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ON HYPERRINGS ASSOCIATED WITH $\mathcal{L}$-FUZZY RELATIONS

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Summary: In this paper we construct a class of hyperrings and a class of H_γ-rings associated with $\mathcal{L}$-fuzzy relations on semigroup. We investigate morphisms of obtained classes and we also establish connection between the constructed hyperring $(H, +_R, \circ_R)$ and the hyperring of multiendomorphisms of hypergroup $(H, +_R)$. Finally, the factorization of a hyperring $(H, +_R, \circ_R)$ is considered.

1 INTRODUCTION

The association between hyperstructures and binary relations had been intensively studied[5,6,10,11,2,4,20]. Chvalina[5,6] and Hort[2] use ordered structures for the construction of semigroups and hypergroups. Rosenberg[4] associated with any binary relation ρ of full domain, a hypergroupoid H_ρ and found conditions on ρ, such that H_ρ is a hypergroup. Corsini and Leoreanu[16] study hypergroups and binary relations.

Connections between algebraic hyperstructures and fuzzy sets have been considered for the first time by Corsini[12]. They were studied afterwards by Corsini[13,14] Corsini and Leoreanu [17,18], Cristea[3], Davvaz[1] and many others.

The interest in fuzzy logic has been rapidly growing recently. Several new algebras playing the role of structures of true values have been introduced and axiomatized. The most general structure considered here is that of a residuated lattice. In this paper we will use complete residuated lattice $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ as the structure of truth values. An $\mathcal{L}$-fuzzy relation on a set A is any function from $A \times A$ into L. $\mathcal{L}$-fuzzy relations and fuzzy homomorphisms of algebras have been studied by Ignjatović, Ćirić and Bogdanović[7,8] and many others.

In section 3 of this paper we associate a hyperring (H_γ-ring) to every semigroup $(H, \cdot)$ equipped with reflexive and transitive $\mathcal{L}$-fuzzy relation R which is compatible with operation $\cdot$. Then we show that under certain condition there exists (inclusion) monomorphism between the constructed hyperring $(H, +_R, \circ_R)$ and the hyperring of multiendomorphisms of hypergroup $(H, +_R)$. We investigate morphisms of hyperrings associated with $\mathcal{L}$-fuzzy relations, too. Also, we show that if the hyperring $(H, +_F, \circ_F)$ is associated with fuzzy congruence F on the semigroup $(H_2, \cdot)$ then there exists a class of functions $\Psi: H_1 \rightarrow H_2$ which are not necessary homomorphisms of semigroups $(H_1, \cdot)$ and $(H_2, \cdot)$ such that Ψ is a strong homomorphism of hyperrings $(H_1, +_E, \circ_E)$ and $(H_2, +_F, \circ_F)$ for the responsive fuzzy – congruence E and it is valid for the responsive classes of associated H_γ-rings, too. Finally, we show that if $(H, +_R, \circ_R)$ is a hyperring associated with $\mathcal{L}$-fuzzy relation R on a semigroup $(H, \cdot)$, then there exists a class of congruences Q on that hyperring, such that the factor-hyperring $(H/Q, \textcircled{+_R}, \textcircled{\circ_R})$ is isomorphic to the hyperring associated with responsive $\mathcal{L}$-fuzzy relation of the factor –semigroup.

2 PRELIMINARIES

Let's first recall some basic definitions[9,21]. A hypergroupoid $(H,\circ)$ is a non-empty set H endowed with a function $\circ: H \times H \to P^*(H)$ called hyperoperation, where $P^*(H)$ denotes the set of all non-empty subsets of H. A hypergroupoid $(H,\circ)$ is called a semihypergroup if for all $(x,y,z) \in H^3: x \circ (y \circ z) = (x \circ y) \circ z$. A semihypergroup $(H,\circ)$, is called a hypergroup if the reproduction axiom is valid, i.e.: $a \circ H = H \circ a = H$, for all $a \in H$. In the above definitions, if $A, B \in P^*(H)$, then $A \circ B$ is given by:

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

$x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$.

An H_γ-semigroup is a hypergroupoid $(H,\circ)$ such that for all $(x,y,z) \in H^3: (x \circ y) \circ z \cap x \circ (y \circ z) \neq \emptyset$. An H_γ-semigroup is called an H_γ-group if the reproduction axiom is valid.

Definition 2.1: A multivalued system $(H, +, \cdot)$ is a hyperring if:

1) $(H, +)$ is a hypergroup.
2) $(H, \cdot)$ is a semihypergroup.
3) $(\cdot)$ is distributive with respect to $(+)$, i.e. for all $(x,y,z) \in H^3$ we have:

(a) $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$

and

(b) $(y + z) \cdot x \subseteq y \cdot x + z \cdot x$

If in the conditions 3a) and 3b) the equality is valid, then the hyperring is called strongly distributive.

Definition 2.2: A multivalued system $(H, +, \circ)$ is called an H_γ-ring if:

1) $(H, +)$ is an H_γ-group.
2) $(H, \circ)$ is an H_γ- semigroup.
3) for all $(x,y,z) \in H^3$ we have:

$$x \circ (y + z) \cap (x \circ y + x \circ z) \neq \emptyset$$

and

$$(y + z) \circ x \cap (y \circ x + z \circ x) \neq \emptyset.$$

Definition 2.3: Let $(A, +, \cdot)$ and $(B, +', \cdot')$ be two hyperrings (H_γ-rings). A map $f: A \to B$ is called an inclusion homomorphism if the following conditions are satisfied:

$$f(x + y) \subseteq f(x) +' f(y) \text{ and } f(x \cdot y) \subseteq f(x) \cdot' f(y).$$

for all $x, y \in A$.

A map f is called a strong homomorphism if in the the previous conditions the equality is valid.

Theorem and definition 2.4. Let $(H, +)$ be a commutative hypergroup, and $F(H)$ the set of multiendomorphisms of H, i.e.

$$F(H)=\left\{h:H\to P^{*}(H)\,\middle|\,(\forall x,y\in H)\bigcup_{u\in x+y}h(u)\subseteq h(x)+h(y)\right\}$$

Let's define for all $(f,g)\in F(H)\times F(H)$:

$$f\oplus g=\{h\in F(H)|(\forall x\in H)h(x)\subseteq f(x)\oplus g(x)\}$$

$$f\odot g=\left\{h\in F(H)\,\middle|\,(\forall x\in H)h(x)\subseteq f\big(g(x)\big)=\bigcup_{u\in g(x)}f(u)\right\}$$

Then the structure $(F(H),\oplus_F,\odot_F)$ is a hyperring. This hyperring is called a hyperring of multiendomorphisms of a hypergroup $(H,+)$.

To each binary relation ρ on a set H, a partial hypergroupoid $H_\rho=(H;\circ)$ is associated[4], as follows:

$$\forall(x,z)\in H^2,\, x\circ x=\{y\in H|(x,y)\in\rho\},\ x\circ z=x\circ x\cup z\circ z.$$

By a partial hypergroupoid we mean a non-empty set H endowed with a function from $H\times H$ to the set of subsets of H.

Let

$$D(\rho)=\{x\in H|\exists y\in H:(x,y)\in\rho\}$$
$$R(\rho)=\{x\in H|\exists z\in H:(z,x)\in\rho\}$$

An element $z\in H$ is called an outer element of ρ if there exists $y\in H$ such that $(y,z)\notin\rho^2$.

Theorem 2.5. ([4], ∮2) H_ρ is a hypergroup if and only if:

1. $H=D(\rho)$;
2. $H=R(\rho)$;
3. $\rho\subseteq\rho^2$;
4. if z is an outer element of ρ, then $\forall x\in H,\ (x,z)\in\rho^2\Longrightarrow(x,z)\in\rho$.

We recall the following elementary background[8].

A residuated lattice is an algebra $\mathcal{L}=(L,\wedge,\vee,\otimes,\to,0,1)$ such that:

(1) $(L,\wedge,\vee,0,1)$ is a lattice with the least element 0 and the greatest element 1,

(2) $(L,\otimes,1)$ is a commutative monoid with the unit 1,

(3) $\otimes$ and $\to$ form an adjoint pair, i.e., they satisfy the adjunction property: for all $x,y,z\in L$,

$$x\otimes y\le z\Leftrightarrow x\le y\to z.$$

By $\le$ is denoted the order associated with lattice $(L,\wedge,\vee)$.

If, in addition $(L,\wedge,\vee,0,1)$ is a complete lattice, then $\mathcal{L}$ is called a complete residuated lattice.

It can be easily verified that with respect to $\le,\otimes$ is isotonic in both arguments, and $\to$ is isotonic in the second and antitonic in the first argument. Also, for any $\{x_i\}_{i\in I}\subseteq L$, the following hold:

$$\left(\bigvee_{i\in I} x_i\right)\otimes x = \bigvee_{i\in I}(x_i\otimes x).$$

Some examples of these structures are:

1) The Lukasewicz structure defined on the unit interval $L=[0,1]$ with $x\wedge y = min(x,y)$ and $x\vee y = max(x,y)$, while $x\otimes y = max(x+y-1,0)$, $x\to y = min(1-x+y,1)$.
2) The product structure defined on the $L=[0,1]$ with $x\wedge y = min(x,y)$, $x\vee y = max(x,y)$, $x\otimes y = x\cdot y$, and $x\to y = 1$ if $x\le y$ and $x\to y = {}^{y}/_{x}$ otherwise.
3) The Gödel structure defined on the $L=[0,1]$ with $x\wedge y = min(x,y)$, $x\vee y = max(x,y)$, $x\otimes y = min(x,y)$, $x\to y = 1$ if $x\le y$ and $x\to y = y$, otherwise.
4) Two-element Boolean algebra of classical logic with the support $\{0,1\}$ and adjoint pair which consist of the classical conjunction and implication operations.

In the further text $\mathcal{L}$ will be a complete residuated lattice. An $\mathcal{L}$-fuzzy subset of a set A is any function from A into L. The set of all $\mathcal{L}$-fuzzy subsets of A is denoted by L^A.

An $\mathcal{L}$-fuzzy relation on A is any function from $A\times A$ into L, that is to say, any $\mathcal{L}$-fuzzy subset of $A\times A$.

An $\mathcal{L}$-fuzzy relation R on A is said to be:

(R) reflexive (or fuzzy reflexive) if $R(x,x)=1$, for every $x\in A$;

(S) symmetric (or fuzzy symmetric) if $R(x,y)=R(y,x)$, for all $x,y\in A$;

(T) transitive (or fuzzy transitive) if for all $x,a,y\in A$ we have:

$$R(x,a)\otimes R(a,y)\le R(x,y).$$

A reflexive, symmetric and transitive $\mathcal{L}$-fuzzy relation on A is called an $\mathcal{L}$-fuzzy equivalence. An $\mathcal{L}$-fuzzy equivalence E on A is called an $\mathcal{L}$-fuzzy equality if for any $x,y\in A$, $E(x,y)=1$ implies $x=y$.

Let A be an algebra of type τ. An $\mathcal{L}$-fuzzy relation R on A is called compatible if for any $f\in\tau_n$, $n\ge 1$ and any elements $x_i,y_i\in A$, $1\le i\le n$, we have:

$$R(x_1,y_1)\otimes \ldots \otimes R(x_n,y_n)\le R\big(f(x_1,\ldots,x_n),f(y_1,\ldots,y_n)\big).$$

By τ_n is denoted the set of n-ary operations of an algebra A.

A compatible $\mathcal{L}$-fuzzy equivalence on A is called a fuzzy congruence.

Let A and B be algebras of type τ and let E be an $\mathcal{L}$-fuzzy equivalence on B.

(1) A function $\Psi\colon A\longrightarrow B$ is called an E -homomorphism if for any $f\in\tau_n$, $n\ge 1$ and any $x_i\in A$, $1\le i\le n$, we have:

$$E\Big(\Psi\big(f^A(x_1,\ldots,x_n)\big), f^B\big(\Psi(x_1),\ldots,\Psi(x_n)\big)\Big)=1$$

and for any $f\in\tau_o$ we have $E(\Psi(f^A),f^B)=1$.

(2) A function $\Psi\colon A\to B$ is called E- surjective if for any $p\in B$ there exists $x\in A$ such that $E(\Psi(x),p)=1$.

(3) If $\Psi\colon A\to B$ is both E- surjective and E–homomorphism, then Ψ is called an E–epimorphism of A onto B.

Clearly, if E is an $\mathcal{L}$-fuzzy equality, then Ψ is an E –homomorphism if and only if it is an ordinary homomorphisms. Also, if $\Psi: A \to B$ is surjective, then Ψ is E-surjective for any $\mathcal{L}$-fuzzy equivalence E.

3 HYPERRINGS AND H_γ-RINGS ASSOCIATED WITH $\mathcal{L}$ -FUZZY RELATIONS ON SEMIGROUP

Let $\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ be a complete residuated lattice.

Theorem 3.1: Let $(H, \cdot)$ be a semigroup and let R be reflexive and transitive $\mathcal{L}$-fuzzy relation on H, such that for all $x_1, y_1, x_2, y_2 \in H$ it holds:

$$R(x_1, y_1) \otimes R(x_2, y_2) \leq R(x_1 \cdot x_2, y_1 \cdot y_2). \quad (1)$$

We define hyperoperations $+_R$ and $\circ_R$ on H, as follows:

$$x +_R y = \{z | R(x, z) = 1 \text{ or } R(y, z) = 1\}$$

and

$$x \circ_R y = \{z | \ R(x \cdot y, z) = 1\}$$

for all $(x, y) \in H \times H$.

The structure $(H, +_R, \circ_R)$ is a strong distributive hyperring.

Proof: Let ρ be a binary relation on a set H defined by: $x \rho y \ iff \ R(x, y) = 1$, for all $(x, y) \in H \times H$. It is easy to see that ρ is reflexive and transitive relation such that for all $a, b, x \in H$, $(a, b) \in \rho$ implies $(a \cdot x, b \cdot x) \in \rho$ and $(x \cdot a, x \cdot b) \in \rho$. Obviously, $x +_R y = \{z | (x, z) \in \rho \text{ or } (y, z) \in \rho\}$ and $x \circ_R y = \{z | (x \cdot y, z) \in \rho\}$. By Theorem 2.5, $(H, +_R)$ is a hypergroup. By well know "Ends lemma"[6], $(H, \circ_R)$ is a semihypergroup.

Now, we prove the right distributivity of $\circ_R$ with respect to $+_R$. Let $x, y, z \in H$. Set:

$$C = (x +_R y) \circ_R z = \bigcup u \circ_R z, \text{ while } u \in x +_R y.$$

and

$$D = (x \circ_R z) +_R (y \circ_R z) = \bigcup a +_R b, \text{ while } a \in x \circ_R z \text{ and } b \in y \circ_R z.$$

If $w \in C$, then $R(u \cdot z, w) = 1$ for some $u \in x +_R y$ and we have two possibilities:

(i) If $R(x, u) = 1$ then $R(x \cdot z, u \cdot z) = 1$. As $R(u \cdot z, w) = 1$ from transivity of R, it follows $R(x \cdot z, w) = 1$. Thus, $w \in x \circ_R z$. For any $b \in y \circ_R z$ it holds $w \in w +_R b$ and so $w \in D$.

(ii) If $R(y, u) = 1$ we can similarly prove that $w \in D$. Thus, $C \subseteq D$.

Suppose now $w \in D$. Then there exist $a, b \in H$ such that $w \in a +_R b$ while $a \in x \circ_R z$ and $b \in y \circ_R z$. That means, $R(a, w) = 1$ or $R(b, w) = 1$ while $R(x \cdot z, a) = 1$ and $R(y \cdot z, b) = 1$.

If $R(a, w) = 1$ since $R(x \cdot z, a) = 1$ we obtain $R(x \cdot z, w) = 1$ i.e. , $w \in x \circ_R z$. As $x \in x +_R y$ and $w \in x \circ_R z$, we obtain $w \in C$.

If $R(b, w) = 1$ similarly we obtain $w \in C$. Thus, $D \subseteq C$.

The left distributivity of $\circ_R$ with respect to $+_R$ can be proved in a similar way. Thus $(H, +_R, \circ_R)$ is a hyperring.

Theorem 3.2. Let $(H,\cdot)$ be a semigroup and let R be an $\mathcal{L}$-fuzzy relation on H, such that R satisfies conditions of Theorem 3.1. If we define hyperoperations $+^R$ and $\circ_R$ on H, as follows:

$$x+^R y = \{z | R(x,z) \vee R(y,z) = 1\}$$

and

$$x \circ_R y = \{z | R(x\cdot y, z) = 1\}$$

then:

1) $(H,+^R)$ is an H_γ-group.
2) $(H,\circ_R)$ is a semihypergroup.
3) for all $(x,y,z) \in H^3$ it holds:
   a) $(x+^R y) \circ_R z \subseteq (x \circ_R z)+^R (y \circ_R z)$.
   b) $x \circ_R (y+^R z) \subseteq (x \circ_R y)+^R (x \circ_R z)$.

Proof: 1) As $\{x,y\} \subseteq x+^R y$ for all $x,y \in H$, it is easy to verify that $(H,+^R)$ is an H_γ-group.

2) It is proved in Theorem 3.1.

3) a) Let $(x,y,z) \in H^3$. Set:

$$C = (x+^R y) \circ_R z = \bigcup u \circ_R z \text{ while } R(x,u) \vee R(y,u) = 1$$

and

$$D = (x \circ_R z)+^R (y \circ_R z).$$

If $w \in C$, then there exists $u \in H$ such that $w \in u \circ_R z$, while $R(x,u) \vee R(y,u) = 1$. It follows $R(u\cdot z, w) = 1$ while $R(x,u) \vee R(y,u) = 1$. As R satisfies condition (1) in Theorem 3.1., we obtain:

$$1 = \big(R(x,u) \vee R(y,u)\big) \otimes R(z,z) = \big(R(x,u) \otimes R(z,z)\big) \vee \big(R(y,u) \otimes R(z,z)\big) \leq$$
$$\leq R(x\cdot z, u\cdot z) \vee R(y\cdot z, u\cdot z).$$

Thus,

$$1 = \big(R(x\cdot z, u\cdot z) \vee R(y\cdot z, u\cdot z)\big) \otimes R(u\cdot z, w) = \big(R(x\cdot z, u\cdot z) \otimes R(u\cdot z, w)\big) \vee$$
$$\vee \big(R(y\cdot z, u\cdot z) \otimes R(u\cdot z, w)\big) \leq R(x\cdot z, w) \vee R(y\cdot z, w).$$

Therefore, $w \in (x\cdot z)+^R (y\cdot z) \subseteq (x \cdot_R z)+^R (y \cdot_R z) = D$. So, $C \subseteq D$. The inclusion in 3b) can be proved in a similar way.

Corollary 3.3. The structure $(H,+^R,\circ_R)$ is an H_γ-ring. (with inclusive distributivity).

Remark 3.4. Let $\mathcal{B}$ be two-element Boolean algebra of classical logic. If R is a $\mathcal{B}$-fuzzy relation on semigroup H such that R satisfies conditions of Theorem 3.1, then obviously $(H,+_R,\circ_R)$ and $(H,+^R,\circ_R)$ are the same structures.

If R is an $\mathcal{L}$-fuzzy congruence on semigroup $(H,\cdot)$, then obiviously R satisfies conditions of Theorem 3.1. The next example shows that there exists a class of compatible $\mathcal{L}$-fuzzy reations on semigroup H, that generates the fuzzy congruences on H.

Example 3.5: Let R be a compatible $\mathcal{L}$-fuzzy relation on a semigroup $(H,\cdot)$, such that:

(1) For any $x \in H$ there exists $a \in H$ such that $R(x,a) = 1$

(2) For any $x, y, p, q \in H$ it holds:

$$R(x,p) \otimes R(y,p) \otimes R(x,q) \leq R(y,q).$$

If we define an $\mathcal{L}$-fuzzy relation E on H as follows:

$$E(x,y) = \bigvee_{p \in H} R(x,p) \otimes R(y,p),$$

then E is an $\mathcal{L}$-fuzzy congruence on a semigroup H.

Indeed, since for any $x \in H$ there exists $a \in H$ such that $R(x,a) = 1$, then $1 = R(x,a) \otimes R(x,a) \leq E(x,x)$ i.e. $E(x,x) = 1$. It is clear that $E(x,y) = E(y,x)$ for all $x, y \in H$. Let $x, y, z \in H$. Then:

$$E(x,y) \otimes E(y,z) = \left(\bigvee_{p \in H} R(x,p) \otimes R(y,p) \right) \otimes \left(\bigvee_{q \in H} R(y,q) \otimes R(z,q) \right) =$$

$$= \bigvee_{p \in H} \left(\bigvee_{q \in H} R(x,p) \otimes R(y,p) \otimes R(y,q) \otimes R(z,q) \right) =$$

$$= \bigvee_{p \in H} \left(\bigvee_{q \in H} R(y,p) \otimes R(x,p) \otimes R(y,q) \otimes R(z,q) \right) \leq$$

$$\overset{(2)}{\leq} \bigvee_{p \in H} \left(\bigvee_{q \in H} R(x,q) \otimes R(z,q) \right) = \bigvee_{q \in H} R(x,q) \otimes R(z,q) = E(x,z).$$

Thus, E is an $\mathcal{L}$-fuzzy equivalence on H. Similarly, one can verify that for any $x_1, y_1, x_2, y_2 \in H$ it holds:

$$E(x_1,y_1) \otimes E(x_2,y_2) = \bigvee_{p \in H} \Bigg(\bigvee_{q \in H} R(x_1,p) \otimes R(y_1,p) \otimes R(x_2,q) \otimes$$

$$R(y_2,q) \Bigg) \leq \bigvee_{p,q \in H} R(x_1 \cdot x_2, p \cdot q) \otimes R(y_1 \cdot y_2, p \cdot q) \leq E(x_1 \cdot x_2, y_1 \cdot y_2).$$

Throughout the following text the triple $(H, \cdot, R)$ will denote a semigroup $(H, \cdot)$ equipped with an $\mathcal{L}$-fuzzy relation R on H, such that R satisfies conditions of Theorem 3.1. By a hyperring associated with $(H, \cdot, R)$ we mean the hyperring $(H, +_R, \circ_R)$ constructed in Theorem 3.1. and by an H_γ-ring associated with $(H, \cdot, R)$ we mean the structure $(H, +^R, \circ_R)$ constructed in Theorem 3.2.

Theorem 3.6. Let $(H_1, \cdot)$ and $(H_2, \cdot)$ be semigroups and let F be an $\mathcal{L}$-fuzzy congruence on H_2.

(a) If $\varphi: H_1 \to H_2$ is an F-epimorphism, then an $\mathcal{L}$-fuzzy relation E on H_1 defined by:

$$E(x,y) = F(\varphi(x),\varphi(y)), \text{ for all } (x,y) \in H_1 \times H_1$$

is an $\mathcal{L}$-fuzzy congruence on H_1 and φ is an inclusion homomorphism of associated hyperrings $(H_1, +_E, \circ_E)$ and $(H_2, +_F, \circ_F)$. Also, φ is an inclusion homomorphism of associated H_γ-rings $(H_1, +^E, \circ_E)$ and $(H_2, +^F, \circ_F)$.

(b) If $\varphi: H_1 \to H_2$ is an F-homomorphism and φ is surjective then φ is a strong epimorphism of a hyperring $(H_1, +_E, \circ_E)$ onto $(H_2, +_F, \circ_F)$. Also, φ is a strong epimorphism of H_γ-ring $(H_1, +^E, \circ_E)$ onto $(H_2, +^F, \circ_F)$.

Proof: By Theorem 4.4. (a)[8], E is an $\mathcal{L}$-fuzzy congruence on H_1. Let $(x,y) \in H_1 \times H_1$. It is easy to verify that $\varphi(x +_E y) \subseteq \varphi(x) +_F \varphi(y)$ and $\varphi(x +^E y) \subseteq \varphi(x) +^F \varphi(y)$. Suppose now $w \in \varphi(x \circ_E y)$. Then, $w = \varphi(u)$ for some $u \in H_1$, while $E(x \cdot y, u) = 1$. Thus, $F(\varphi(x \cdot y), \varphi(u)) = 1$. As φ is an F-homomorphism, we have:

$$F(\varphi(x \cdot y), \varphi(x) \cdot \varphi(y)) = 1$$

Thus, $1 = F(\varphi(x) \cdot \varphi(y), \varphi(x \cdot y)) \otimes F(\varphi(x \cdot y), w) \leq F(\varphi(x) \cdot \varphi(y), w)$ i.e. $w \in \varphi(x) \circ_F \varphi(y)$.

Therefore, $\varphi(x \circ_E y) \subseteq \varphi(x) \circ_F \varphi(y)$.

(b) Since φ is is surjective, then φ is F –surjective. So, φ is an F-epimorphism and by (a), φ is an inclusion homomorphism of associated hyperrings.

Suppose now $x, y \in H_1$ and $w \in \varphi(x) +_F \varphi(y)$. Then, there exists $u \in H_1$ such that $w = \varphi(\mathrm{u})$ while $F(\varphi(x), \varphi(u)) = 1$ or $F(\varphi(y), \varphi(u)) = 1$. Thus, $w = \varphi(\mathrm{u})$ while $E(x,u) = 1$ or $E(y,u) = 1$ i.e. $u \in x +_E y$ and so $w \in \varphi(x +_E y)$. So, $\varphi(x) +_F \varphi(y) \subseteq \varphi(x +_E y)$. The reverse inclusion is valid by (a) and thus $\varphi(x +_E y) = \varphi(x) +_F \varphi(y)$. Similarly, we prove that $\varphi(x +^E y) = \varphi(x) +^F \varphi(y)$.
Also, due to (a) it holds $\varphi(x \circ_E y) \subseteq \varphi(x) \circ_F \varphi(y)$.

If $w \in \varphi(x) \circ_F \varphi(y)$, then $w = \varphi(\upsilon)$ for some $v \in H_1$ and $F(\varphi(x) \cdot \varphi(y), \varphi(v)) = 1$. Since φ is an F-homorphism, by transitivity of F we obtain $F(\varphi(x \cdot y), \varphi(v)) = 1$ i.e. $E(x \cdot y, v) = 1$. Thus, $w = \varphi(v) \in \varphi(x \circ_E y)$. Therefore, φ is a strong epimorphism of a hyperring $(H_1, +_E, \circ_E)$ onto $(H_2, +_F, \circ_F)$ and the same is valid for the responsive H_γ-rings.

Definition 3.7. Let the pair (H_i, R_i) denote the nonempty set H_i equipped with $\mathcal{L}$-fuzzy relation R_i for $i \in \{1,2\}$.

(1) The function $f: H_1 \to H_2$ is said to be isotone if $R_1(x,y) = 1 \Rightarrow R_2(f(x), f(y)) = 1$, for all $(x,y) \in H_1 \times H_1$.

(2) The function $f: H_1 \to H_2$ is said to be strongly isotone if: $R_2(f(x), z) = 1 \Leftrightarrow (\exists y \in H_1) R_1(x,y) = 1 \wedge f(y) = z$ for all $(x,z) \in H_1 \times H_2$.

Theorem 3.8. Let $(H_i, +_{R_i}, \circ_{R_i})$ be a hyperring associated with $(H_i, \cdot, R_i)$, for $i \in \{1,2\}$. If $f: (H_1, \cdot) \to (H_2, \cdot)$ is an isotone (strongly isotone) homomorphism of semigroups $(H_1, \cdot)$ and $(H_2, \cdot)$ then f is an inclusion (strong) homomorphism of a hyperring $(H_1, +_{R_1}, \circ_{R_1})$ into hyperring $(H_2, +_{R_2}, \circ_{R_2})$.

Proof: We can proceed similarly as in the proof of Theorem 3.6.

Theorem 3.9. Let $(H, +_R, \circ_R)$ be a hyperring associated with $(H, \cdot, R)$ and $(F(H), \oplus, \odot)$ be a hyperring of multiendmorphisms of hypergroup $(H, +_R)$. If we define a mapping $\Psi: (H, +_R, \circ_R) \longrightarrow (F(H), \oplus, \odot)$ by:

$$\Psi(a) = f_a, \text{ for all } a \in H,$$

where $f_a : H \to P^*(H)$ is defined by:

$$f_a(x) = a \circ_R x, \text{ for all } x \in H,$$

then:

(1) Ψ is an inclusion homomorphism.

(2) If $R(x,y) = R(y,x) = 1$ implies $x = y$, for all $(x,y) \in H \times H$ and if semigroup $(H,\cdot)$ has at least one reductive element, then φ is injective.

Proof: As $(H,+_R)$ is commutative hypergroup there exists the hyperring of its multiendomorphisms $(F(H),\oplus,\odot)$.

First, we verify that for all $a \in H$, it holds $f_a \in F(H)$. Let $x,y \in H$. Then:

$$\bigcup_{u \in x+_R y} f_a(u) = \bigcup_{u \in x+_R y} a \circ_R u = a \circ_R (x +_R y) = (a \circ_R x) +_R (a \circ_R y) = f_a(x) +_R f_a(y).$$

(1) Let $a,b \in H$. Set:

$$C = \Psi(a +_R b) = \{f_c | R(a,c) = 1 \text{ or } R(b,c) = 1\}$$

and

$$D = \Psi(a) \oplus \Psi(b) = f_a \oplus f_b = \{h | (\forall x) h(x) \subseteq f_a(x) +_R f_b(x)\}.$$

Let $c \in H$ such that $R(a,c) = 1$ and $x \in H$. Then:

$$f_c(x) = c \circ_R x = \{y | R(c \cdot x, y) = 1\}.$$

Since R is compatible, $R(a,c) = 1$ implies $R(a \cdot x, c \cdot x) = 1$. Thus, if $y \in f_c(x)$, by transitivity of R we obtain $R(a \cdot x, y) = 1$. So, $y \in f_a(x) \subseteq f_a(x) +_R f_b(x)$. We proved that $f_c(x) \subseteq f_a(x) +_R f_b(x)$, for all $x \in H$, i.e. $f_c \in D$.

If $c \in H$ such that $R(b,c) = 1$, similarly we obtain $f_c \in D$. Thus, $C \subseteq D$.

Now, assume that:

$$C = \varphi(a \circ_R b) = \{f_c | R(a \cdot b, c) = 1\}.$$

and

$$D = \varphi(a) \odot \varphi(b) = \left\{h \middle| (\forall x) h(x) \subseteq f_a\big(f_b(x)\big)\right\}.$$

Let $c \in H$ such that $R(a \cdot b, c) = 1$ and $x \in H$. Then $f_c(x) = \{y | R(c \cdot x, y) = 1\}$. Since, $R(a \cdot b, c) = 1$, then $R(a \cdot b \cdot x, c \cdot x) = 1$. If $y \in f_c(x)$, then by transitivity of R, it holds $R(a \cdot b \cdot x, y) = 1$, and so $y \in f_a(b \cdot x) \subseteq f_a\big(f_b(x)\big)$. Thus, $f_c(x) \subseteq f_a\big(f_b(x)\big)$ i.e. $f_c \in D$.

Therefore, $C \subseteq D$.

(2) We will prove that Ψ is injective if the conditions in 2) are fulfilled. If $a \neq b$, then for the reductive element x it holds:

$$a \cdot x \neq b \cdot x$$

Thus, $R(a \cdot x, b \cdot x) \neq 1$ or $R(b \cdot x, a \cdot x) \neq 1$. It follows $b \cdot x \notin f_a(x)$ or $a \cdot x \notin f_b(x)$. As $b \cdot x \in f_b(x)$ and $a \cdot x \in f_a(x)$, we obtain $f_b(x) \not\subseteq f_a(x)$ or $f_a(x) \not\subseteq f_b(x)$, i.e. $f_a(x) \neq f_b(x)$. Hence, $f_a \neq f_b$ i.e. $\Psi(a) \neq \Psi(b)$.

4 A FACTORIZATION OF A HYPERRING $(H, +_R, \circ_R)$

Let Q be an equivalence on a set A and $M \subseteq A$. Denote by:

$$Q(M) = \{x \in A | (a, x) \in Q \text{ for some } a \in M\}$$

i.e. $Q(M) = \bigcup_{a \in M} [a]_Q$,

where $[a]_Q$ is the Q-class containing the element a.

We will use the concept of congruence on a hypergroup defined by Chvalina[6], in responsive definition related to hyperrings.

Definition 4.1.

(1) Let $(H,*)$ be a hypergroupoid and Q be an equivalence on H. We call Q to be a congruence on $(H,*)$ if for all $a, b, c, d \in H$ we have:

$$(a, b) \in Q \text{ and } (c, d) \in Q \text{ imply } Q(a * c) = Q(b * d).$$

(2) We call Q to be a congruence on a hyperring $(H, +, \circ)$ if Q is congruence on both hypergroup $(H, +)$ and hypersemigroup $(H, \circ)$.

Theorem 4.2. Let Q be a congruence on a hyperring $(H, +, \circ)$. Denote by h_Q the natural mapping of H onto H/Q defined by $h_Q(a) = [a]_Q$. If we define hyperoperations $\oplus$ and $\odot$ on H/Q as follows:

$$[a]_Q \oplus [b]_Q = h_Q(a + b)$$

$$[a]_Q \odot [b]_Q = h_Q(a \circ b)$$

for all $([a]_Q, [b]_Q) \in (H/Q) \times (H/Q)$, then the structure $(H/Q, \oplus, \odot)$ is a hyperring. This hyperring will be called a factor-hyperring of a hyperring $(H, +, \circ)$ induced by Q.

Proof: Direct verification.

Definition 4.3. Let R be an $\mathcal{L}$-fuzzy relation on a set A and Q be an equivalence on A. Denote by R/Q an $\mathcal{L}$-fuzzy relation on A/Q defined by:

$$(R/Q)([a]_Q, [b]_Q) = \bigvee_{\substack{x \in [a]_Q, \\ y \in [b]_Q}} R(x, y)$$

Denote by $\mathcal{B}$ two-element Boolean algebra of classical logic with the support $\{0,1\}$ and adjoint pair which consist of the classical conjunction and implication operations. Let R be an $\mathcal{L}$-fuzzy relation an a semigroup $(H,\cdot)$ and $(H, +_R, \circ_R)$ be a hyperring associated with $(H, \cdot, R)$. It is easy to see that there exists $\mathcal{B}$-fuzzy relation R_1 on a set H such that the triple $(H, \cdot, R_1)$ satisfies conditions of Theorem 3.1. and that for all $x, y \in H$ it holds $x +_R y = x +_{R_1} y$ and $x \circ_R y = x \circ_{R_1} y$. Thus, in what following by a hyperring $(H, +_R, \circ_R)$ we mean a hyperring associated with the triple $(H, \cdot, R)$ where R is a $\mathcal{B}$-fuzzy relation on a semigroup $(H, \cdot)$.

Notice that if R is a $\mathcal{B}$-fuzzy relation on a set H and Q is an equivalence on H, then by definition 4.3 it holds:

$$(R/Q)([a]_Q,[b]_Q)=1 \text{ iff there exists } x\in[a]_Q,\ y\in[b]_Q \text{ such that } R(x,y)=1.$$

Theorem 4.4. Let Q be a congruence on a semigroup $(H,\cdot)$ and $(H/Q,\cdot_Q)$ be a factor-semigroup induced by Q. Let $(H,+_R,\circ_R)$ be a hyperring associated with $(H,\cdot,R)$. If for all $x,y,z\in H$ it holds:

$$(x,y)\in Q \text{ and } R(y,z)=1 \Longrightarrow (\exists z'\in H)R(x,z')=1 \text{ and } (z,z')\in Q \qquad (2)$$

then Q is a congruence on a hyperring $(H,+_R,\circ_R)$ and the factor-hyperring $(H/Q \bigcirc\!\!\!\!+_R, \bigcirc\!\!\!\!\circ_R)$ is isomorphic to the hyperring associated with $(H/Q,\cdot_Q,R/Q)$.

Proof: By Corollary 2 [19, p.75], it follows that Q is a congruence on a hypergroup $(H,+_R)$. We verify that Q is a congruence on a hypersemigroup $(H,\circ_R)$. Suppose $(a,b)\in Q$ and $(c,d)\in Q$. If $z\in Q(a\circ_R c)$ then there exists z' such that $(z,z')\in Q$ and $R(a\cdot c,z')=1$. As $(a\cdot c,b\cdot d)\in Q$, by condition (2) it results that there exists z'' such that $R(b\cdot d,z'')=1$ while $(z',z'')\in Q$.

Thus, $R(b\cdot d,z'')=1$ and $(z,z'')\in Q$ i.e. $z\in Q(b\circ_R d)$. So, $Q(a\circ_R c)\subseteq Q(b\circ_R d)$. Similarly, we prove the reverse inclusion. Therefore, Q is a congruence on a hyperring $(H,+_R,\circ_R)$.

Now, let's prove that the triple $(H/Q,\cdot_Q,R/Q)$ satisfies the conditions of Theorem 3.1 By Corollary 2 [19, p. 75], it follows that R/Q is reflexive and transitive $\mathcal{B}$-fuzzy relation. We will show that if $(R/Q)([a]_Q,[b]_Q)=1$ and $(R/Q)([c]_Q,[d]_Q)=1$ then $(R/Q)([a\cdot c]_Q,[b\cdot d]_Q)=1$. Indeed, if $(R/Q)([a]_Q,[b]_Q)=1$ and $(R/Q)([c]_Q,[d]_Q)=1$ then there exist $x,y,z,w\in H$ such that:

$$(a,x)\in Q,\ (b,y)\in Q,\ R(x,y)=1 \text{ and } (c,z)\in Q,\ (d,w)\in Q,\ R(z,w)=1.$$

Due to (2), there exist $y',w'\in H$ such that:

$$R(a,y')=1,\ (y,y')\in Q \text{ while } (b,y)\in Q$$

and

$$R(c,w')=1,\ (w,w')\in Q \text{ while } (d,w)\in Q.$$

It follows, $R(a\cdot c,y'\cdot w')=1$ while $(b,y')\in Q$ and $(d,w')\in Q$.Thus, $R(a\cdot c,y'\cdot w')=1$ while $(b\cdot d,y'\cdot w')\in Q$. So, $R(a\cdot c,y'\cdot w')=1, y'\cdot w'\in[b\cdot d]_Q$, $a\cdot c\in[a\cdot c]_Q$ i.e.$(R/Q)([a\cdot c]_Q,[b\cdot d]_Q)=1$.

Therefore, there exists a hyperring $(H/Q,+_{R/Q},\circ_{R/Q})$ associated with $(H/Q,\cdot_Q,R/Q)$.

Let $a,b\in H$. Set:

$$C=[a]_Q \bigcirc\!\!\!\!+_R [b]_Q=h_Q(a+_R b)=\{[c]_Q \mid R(a,c)=1 \text{ or } R(b,c)=1\}$$

and

$$D=[a]_Q+_{R/Q}[b]_Q=\{[c]_Q \mid (R/Q)([a]_Q,[c]_Q)=1 \text{ or } (R/Q)([b]_Q,[c]_Q)=1\}.$$

Clearly, $C\subseteq D$. Suppose now $[c]_Q\in D$. If $(R/Q)([a]_Q,[c]_Q)=1$ then there exist $x,y\in H$ such that:
$(a,x)\in Q$, $(c,y)\in Q$ and $R(x,y)=1$. By the condition (2) if follows that there exists $y'\in H$ such that:

$$R(a, y') = 1,\ (y, y') \in Q \ \text{ while } (c, y) \in Q.$$

Thus, $R(a, y') = 1$ and $[y']_Q = [c]_Q$. So, $[c]_Q = [y']_Q \in C$. If $(R/Q)\big([b]_Q, [c]_Q\big) = 1$, similarly we obtain $[c]_Q \in C$.

Therefore, $D \subseteq C$.

Now, set:

$$C = [a]_Q \bigcirc\!\!\!\!\!\circ_R\, [b]_Q = h_Q(a \circ_R b) = \big\{[c]_Q | R(a \cdot b, c) = 1\big\}$$

and

$$D = [a]_Q \circ_{R/Q} [b]_Q = \big\{[c]_Q \big| (R/Q)\big([a \cdot b]_Q, [c]_Q\big) = 1\big\}$$

Suppose $[c]_Q \in D$. Then there exist $x, y \in H$ such that:

$$(a \cdot b, x) \in Q,\ (c, y) \in Q \text{ and } R(x, y) = 1.$$

Due to (2) it follows that there exists $y' \in H$ such that:

$$R(a \cdot b, y') = 1,\ (y, y') \in Q \text{ while } (c, y) \in Q$$

i.e. $R(a \cdot b, y') = 1$ while $[y']_Q = [c]_Q$. Thus, $[c]_Q \in C$. So, $D \subseteq C$. Obviously, it holds $C \subseteq D$. This completes the proof.

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