ &\geq C \mu(\Delta_k)^{1/p} \eta_k^{-m(n+l-1)}. \end{aligned}$$

The last two estimates combined with (21) give 2°. □

The corresponding result for mixed norm spaces is the following theorem.

Theorem 3. *Let $0 < p, q < \infty$, $0 < s \leq q$, $\beta_i > -1$ for $i = 1, \dots, m$, $0 < t_i \leq s$ for $i = 1, \dots, m$ and let μ be a positive Borel measure on \mathbb{H} . Then the following conditions are equivalent:*

1°. *If $f_i \in B_{(\beta_i+1)/s}^{s, t_i}$, $i = 1, \dots, m$, then*

$$(20) \quad \|(f_1, \dots, f_m)\|_{A(p, q, m, d\mu)} \leq C \prod_{i=1}^m \|f_i\|_{B_{\frac{\beta_i+1}{s}}^{s, t_i}}.$$

2°. *If $f_i \in F_{(\beta_i+1)/s}^{s, t_i}$, $i = 1, \dots, m$, then*

$$(21) \quad \|(f_1, \dots, f_m)\|_{A(p, q, m, d\mu)} \leq C \prod_{i=1}^m \|f_i\|_{F_{\frac{\beta_i+1}{s}}^{s, t_i}}.$$

3°. *The measure μ satisfies a Carleson type condition:*

$$\mu(\Delta_k) \leq C \eta_k^{\sum_{i=1}^m \frac{p(n+1+\beta_i)}{s}}, \quad k \geq 1.$$

Proof. Since $A(s, \beta) = B_{(\beta+1)/s}^{s, s}$, the previous theorem, combined with embeddings $B_{(\beta+1)/s}^{s, t} \hookrightarrow B_{(\beta+1)/s}^{s, s}$ and $F_{(\beta+1)/s}^{s, t} \hookrightarrow B_{(\beta+1)/s}^{s, s}$, valid for $0 < t \leq s$, gives implications $3^\circ \Rightarrow 1^\circ$ and $3^\circ \Rightarrow 2^\circ$.

Now we assume 2° holds. Let us choose $l \in \mathbb{N}_0$ such that $s(n+l-1) > n + \beta + 1$. Set, for $w = (y, \sigma) \in \mathbb{H}$, $f_i(z) = f(z) = f_{w, l}(z)$, $i = 1, \dots, m$, see (17). We have, see [1]:

$$(22) \quad \|f_{w, l}\|_{F_{\frac{\beta+1}{s}}^{s, t}} = C \sigma^{\frac{n}{s} - (n-1+l - \frac{\beta+1}{s})}.$$

Let us fix a cube Δ_k . Using pointwise estimates from below for functions $f_{w, l}$, see [1], we obtain:

$$(23) \quad \begin{aligned} \mu(\Delta_k)^{1/p} \eta_k^{-m(n-1+l)} &\leq C \left(\int_{\Delta_k} |f_{w, l}(z)|^{mp} d\mu(z) \right)^{1/p} \\ &\leq C \|(f_1, \dots, f_m)\|_{A(p, q, m, d\mu)}, \quad k \geq 1. \end{aligned}$$

Now combining (23), (22) with $w = \zeta_k$ and (21) gives $\mu(\Delta_k) \leq C \eta_k^{\sum_{i=1}^m \frac{p(n+1+\beta_i)}{s}}$ and 3° follows.

The implication $1^\circ \Rightarrow 3^\circ$ can be proved using the same test functions $f_{w, l}$ as above, relevant estimates of $B_\alpha^{p, q}$ norm of these functions can be found in [1] and we leave details to the reader. □

For $w = (y, s) \in \mathbb{H}$ we set Q_w to be the cube, with sides parallel to the coordinate axis, centered at w with side length equal to s .

Our next multi functional embedding theorem is motivated by Theorem 3.6 from [12], which deals with the case of a single analytic function on the unit ball in \mathbb{C}^n .

Theorem 4. *Let $0 < p, q < \infty$ and $0 < \sigma_i \leq q$, $-1 < \alpha_i < \infty$ for $i = 1, \dots, m$. Let μ be a positive Borel measure on \mathbb{H} . Then the following two conditions are equivalent:*

1°. *For any m -tuple (f_1, \dots, f_m) of harmonic functions on \mathbb{H} we have*

$$(24) \quad \|(f_1, \dots, f_m)\|_{A(p,q,m,d\mu)}^q \leq C \prod_{i=1}^m \int_{\mathbb{H}} \left(\int_{Q_w} |f_i(z)|^{\sigma_i} t^{\alpha_i} dz \right)^{q/\sigma_i} dw.$$

2°. *The measure μ satisfies the following Carleson-type condition:*

$$(25) \quad \mu(\Delta_k) \leq C \eta_k^{m(n+1)\frac{p}{q} + \sum_{i=1}^m \frac{p(n+1+\alpha_i)}{\sigma_i}}, \quad k \geq 1.$$

Proof. Let us prove $2^\circ \Rightarrow 1^\circ$ assuming $m = 1$. Let $f \in h(\mathbb{H})$, then we have, using Lemma 1, Lemma 2 and Lemma 3:

$$\begin{aligned} \|f\|_{A(p,q,d\mu)}^q &= \sum_{k=1}^{\infty} \left(\int_{\Delta_k} |f(z)|^p d\mu(z) \right)^{q/p} \leq \sum_{k=1}^{\infty} \left(\max_{\Delta_k} |f|^p \mu(\Delta_k) \right)^{q/p} \\ &= \sum_{k=1}^{\infty} \max_{\Delta_k} |f|^q \eta_k^{q(\frac{n+1+\alpha}{\sigma} + \frac{n+1}{q})} \\ &\leq C \sum_{k=1}^{\infty} \left(\int_{\Delta_k^*} |f(z)|^\sigma t^{-n-1} dz \right)^{q/\sigma} \eta_k^{q(\frac{n+1+\alpha}{\sigma} + \frac{n+1}{q})} \\ &\leq C \sum_{k=1}^{\infty} \left(\int_{\Delta_k^*} |f|^\sigma dz \right)^{q/\sigma} \eta_k^{n+1+q\alpha/\sigma}. \end{aligned}$$

We continue this estimation, using Lemma 2 and Lemma 3, and obtain

$$\begin{aligned} \|f\|_{A(p,q,d\mu)}^q &\leq C \sum_{k=1}^{\infty} \left(\int_{\Delta_k^*} \left(\int_{Q_z} |f(w)|^\sigma s^\alpha dw \right) t^{-n-1-\alpha} dz \right)^{q/\sigma} \eta_k^{n+1+q\alpha/\sigma} \\ &\leq C \sum_{k=1}^{\infty} \left(\int_{\Delta_k^*} \left(\int_{Q_z} |f(w)|^\sigma s^\alpha dw \right) t^{-n-1} dz \right)^{q/\sigma} \eta_k^{n+1} \\ (26) \quad &\leq C \sum_{k=1}^{\infty} \int_{\Delta_k^*} \left(\int_{Q_z} |f(w)|^\sigma s^\alpha dw \right)^{q/\sigma} t^{-n-1} dz \eta_k^{n+1} \\ &\leq C \sum_{k=1}^{\infty} \int_{\Delta_k^*} \left(\int_{Q_z} |f(w)|^\sigma s^\alpha dw \right)^{q/\sigma} dz \end{aligned}$$

$$\leq C \int_{\mathbb{H}} \left(\int_{Q_z} |f(w)|^\sigma s^\alpha dw \right)^{q/\sigma} dz,$$

arriving at (24) for $m = 1$. Note that at (26) we used Holder's inequality and at the last step we used finite overlapping property of the cubes Δ_k^* .

Now we assume 1° , and again we restrict ourselves to the case $m = 1$. Let us choose $l \in \mathbb{N}_0$ such that $q(n - 1 + l) > q(n + 1 + \alpha)/\sigma + n + 1$. Let us fix a cube Δ_k centered at $\zeta_k = (\xi_k, \eta_k)$. We use a construction from [1], where interested reader can find more details. Namely, there are constants $c > 0$ and $\delta > 0$ such that for all $w \in \mathbb{H}$ we have

$$(27) \quad |T_w| = c|Q_w|, \quad T_w = \{z \in Q_w : |f_{w,l}(z)| > c|z - \bar{w}|^{-(n-1+l)}\}.$$

Next, using a compactness argument, one shows that for $w = (y, s) \in \mathbb{H}$ there are points $w_j = (y_j, s_j) \in \mathbb{H}$, $1 \leq j \leq N$, such that $s_j \asymp s$ and $Q_w = \cup_{j=1}^N T_{w_j} \cap Q_w$. Here N depends only on n and l . We apply this argument to $w = \zeta_k$, it easily follows that there is $\theta_k = (u_k, v_k) \in \mathbb{H}$ such that $\mu(T_{\theta_k} \cap \Delta_k) \geq N^{-1}\mu(\Delta_k)$ and $v_k \asymp \eta_k$. Now we choose $f = f_{\theta_k, l}$ as a test function. We have

$$(28) \quad \begin{aligned} \|f\|_{A(p,q,d\mu)}^q &\geq \left(\int_{\Delta_k} |f(z)|^p d\mu(z) \right)^{q/p} \geq \left(\int_{\Delta_k \cap T_{\theta_k}} |f(z)|^p d\mu(z) \right)^{q/p} \\ &\geq C\mu(\Delta_k \cap T_{\theta_k})^{q/p} v_k^{-q(n-1+l)} \\ &\geq C\mu(\Delta_k) \eta_k^{-q(n-1+l)}. \end{aligned}$$

On the other hand, using Lemma 3 and Lemma 4, we have

$$\begin{aligned} \int_{\mathbb{H}} \left(\int_{Q_w} |f(z)|^\sigma dm_\alpha(z) \right)^{q/\sigma} dw &\leq C \int_{\mathbb{H}} \left(\frac{s^{n+1+\alpha}}{|w - \bar{\theta}_k|^{\sigma(n-1+l)}} \right)^{q/\sigma} dw \\ &\leq C v_k^{\frac{q}{\sigma}(n+1+\alpha) - q(n-1+l) + n+1} \\ &\leq C \eta_k^{\frac{q}{\sigma}(n+1+\alpha) - q(n-1+l) + n+1}. \end{aligned}$$

This estimate, combined with (28) and (24) (with $m = 1$), gives desired estimate (25) for $m = 1$.

The general case $m > 1$ can be proved along the same lines, multiple sums as in the proof of Theorem 2 appear. Since no new ideas are involved we can leave details to the interested reader. \square

As a preparation for our next result we state and prove the following lemma.

Lemma 6. *Let $\alpha > -1$, $0 < \tau < \infty$. Then there is a constant C such that for any measurable function $u(z) \geq 0$ on \mathbb{H} we have*

$$(29) \quad \eta_k^{n+1} \left(\int_{\Delta_k^*} u(z) dm_\alpha(z) \right)^\tau \leq C \int_{\Delta_k^*} \left(\int_{Q_w} u(z) dm_\alpha(z) \right)^\tau dw, \quad k \geq 1.$$

Proof. Let us denote the left hand side by L_α and the right hand side, without constant C , by D_α . Since $L_\alpha \asymp \eta_k^\alpha L_0$ and $D_\alpha \asymp \eta_k^\alpha D_0$ it suffices to consider the special case $\alpha = 0$. Let q_w denote the cube centered at $w = (y, s) \in \mathbb{H}$, with side length equal $4s/5$. Then we

have $\Delta_k^* = \cup_{i=1}^N q_{w_i}$ for suitable $w_1, \dots, w_N \in \Delta_k^*$ where N depends only on n . Therefore (29) reduces to the following estimate:

$$(30) \quad \eta_k^{n+1} \left(\int_{q_w} u(z) dm(z) \right)^\tau \leq C \int_{\Delta_k^*} \left(\int_{Q_{\tilde{w}}} u(z) dm(z) \right)^\tau d\tilde{w}, \quad w \in \Delta_k^*.$$

Now we fix $w \in \Delta_k^*$. Clearly, $\int_{q_w} u(z) dm(z) \leq \int_{Q_{\tilde{w}}} u(z) dm(z)$ whenever $q_w \subset Q_{\tilde{w}}$, and the estimate (30) follows from the simple observation, based on the sizes of q_w and Q_w , that

$$|E_k| \geq c\eta_k^{n+1}, \quad E_k = \{\tilde{w} \in \Delta_k^* : q_w \subset Q_{\tilde{w}}\}. \quad \square$$

In the following theorem we allow for different exponents p_i .

Theorem 5. *Let $0 < p_i, \sigma_i < \infty$, $-1 < \alpha_i < \infty$ for $i = 1, \dots, m$. Let μ be a positive Borel measure on \mathbb{H} . Then the following two conditions are equivalent:*

1°. *For any m -tuple (f_1, \dots, f_m) of harmonic functions on \mathbb{H} we have*

$$(31) \quad \int_{\mathbb{H}} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \leq C \prod_{i=1}^m \int_{\mathbb{H}} \left(\int_{Q_w} |f_i(z)|^{\sigma_i} dm_{\alpha_i}(z) \right)^{p_i/\sigma_i} dw.$$

2°. *The measure μ satisfies the following Carleson-type condition:*

$$(32) \quad \mu(\Delta_k) \leq C\eta_k^{m(n+1) + \sum_{i=1}^m \frac{p_i(n+1+\alpha_i)}{\sigma_i}}, \quad k \geq 1.$$

Proof. Let us assume (31) holds. We choose $l \in \mathbb{N}$ such that

$$p_i(n-1+l) > n+1 + \frac{p_i}{\sigma_i}(n+1+\alpha_i) \quad i = 1, \dots, m.$$

We fix a cube Δ_k and take as test functions $f_i = f = f_{\theta_k, l}$, $i = 1, \dots, m$, where $\theta_k = (u_k, v_k)$ is chosen as in Theorem 4. Then we have, using Lemma 4:

$$\begin{aligned} \eta_k^{-(n-1+l) \sum_{i=1}^m p_i} \mu(\Delta_k) &\leq C \int_{T_{\theta_k} \cap \Delta_k} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \leq C \int_{\mathbb{H}} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \\ &\leq C \prod_{i=1}^m \int_{\mathbb{H}} \left(\int_{Q_w} |f_i(z)|^{\sigma_i} dm_{\alpha_i}(z) \right)^{p_i/\sigma_i} dw \\ &\leq C \prod_{i=1}^m \int_{\mathbb{H}} \left(\frac{s^{n+1+\alpha_i}}{|\bar{\theta}_k - w|^{\sigma_i(n-1+l)}} \right)^{p_i/\sigma_i} dw \\ &\leq C v_k^{m(n+1) + \sum_{i=1}^m \frac{p_i(n+1+\alpha_i)}{\sigma_i} - (n-1+l) \sum_{i=1}^m p_i} \\ &\leq C\eta_k^{m(n+1) + \sum_{i=1}^m \frac{p_i(n+1+\alpha_i)}{\sigma_i} - (n-1+l) \sum_{i=1}^m p_i} \end{aligned}$$

which gives (32). Conversely, we assume now (32). Using finite overlapping property of Δ_k^* we obtain

$$\prod_{i=1}^m \int_{\mathbb{H}} \left(\int_{Q_w} |f_i(z)|^{\sigma_i} dm_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dw \geq C \sum_{k=1}^{\infty} \prod_{i=1}^m \int_{\Delta_k^*} \left(\int_{Q_w} |f_i(z)|^{\sigma_i} dm_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dw$$

and we have also an obvious estimate

$$\begin{aligned} \int_{\Delta_k} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) &\leq \mu(\Delta_k) \prod_{i=1}^m \max_{z \in \Delta_k} |f_i(z)|^{p_i} \\ &\leq C \eta_k^{m(n+1) + \sum_{i=1}^m \frac{p_i(n+1+\alpha_i)}{\sigma_i}} \prod_{i=1}^m \max_{z \in \Delta_k} |f_i(z)|^{p_i}. \end{aligned}$$

Therefore it suffices to prove, for each $k \geq 1$ and $1 \leq i \leq m$:

$$\eta_k^{n+1 + \frac{p_i(n+1+\alpha_i)}{\sigma_i}} \max_{z \in \Delta_k} |f_i(z)|^{p_i} \leq C \int_{\Delta_k^*} \left(\int_{Q_w} |f_i(z)|^{\sigma_i} dm_{\alpha_i}(z) \right)^{p_i/\sigma_i} dw.$$

However, for $k \geq 1$ and $1 \leq i \leq m$, using Lemma 2 and Lemma 6 we obtain:

$$\begin{aligned} \eta_k^{n+1 + \frac{p_i(n+1+\alpha_i)}{\sigma_i}} \max_{z \in \Delta_k} |f_i(z)|^{p_i} &\leq C \eta_k^{n+1} \left(\int_{\Delta_k^*} |f_i(z)|^{\sigma_i} dm_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} \\ &\leq C \int_{\Delta_k^*} \left(\int_{Q_w} |f(z)|^{\sigma_i} dm_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dw \end{aligned}$$

and the proof is completed. \square

The following theorem is motivated by Theorem A from [13]. It gives a very general and flexible method of obtaining multi functional results starting from a uniform estimate for a single function.

Theorem 6. *Let μ be a positive Borel measure on an open proper subset G of \mathbb{R}^n and let $(X_i, \|\cdot\|_{X_i})$ be a (quasi)-normed space of functions harmonic in G , $1 \leq i \leq k$. Let $d(x) = \text{dist}(x, \partial G)$, $x \in G$ and $0 < q_i < \infty$, $-1 < \beta_i < \infty$ for $i = 1, \dots, k$. Assume*

$$(33) \quad \sup_{x \in G} |f_i(x)|^{q_i} d(x)^{\beta_i} \leq C \|f_i\|_{X_i}^{q_i}, \quad f_i \in X_i, \quad i = 2, \dots, k$$

and

$$(34) \quad \int_G |f_1(x)|^{q_1} d\mu(x) \leq C \|f_1\|_{X_1}^{q_1}, \quad f_1 \in X_1.$$

Then

$$(35) \quad \int_G \prod_{i=1}^k |f_i(x)|^{q_i} d(x)^{\beta_2 + \dots + \beta_k} d\mu(x) \leq C \prod_{i=1}^k \|f_i\|_{X_i}^{q_i}, \quad f_i \in X_i.$$

We refer the reader for the proof by induction to [13], where arguments readily extend to the present situation. We also note that the above theorem can be extended to weights more general than $d(x)$, however it is precisely the case of $d(x)$ where uniform estimates of type (33) are available. For example, if $G = \mathbb{H}$, then $d(z) = t$ and we have an estimate

$$|f(z)|^p d(z)^{n+1+\alpha} \leq C \|f\|_{A(p,\alpha)}^p, \quad f \in A(p,\alpha), \quad z \in \mathbb{H}.$$

Therefore, taking $d\mu(z) = t^{s_1} dx dt$ we obtain the following corollary.

Corollary 1. *If $0 < p_i < \infty$, $-1 < s_i < \infty$, $f_i \in A(p_i, s_i)(\mathbb{H})$ for $i = 1, \dots, m$ and $\alpha > -1$, then*

$$(36) \quad \int_{\mathbb{H}} \prod_{i=1}^m |f_i(z)|^{p_i} t^{(n+1)(m-1) + \sum_{i=1}^m s_i} dz \leq C \prod_{i=1}^m \|f_i\|_{A(p_i, s_i)}^{p_i}$$

Using embedding $B_\alpha^{p,q} \hookrightarrow B_\alpha^{p,p}$, $0 < q \leq p < \infty$, see [7], we deduce another corollary.

Corollary 2. *If $0 < q_i \leq p_i < \infty$, $-1 < s_i < \infty$ and $f_i \in B_{s_i}^{p_i, q_i}$ for $1 \leq i \leq m$, then*

$$(37) \quad \int_{\mathbb{H}} \prod_{i=1}^m |f_i(z)|^{p_i} t^{n(m-1) + \sum_{i=1}^m s_i p_i - 1} dz \leq C \prod_{i=1}^m \|f_i\|_{B_{s_i}^{p_i, q_i}}^{p_i}.$$

Let $\mathcal{D}^\beta f$ denote the Riesz potential of $f \in h(\mathbb{H})$ with respect to t variable. Using estimate

$$(38) \quad \int_{\mathbb{H}} |f(z)|^p t^{\alpha p - 1} dz \leq C \int_{\mathbb{H}} |\mathcal{D}^\beta f(z)|^p t^{\alpha p - 1 + \beta p} dz = N_{\alpha, \beta, p}(f), \quad f \in h(\mathbb{H})$$

which is valid for $0 < p < \infty$, $\alpha, \beta > 0$, see Theorem 7 from [3], we obtain the following corollary.

Corollary 3. *Let $0 < s_i, p_i < \infty$ and $f_i \in A(p_i, s_i)$ for $i = 1, \dots, m$. Then we have*

$$(39) \quad \int_{\mathbb{H}} \prod_{i=1}^m |f_i(z)|^{p_i} t^{(n+1)(m-1) + \sum_{i=1}^m s_i} dz \leq C \prod_{i=1}^m N_{s_i, \beta, p_i}(f_i).$$

Similar assertions can be obtained easily also in the unit ball \mathbb{B} .

Let $H^p(\mathbb{B})$, $1 < p < \infty$, stand for the harmonic Hardy space on the open unit ball.

Proposition 2. *Let $f_i \in H^{p_i}(\mathbb{B})$ where $1 < p_i < \infty$ for $i = 1, \dots, m$. Then we have*

$$(40) \quad \int_{\mathbb{S}} \prod_{j=1}^m |f_j(r\xi)|^{p_j} d\sigma(\xi) (1-r)^{(m-1)n} \leq C \prod_{j=1}^m \|f_j\|_{H^{p_j}}^{p_j}.$$

Similarly, if $f_i \in H^{p_i}(\mathbb{H})$ where $1 < p_i < \infty$ for $i = 1, \dots, m$ then we have

$$(41) \quad \int_{\mathbb{R}^n} \prod_{j=1}^m |f_j(x, t)|^{p_j} dx t^{(m-1)n} \leq C \prod_{j=1}^m \|f_j\|_{H^{p_j}}^{p_j}.$$

The proof is by induction on m , using estimate

$$(42) \quad |f(x)|^p (1-|x|)^n \leq C \|f\|_{H^p}^p, \quad f \in H^p(\mathbb{B}),$$

and an analogous one for the case of the upper half space.

Lemma 7. *Let $\beta > 0$ and $1 < p < \infty$. Then the operator S defined by*

$$(43) \quad Sg(z) = \int_{\mathbb{H}} \left(\frac{t^n}{|z - \bar{w}|^{2n}} \right)^{1+\beta} g(w) dm_{\beta n-1}(w), \quad z \in \mathbb{H},$$

is bounded on $L^p(\mathbb{H}, dm_{\beta n-1})$.

For a similar result for analytic functions in the unit ball see [19].

Proof. We use Schur's test, see [19] for statement and applications to related problems. Let $1/p + 1/q = 1$ and set $h(z) = t^\lambda$, where λ will be subject to additional conditions. Using Lemma 4 we obtain

$$(44) \quad \int_{\mathbb{H}} \left(\frac{t^n}{|z - \bar{w}|^{2n}} \right)^{1+\beta} s^{\lambda q} dm_{\beta n-1}(w) \leq Ct^{\lambda q},$$

provided $-\beta n < \lambda q < \beta n + n$. Similarly we have

$$(45) \quad \int_{\mathbb{H}} \left(\frac{t^n}{|z - \bar{w}|^{2n}} \right)^{1+\beta} t^{\lambda p} dm_{\beta n-1}(w) \leq Ct^{\lambda p},$$

provided $-n - 2\beta n < \lambda p < 0$. Clearly, a parameter λ satisfying all the inequalities exists, it suffices to choose $\max(-\beta n/q, (-n - 2\beta n)/p) < \lambda < 0$. Schur's test implies that S is bounded on $L^p(\mathbb{H}, dm_{\beta n-1})$ for all $1 < p < \infty$. \square

The following result was proved for analytic functions in [12] (Theorem 3.3).

Theorem 7. *Let $0 < p < q < \infty$, $\alpha > 0$ and let μ be a positive Borel measure on \mathbb{H} . Then the following two conditions are equivalent:*

1^o.

$$(46) \quad \int_{\mathbb{H}} \left(\int_{\mathbb{H}} \left(\frac{t^n}{|z - \bar{w}|^{2n}} \right)^{1+\alpha q} d\mu(z) \right)^{\frac{q}{q-p}} dm_{\alpha q n-1}(w) < \infty.$$

2^o.

$$(47) \quad \sum_{k=1}^{\infty} \eta_k^{-(\alpha q n + n) \frac{p}{q-p}} \mu(\Delta_k)^{\frac{q}{q-p}} < \infty.$$

Proof. Let us prove implication 1^o \Rightarrow 2^o. We have

$$\begin{aligned} \infty &> \int_{\mathbb{H}} \left(\int_{\mathbb{H}} \left(\frac{t^n}{|z - \bar{w}|^{2n}} \right)^{1+\alpha q} d\mu(z) \right)^{\frac{q}{q-p}} dm_{\alpha q n-1}(w) \\ &\geq C \sum_{k=1}^{\infty} \int_{\Delta_k} \left(\sum_{j=1}^{\infty} \int_{\Delta_j} \left(\frac{t^n}{|z - \bar{w}|^{2n}} \right)^{1+\alpha q} d\mu(z) \right)^{\frac{q}{q-p}} dm_{\alpha q n-1}(w) \\ &\geq C \sum_{k=1}^{\infty} \int_{\Delta_k} \left(\int_{\Delta_k} \left(\frac{t^n}{|z - \bar{w}|^{2n}} \right)^{1+\alpha q} d\mu(z) \right)^{\frac{q}{q-p}} dm_{\alpha q n-1}(w) \\ &\geq C \sum_{k=1}^{\infty} \eta_k^{-(\alpha q n + n) \frac{p}{q-p}} \mu(\Delta_k)^{\frac{q}{q-p}}, \end{aligned}$$

where in the first estimate we used Lemma 1 and in last one we used Lemma 3.

Next we prove 2^o \Rightarrow 1^o. Assume $f : \mathbb{H} \rightarrow \mathbb{R}$ is subharmonic. Then $|f|^{q/p}$ is also subharmonic and we have, using Lemma 2,

$$\int_{\mathbb{H}} |f(z)| d\mu(z) = \sum_{k=1}^{\infty} \int_{\Delta_k} |f(z)| d\mu(z) \leq \sum_{k=1}^{\infty} \sup_{z \in \Delta_k} |f(z)| \mu(\Delta_k)$$

$$\leq \sum_{k=1}^{\infty} \left(\int_{\Delta_k^*} |f(z)|^{q/p} dm_{\alpha q n-1}(z) \right)^{p/q} \eta_k^{\frac{-p(n+\alpha q n)}{q}} \mu(\Delta_k).$$

Note that q/p and $q/(q-p)$ are a pair of conjugate exponents, hence an application of Holder's inequality gives

$$(48) \quad \int_{\mathbb{H}} |f(z)| d\mu(z) \leq C \|f\|_{L^{q/p}(\mathbb{H}, dm_{\alpha q n-1})} \left[\sum_{k=1}^{\infty} \eta_k^{\frac{-(n+\alpha q n)p}{q-p}} \mu(\Delta_k)^{\frac{q}{q-p}} \right]^{\frac{q-p}{q}}.$$

Note that here we used finite overlapping property of the family Δ_k^* . Let us choose $g \in L^{q/p}(\mathbb{H}, dm_{\alpha q n-1})$, $g \geq 0$, and set

$$f(z) = (Sg)(z) = \int_{\mathbb{H}} \left(\frac{t^n}{|z - \bar{w}|^{2n}} \right)^{1+\alpha q} g(w) s^{\alpha q n-1} dw.$$

Since, for any fixed $w \in \mathbb{H}$, the function $t^n/|z - \bar{w}|^{2n}$ is subharmonic it easily follows that f is subharmonic. Moreover, the operator S is bounded on $L^{q/p}(\mathbb{H}, dm_{\alpha q n-1})$ by Lemma 7, i.e. $\|Sg\|_{L^{q/p}(\mathbb{H}, dm_{\alpha q n-1})} \leq C \|g\|_{L^{q/p}(\mathbb{H}, dm_{\alpha q n-1})}$.

Now we apply (48) to $f = S(g)$ and obtain

$$\begin{aligned} & \int_{\mathbb{H}} \int_{\mathbb{H}} \left(\frac{t^n}{|z - \bar{w}|^{2n}} \right)^{1+\alpha q} g(w) s^{\alpha q n-1} dw d\mu(z) \\ & \leq C \|g\|_{L^{q/p}(\mathbb{H}, dm_{\alpha q n-1})} \left[\sum_{k=1}^{\infty} \eta_k^{\frac{-(n+\alpha q n)p}{q-p}} \mu(\Delta_k)^{\frac{q}{q-p}} \right]^{\frac{q-p}{q}}. \end{aligned}$$

Since the last estimate is true for any non negative $g \in L^{q/p}(\mathbb{H}, dm_{\alpha q n-1})$, 1^o follows by duality. \square

An application of the above theorem to $d\mu(z) = |f(z)|^{q-p} dm_{\alpha}(z)$ gives a characterization of $A(p, q, \alpha)$ spaces for $0 < p < q < \infty$, $\alpha > -1$, see [12].

Set $r_k = 2^{-k}$, $I_k = [r_{k+1}, r_k]$ and $H_k = \mathbb{R}^n \times I_k$ for $k \in \mathbb{Z}$.

Definition 3. Let μ be a positive Borel measure on \mathbb{H} and let $0 < p, q < \infty$. We define the space $K_q^p(\mu)$ as the space of $f \in h(\mathbb{H})$ such that

$$(49) \quad \|f\|_{K_q^p(\mu)}^q = \sum_{k=-\infty}^{\infty} \left(\int_{H_k} |f(z)|^p d\mu(z) \right)^{q/p} < \infty.$$

The spaces $K_q^p(\mu)$ are harmonic Herz type spaces, related classes of spaces appeared in literature, for example in [10]. Clearly these spaces include weighted Bergman spaces: $K_p^p(m_s) = A(p, s)$, $0 < p < \infty$, $s > -1$.

Note that the $K_q^p(\mu)$ (quasi) norm of f is the l^q norm of the sequence $\|f\|_{L^p(H_k, d\mu)}$, $-\infty < k < \infty$. Therefore we have:

$$(50) \quad \|f\|_{K_{q_2}^p(d\mu)} \leq \|f\|_{K_{q_1}^p(d\mu)}, \quad 0 < q_1 \leq q_2 < \infty.$$

Theorem 8. *Let $\alpha > -1$, $0 < p \leq q < \infty$ and let μ be a positive Borel measure on \mathbb{H} . Then the following two conditions are equivalent.*

1°. $A(p, \alpha) \hookrightarrow K_q^p(\mu)$.

2°. The measure μ satisfies the following Carleson type condition:

$$(51) \quad \frac{\mu(\Delta_j)}{|\Delta_j|^{1+\frac{\alpha}{n+1}}} \leq C, \quad j \geq 1.$$

Proof. Sufficiency, for $p = q$, is proved in [1], and the general case follows from the embedding (50).

Now we prove necessity, i.e. we assume 1° holds. We choose $l \in \mathbb{N}_0$ such that $p(n-1+l) > \alpha + n + 1$. Let us fix a cube Δ_j and choose $k \in \mathbb{Z}$ such that $m(H_k \cap \Delta_j) \geq |\Delta_j|/2$. We use a test function $f = f_{\zeta_j, l}$:

$$(52) \quad \begin{aligned} \|f_{\zeta_j, l}\|_{K_q^p(\mu)}^q &\geq \left(\int_{H_k \cap \Delta_j} |f_{\zeta_j, l}(z)|^p d\mu(z) \right)^{q/p} \geq C [\mu(\Delta_j) \eta_j^{-p(n-1+l)}]^{q/p} \\ &= C \mu(\Delta_j)^{q/p} \eta_j^{-q(n-1+l)}. \end{aligned}$$

Next we estimate $A(p, \alpha)$ -norm of $f_{\zeta_j, l}$ using Lemma 4:

$$(53) \quad \|f\|_{A(p, \alpha)}^p = \int_{\mathbb{H}} |f_{w, l}(z)|^p t^\alpha dz \leq C \int_{\mathbb{H}} \frac{t^\alpha dz}{|z - \bar{\zeta}_j|^{p(n-1+l)}} \leq C \eta_j^{\alpha+n+1-p(n-1+l)}.$$

Combining our assumption $\|f\|_{K_q^p(\mu)} \leq C \|f\|_{A(p, \alpha)}$ with (52) and (53) gives 2°. \square

Remark 1. Sharp embedding theorems of the spaces $K_q^p(\mu)$ into $B_\alpha^{p, q}$ and $K_q^p(\mu)$ into $F_\alpha^{p, q}$ with some restriction on the indexes are also true and they follow from Theorem 8 and here our arguments repeat arguments of proof of Theorem 3 which was obtained from linear case (Theorem 2) and appropriate embedding results for $B_\alpha^{p, q}$ and $F_\alpha^{p, q}$ spaces.

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