SOME PROPERTIES OF $k$-GENERALIZED FIBONACCI NUMBERS

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DOI: 10.20948/mathmontis-2021-50-7

Summary. In the present paper, we propose some properties of the new family $k$-generalized Fibonacci numbers which related to generalized Fibonacci numbers. Moreover, we give some identities involving binomial coefficients for $k$-generalized Fibonacci numbers.

1 INTRODUCTION

Fibonacci numbers have a great importance in mathematics. It is one of the most popular sequences that have a lot of applications in many branch of mathematics as in diverse sciences [1, 2, 6, 7, 10-13, 16-20]. The Fibonacci numbers $F_n$ are given by the recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. Koshy [9] written one of the most popular books of Fibonacci and Lucas numbers, and gave numerous recurrence relations, generalizations and applications of Fibonacci and Lucas numbers. For $a, b \in \mathbb{R}$ and $n \geq 1$, the well-known generalized Fibonacci numbers are defined

$$G_{n+1} = G_n + G_{n-1}$$

where $G_0 = a$ and $G_1 = b$.


El-Mikkawy and Sogabe [3] proposed a new family of $k$-Fibonacci numbers and gave the relationship between the $k$-Fibonacci numbers and Fibonacci numbers as follow:

$$F_{n}^{(k)} = (F_{m})^{k-r}(F_{m+1})^r, \quad n = mk + r.$$

In [14], Özkan et al. defined a new family of $k$-Lucas numbers and gave some identities of the new family of $k$-Fibonacci and $k$-Lucas numbers. Özkan et al. [15] introduced some identities of the new family of $k$-Fibonacci numbers.

In this study, we present some identities of the new family of $k$-generalized Fibonacci numbers. We give relationships between the new family of $k$-Fibonacci numbers and $k$-generalized Fibonacci numbers. Also, we introduce Cassini formulas of $k$-generalized Fibonacci numbers and some properties involving binomial coefficients. The rest of the paper is organized as follows: In Section 2 (Preliminaries), the fundamental definitions and theorems are given. Then main theorems and proofs are introduced in Section 3.

2010 Mathematics Subject Classification: 11B39.
Key words and Phrases: Fibonacci numbers; generalized Fibonacci numbers; generalized $k$-Fibonacci numbers.
2 PRELIMINARIES

Definition 2.1. [21] For \( n, k \ (k \neq 0) \in \mathbb{N} \), the new family of \( k \)-generalized Fibonacci numbers are defined by

\[
G^{(k)}_n = \frac{1}{\sqrt{5}}(\alpha + b \alpha)^m - (\alpha + b \beta)\beta^m = \sum_{r=0}^{\lfloor k \rfloor} \binom{k}{r} G_0 G_{r-1} B_r
\]

where \( n = mk + r, 0 \leq r < k \) and \( m \in \mathbb{N} \).

It is clear that for \( a = 0 \) and \( b = 1, G^{(k)}_n = F^{(k)}_n \) and for \( k = 1, r = 0 \) and \( n = m, G^{(1)}_n = G_n \).

Then they gave the relationship of between the new family of \( k \)-generalized Fibonacci numbers and generalized Fibonacci numbers as follow:

\[
G^{(k)}_n = (G_m)^k - r(G_{m+1})^r, \quad n = mk + r. \tag{2.1}
\]

Theorem 2.2. [9]

i. \( G^{3}_{n+1} - G_{n+1} - G^{3}_{n-1} = 3G_{n+1}G_{n} - G_{n-1} \)

ii. \( \sum_{i=1}^{n} F_i G_{3i} = F_n F_{n+1} G_{2n+1} \)

iii. \( G^2_n + G^2_{n+1} = (3a - b)G_{2n+1} - (a^2 + ab - b^2)F_{2n+1} \)

iv. \( F_{2n+1} = F_{n+1}^2 + F_n^2 \)

v. \( G^6_{n-1} + G^6_n + G^6_{n+1} = 2[2G^2_n + (a^2 + ab - b^2)(-1)^n] + 3G^2_{n-1}G^2_n G^2_{n+1} \)

vi. \( G_{nt}G_{n+t-2} - G^{2}_{n+t-1} = (a^2 + ab - b^2)(-1)^{n+t-1}F_k^2 \)

Theorem 2.3. [15]

\[
\sum_{i=1}^{n} F_i F_{3i} = F_{2n+1}^{(2)}(F_{2n+3}^{(2)} - F_{2n+1}^{(2)})
\]

Theorem 2.4. [3]

i. \( \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} F^{(k)}_{mk+i} = (-1)^k F_{m} F^{(k-1)}_{(m-1)(k-1)} \)

ii. \( \sum_{i=0}^{k-1} \binom{k-1}{i} F^{(k)}_{mk+i} = F_{m} F^{(k-1)}_{(m+2)(k-1)} \).

3 MAIN RESULTS

In this section, we present some properties of the new family of \( k \)-generalized Fibonacci numbers.

**Theorem 3.1.** For \( n \geq 1 \), we have

\[
G^{(2)}_{2n+2} + G^{(2)}_{2n} = 2G^{(2)}_{2n+1} + G^{(2)}_{2n-2}.
\]

**Proof.** Using Theorem 2.2 (i), we have
\[
G_{n+1}^3 - G_n^3 = G_{n+1}^3 + 3G_{n+1}G_nG_{n-1}
\]
\[
(G_{n+1} - G_n)(G_{n+1}^2 + G_{n+1}G_n + G_n^2) = G_{n+1}^2 + G_{n+1}G_n^2 + 3G_{n+1}G_nG_{n-1}
\]
\[
G_{n-1}(G_{2n+2}^{(2)} + G_{2n+1}^{(2)} + G_{2n}^{(2)}) = G_{n-1}(G_{2n-2}^{(2)} + 3G_{2n+1}^{(2)}
\]
\[
G_{2n+2}^{(2)} + G_{2n+1}^{(2)} + G_{2n}^{(2)} = G_{2n-2}^{(2)} + 3G_{2n+1}^{(2)}
\]
\[
G_{2n+2}^{(2)} + G_{2n}^{(2)} = 2G_{2n+1}^{(2)} + G_{2n-2}^{(2)}
\]

**Theorem 3.2.** For \( n \geq 1 \), we have

\[
(3a - b) \sum_{i=1}^{n} F_iG_{3i} = (3a - b)F_nF_{n+1}G_{2n+1}
\]

**Proof.** Using Theorem 2.2 (ii), (iii), (iv) and Theorem 2.3, we have

\[
(3a - b) \sum_{i=1}^{n} F_iG_{3i} = (3a - b)F_nF_{n+1}G_{2n+1}
\]

\[
= F_nF_{n+1}(G_n^2 + G_{n+1}^2 + (a^2 + ab - b^2)F_{2n+1})
\]

\[
= F_nF_{n+1}(G_n(G_{n+1} - G_{n-1}) + G_{n+1}(G_{n+2} - G_n))
\]

\[
+ (a^2 + ab - b^2)(F_{n+1}^2 + F_n^2)
\]

\[
= F_nF_{n+1}(-G_nG_{n-1} + G_{n+1}G_{n+2})
\]

\[
+ (a^2 + ab - b^2)(F_{n+2}F_{n+1} - F_nF_{n-1})
\]

\[
= F_{2n+1}(G_{2n+3}^{(2)} - G_{2n-1}^{(2)} + (a^2 + ab - b^2)(F_{2n+3}^{(2)} - F_{2n-1}^{(2)}))
\]

\[
= F_{2n+1}(G_{2n+3}^{(2)} - G_{2n-1}^{(2)}) + (a^2 + ab - b^2)F_{2n+1}(F_{2n+3}^{(2)} - F_{2n-1}^{(2)})
\]

\[
= F_{2n+1}(G_{2n+3}^{(2)} - G_{2n-1}^{(2)}) + (a^2 + ab - b^2) \sum_{i=1}^{n} F_iF_{3i}.
\]

**Theorem 3.3.** For \( n \geq 1 \), we have

\[
(G_{2n-2}^{(2)})^3 + (G_{2n}^{(2)})^3 + (G_{2n+2}^{(2)})^3 = 2[2G_{2n}^{(2)} + (a^2 + ab - b^2)(-1)^n] + 3G_{2n-2}^{(2)}G_{2n}^{(2)}G_{2n+2}^{(2)}.
\]

**Proof.** Using Theorem 2.2 (v), we get

\[
(G_{2n-2}^{(2)})^3 + (G_{2n}^{(2)})^3 + (G_{2n+2}^{(2)})^3 = (G_{n-1}^{(2)})^3 + (G_n^{(2)})^3 + (G_{n+1}^{(2)})^3
\]

\[
= G_{n-1}^{6} + G_n^{6} + G_{n+1}^{6}
\]

\[
= 2[2G_n^{2} + (a^2 + ab - b^2)(-1)^n] + 3G_{n-1}^{2}G_n^{2}G_{n+1}^{2}
\]

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\[ = 2\left[2G_{2n}^{(2)} + (a^2 + ab - b^2)(-1)^n\right]^3 + 3G_{2n-2}^{(2)} G_{2n}^{(2)} G_{2n+2}^{(2)}. \]

**Theorem 3.4.** For \( n \geq 1 \), we have
\[ G_{2n+2}^{(2)} - G_{2n}^{(2)} = G_{2n-2}^{(2)} + 2 G_{2n-1}^{(2)}. \]

**Proof.** From equation (2.1) and recurrence relation of generalized Fibonacci numbers, we get
\[ G_{2n+2}^{(2)} - G_{2n}^{(2)} = G_{n+1}^2 - G_n^2 = (G_{n+1} - G_n)(G_{n+1} + G_n) = G_{n-1}(G_{n+1} + G_n) = G_{n-1}G_{n+1} + G_{n-1}G_n = G_{n-1}(G_n + G_{n-1}) + G_{n-1}G_n = G_{n-1}^2 + 2G_{n-1}G_n = G_{2n-2}^{(2)} + 2 G_{2n-1}^{(2)}. \]

**Theorem 3.5.** For \( n \geq 1 \), we have
\[ G_{2n-2}^{(2)} + G_{2n-1}^{(2)} = G_{2n}^{(2)} + (a^2 + ab - b^2)(-1)^n. \]

**Proof.** Using Theorem 2.2 (vi), we have
\[ G_{2n-2}^{(2)} + G_{2n-1}^{(2)} = G_{n-1}^2 + G_nG_{n-1} = G_{n-1}(G_n + G_n) = G_{n-1}G_{n+1} = G_{n}^2 + (a^2 + ab - b^2)(-1)^n = G_{2n}^{(2)} + (a^2 + ab - b^2)(-1)^n. \]

**Theorem 3.6.** For \( n \geq 1 \), we have
\[ G_{4n+5}^{(4)} = (G_{2n}^{(2)})^3 + G_{4n+1}^{(4)} + 2G_{4n-3}^{(4)} + (G_{2n-2}^{(2)})^2 + 3G_{2n+3}^{(2)} G_{2n-1}^{(2)}. \]

**Proof.** Using Theorem 2.2 (i), we have
\[ G_{4n+5}^{(4)} = (G_{n+1}^{3})^3 G_{n+2} = (G_{n}^{3} + G_{n+1}^{3} + 3G_{n+1}G_{n}G_{n-1})G_{n+2} = G_{n}^{3}G_{n+2} + G_{n+1}^{3}G_{n+2} + 3G_{n+1}G_{n}G_{n-1} = G_{n}^{3}(G_{n} + G_{n+1})G_{n-1}^3 (2G_{n} + G_{n-1}) + 3G_{2n+3}^{(2)} G_{2n-1}^{(2)} = (G_{2n}^{(2)})^2 + G_{4n+1}^{(4)} + 2G_{4n-3}^{(4)} + (G_{2n-2}^{(2)})^2 + 3G_{2n+3}^{(2)} G_{2n-1}^{(2)}. \]
Theorem 3.7. For \( k, n, t \geq 1 \), we have
\[
G_{kn+t}^{(k)}G_{kn+t-2}^{(k)} - \left( G_{kn+t-1}^{(k)} \right)^2 = \left\{ \begin{array}{ll} G_n^{2k-2}(-1)^n(a^2 + ab - b^2), & t = 1 \\ 0, & t \neq 1 \end{array} \right.
\]

Proof. For \( t = 1 \), we get
\[
G_{kn+1}^{(k)}g_{kn-1}^{(k)} - \left( G_{kn}^{(k)} \right)^2 = (G_{n-1}^{k-1}G_{n+1}^{k-1}) - (G_n^{k})^2 = G_n^{k-1}G_{n-1}^{k} - G_n^{k} = G_n^{2k-2}[G_{n-1}G_{n+1} - G_n^2] = G_n^{2k-2}(-1)^n(a^2 + ab - b^2).
\]

For \( t \neq 1 \), we get
\[
G_{kn+t}^{(k)}G_{kn+t-2}^{(k)} - \left( G_{kn+t-1}^{(k)} \right)^2 = (G_{n}^{k-t}G_{n+1}^{k-t}) - (G_{n}^{k-t+1}G_{n+1}^{t-1}) = G_n^{2k-2t+2}G_{n+1}^{2t-2} - G_n^{2k-2t-2}G_{n+1}^{2t-2} = 0.
\]

Theorem 3.8. For \( n \geq 1 \), we have
\[
G_{2(n+s-1)}^{(2)} - G_{n+s}G_{n+s-2} = (-1)^{n+s}(a^2 + ab - b^2).
\]

Proof. From the equation (2.1) and Theorem 2.2. (vi), we acquire
\[
G_{2(n+s-1)}^{(2)} - G_{n+s}G_{n+s-2} = G_n^2 - G_{n+s}G_{n+s-2} = -(G_{n+s}G_{n+s-2} - G_n^2) = -( (-1)^{n+s-1}(a^2 + ab - b^2)) = (-1)^{n+s}(a^2 + ab - b^2).
\]

Theorem 3.9. For \( n \geq 1 \), we have
\[
\sum_{i=1}^{k}(-1)^i\binom{k-1}{i}G_{mk+i}^{(k)} = (-1)^{k-1}G_m^{(k-1)}G_{(m-1)(k-1)}.
\]

Proof. By using the equation (2.1) and the well known binomial property, we obtain
\[
\sum_{i=1}^{k}(-1)^i\binom{k-1}{i}G_{mk+i}^{(k)} = (-1)^{k-1}\sum_{i=1}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}G_{m}^{k-i}G_{m+1}^{i} = (-1)^{k-1}G_m\sum_{i=1}^{k-1}\binom{k-1}{i}(-G_m)^{k-i-1}G_{m+1}^{i}
\]

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\[(−1)^{k−1}G_m(G_{m+1} - G_m)^{k−1}\]

\[= (−1)^{k−1}G_mG_{m−1}\]

\[= (−1)^{k−1}G_mG_{m(m−1)(k−1)}^{(k−1)} \cdot\]

**Theorem 3.10.** For \(n \geq 1\), we have

\[\sum_{i=1}^{k−1} \binom{k−1}{i}G_{mk+i}^{(k)} = G_mG_{(m+2)(k−1)}^{(k−1)} \cdot\]

**Proof.** By taking account the equation (2.1) and the well known binomial property, we get

\[\sum_{i=1}^{k−1} \binom{k−1}{i}G_{mk+i}^{(k)} = \sum_{i=1}^{k−1} \binom{k−1}{i}G_m^{k−i}G_{m+1}^i\]

\[= G_m \sum_{i=1}^{k−1} \binom{k−1}{i}G_{m+1}^i(G_m)^{k−i−1}\]

\[= G_m(G_{m+1} + G_m)^{k−1}\]

\[= G_mG_{k+2}^{k−1}\]

\[= G_mG_{(m+2)(k−1)}^{(k−1)} \cdot\]

4 **CONCLUSIONS**

In this study, we prove that some identities of the new family of \(k\)-generalized Fibonacci numbers. Then, we show that some properties of the new family of \(k\)-generalized Fibonacci numbers related to generalized Fibonacci numbers. Furthermore, we extend Cassini’s formula to the new family of \(k\)-generalized Fibonacci numbers and present identities comprising binomial coefficients for the new family of \(k\)-generalized Fibonacci numbers.

**REFERENCES**


