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SOME CHARACTERIZATIONS OF RIGHT WEAKLY PRIME Γ-HYPERIDEALS OF ORDERED Γ-SEMIHYPERGROUPS

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Summary. In this paper, we deal with ordered Γ-semihypergroups. In particular, we study right weakly prime Γ-hyperideals and maximal Γ-hyperideals in ordered Γ-semihypergroups. Moreover, we give some results on ordered Γ-semihypergroups.

1 INTRODUCTION AND PREREQUISITES

Hyperstructure theory was first introduced in Marty's classical paper [1]. Semihypergroup is the generalization of semigroup theory with the wide range of usages in theory of hyperstructures [2,3]. In [4], Heidari and Davvaz studied a semihypergroup $(S,\bullet)$ besides a binary relation $\leq$, where $\leq$ is a partial orderrelation such that satisfies the monotone condition. This structure is called an ordered semihypergroup. As a reference for more definitions and results on ordered semihypergroups we refer to [3,5,6]. Omidi and Davvaz [7] investigated on the relation N in ordered semihypergroups. We refer to [5] for a survey of some results on the pseudo orders of ordered semihypergroups. Omidi et al. [8] discussed quasi-Γ-hyperideals and hyperfilters in ordered Γ-semihypergroups. Tang et al. [9] studied fuzzy quasi-Γ-hyperideals in ordered Γ-semihypergroups. In 2016, Omidi and Davvaz [10,11,12] studied some properties of hyperideals and k-hyperideals in ordered semihyperrigs and hyperrings. The study of weakly prime ideals of ordered Γ-semigroups was started by the pioneering work of Kwon and Lee [13]. In 2013, Changphas [14] defined right prime ideals and maximal right ideals in ordered semigroups. Weakly prime ideals in involution po-Γ-semigroups discussed by Abbasi and Basar [15].

Let S be a non-empty set. A mapping $\bullet: S \times S \rightarrow P^*(S)$, where $P^*(S)$ denotes the family of all non-empty subsets of S, is called a hyperoperation on S. By a hypergroupoid we mean a non-empty set S endowed with a hyperoperation $\bullet$. In the above definition, if A and B are two non-empty subsets of S and $x \epsilon S$, then we denote $A \bullet B$ the union of $a \bullet b$, where $a \epsilon A$ and $b \epsilon B$. Moreover, for every $x \epsilon S$,

$$A \bullet x = A \bullet \{x\} \text{ and } x \bullet B = \{x\} \bullet B.$$

A hypergroupoid $(S,\bullet)$ is called a semihypergroup if for all x,y,z of S, we have

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$$x\bullet(y\bullet z)=(x\bullet y)\bullet z,$$

which means that the union of $x\bullet u$, where $u\epsilon y\bullet z$ is equal to the union of $v\bullet z$, where $v\epsilon x\bullet y$.

A non-empty subset A of a semihypergroup $(S,\bullet)$ is called a subsemihypergroup of S if $A\bullet A$ is a subset of A. Let $(S,\bullet)$ be a semihypergroup. Then, S is called a hypergroup if it satisfies the reproduction axiom, for all $x\epsilon S$, $x\bullet S=S=S\bullet x$. A non-empty subset K of S is a subhypergroup of S if $a\bullet K=K=K\bullet a$, for every $a\epsilon K$.

Let S and Γ be two non-empty sets. Then, S is called a Γ-semihypergroup [16] if every $\gamma\epsilon\Gamma$ is a hyperoperation on S, i.e., $x\gamma y$ is a subset of S for every $x,y\epsilon S$ and for every $\alpha,\beta\epsilon\Gamma$ and $x,y,z\epsilon S$, we have

$$x\alpha(y\beta z)=(x\alpha y)\beta z.$$

A Γ-semihypergroup S is called commutative if for all $x,y\epsilon S$ and $\gamma\epsilon\Gamma$, we have $x\gamma y=y\gamma x$. For some properties of Γ-semihypergroups, readers can see [16]. The following concepts are adapted from [8,9].

An ordered semihypergroup $(S,\bullet,\leq)$ is a semihypergroup $(S,\bullet)$ together with a partial order $\leq$ that is compatible with the hyperoperation $\bullet$, meaning that for any $x,y,z\epsilon S$,

$$x\leq y \text{ implies that } z\bullet x\leq z\bullet y \text{ and } x\bullet z\leq y\bullet z.$$

Here, $z\bullet x\leq z\bullet y$ means for any $a\epsilon z\bullet x$ there exists $b\epsilon z\bullet y$ such that $a\leq b$. The case $x\bullet z\leq y\bullet z$ is defined similarly.

Definition 1.1 An algebraic hyperstructure $(S,\Gamma,\leq)$ is called an ordered Γ-semihypergroup if (S,Γ) is a Γ-semihypergroup and $(S,\leq)$ is a partially ordered set such that for any $x,y,z\epsilon S$ and $\gamma\epsilon\Gamma$, we have

$$x\leq y \text{ implies that } z\gamma x\leq z\gamma y \text{ and } x\gamma z\leq y\gamma z.$$

Here, $z\gamma x\leq z\gamma y$ means for any $a\epsilon z\gamma x$ there exists $b\epsilon z\gamma y$ such that $a\leq b$. The case $x\gamma z\leq y\gamma z$ is defined similarly.

See to [8,9] for the examples of the ordered Γ-semihypergroups. For a non-empty subset A of an ordered Γ-semihypergroup S, we denote

$$(A]=\{x\epsilon S|\ x\leq a \text{ for some } a\epsilon A\}.$$

Definition 1.2 A non-empty subset I of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is called a left (resp. right) Γ-hyperideal of S if

(1) $S\Gamma I\subseteq I$ (resp. $I\Gamma S\subseteq I$);

(2) when $x\epsilon I$ and $y\epsilon S$ such that $y\leq x$, imply that $y\epsilon I$.

Note that the condition (2) in Definition 1.1 is equivalent to $(I] \subseteq I$. A non-empty subset I of S is called a Γ-hyperideal of S if it is a right and left Γ-hyperideal of S. A Γ-hyperideal T of S is said to be proper if $T \neq S$.

Theorem 1.3 [9] Let $(S,\Gamma,\leq)$ be an ordered Γ-semihypergroup. Then,

(1) $A \subseteq (A]$ for any $A \subseteq S$.

(2) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.

(3) $(A]\Gamma(B] \subseteq (A\Gamma B]$ and $((A]\Gamma B]] = (A\Gamma B]$ for any $A,B \subseteq S$.

Now, we present two examples of ordered Γ-semihypergroups. We refer the readers to see more examples of ordered Γ-semihypergroups in [9,17].

Example 1.4 Let $S=[0,1]$ and $\Gamma=N$. For every $x,y \in S$ and $\gamma \in \Gamma$, we define $\gamma: S \times \Gamma \times S \to P^*(S)$ by $x\gamma y = [0, \frac{xy}{\gamma}]$. Then, γ is a hyperoperation. For every $x,y,z \in S$ and $\alpha,\beta \in \Gamma$, we have

$$(x\alpha y)\beta z = [0, \frac{xyz}{\alpha\beta}] = x\alpha(y\beta z).$$

This means that S is a Γ-semihypergroup [16]. Consider S as a poset with the natural ordering.

Thus, $(S,\Gamma,\leq)$ is an ordered Γ-semihypergroup.

Example 1.5 Let $S=\{a,b,c,d\}$ and $\Gamma=\{\gamma,\beta\}$ be the sets of binary hyperoperations defined as follows:

γ	a	b	c	d
a	a	$\{a,b\}$	$\{c,d\}$	d
b	$\{a,b\}$	b	$\{c,d\}$	d
c	$\{c,d\}$	$\{c,d\}$	c	d
d	d	d	d	d

β	a	b	c	d
a	a	$\{a,b\}$	$\{c,d\}$	d
b	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	d
c	$\{c,d\}$	$\{c,d\}$	c	d
d	d	d	d	d

Clearly, S is a Γ-semihypergroup. We have $(S,\Gamma,\leq)$ is an ordered Γ-semihypergroup where the order relation ≤ is defined by:

$$\leq := \{(a,a), (a,b), (b,b), (c,b), (c,c), (c,d), (d,b), (d,d)\}.$$

2 MAIN RESULTS

For the sake of simplicity, throughout this paper, we denote $I^2=I\Gamma I$. Let A be a non-empty subset of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$. We denote by $I(A)$ the Γ-hyperideal of S generated by A. One can easily prove that

$$I(A)=(A\bigcup S\Gamma A\bigcup A\Gamma S\bigcup S\Gamma A\Gamma S].$$

A non-empty subset I of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is called prime if for every $A,B\subseteq S$ such that $A\Gamma B\subseteq I$, we have $A\subseteq I$ or $B\subseteq I$.

Definition 2.1 A non-empty subset I of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is called weakly prime if for all Γ-hyperideals A,B of S such that $A\Gamma B\subseteq I$, we have $A\subseteq I$ or $B\subseteq I$. Also, I is called a weally prime Γ-hyperideal if I is a Γ-hyperideal which is weakly prime. A Γ-hyperideal I of S is said to be maximal if for any proper Γ-hyperideal K of S, $I\subseteq K$ implies that $I=K$.

Remark 2.2 It is easy to see that every prime Γ-hyperideal is weakly prime.

Theorem 2.3 Let $(S,\Gamma,\leq)$ be a commutative ordered Γ-semihypergroup. If P is a weakly prime Γ-hyperideal of S, then P is prime.

Proof. Assume that A,B are non-empty subsets of S such that $A\Gamma B\subseteq P$. We have

$$\begin{aligned}I(A)\Gamma I(B)&=(A\bigcup S\Gamma A\bigcup A\Gamma S\bigcup S\Gamma A\Gamma S]\Gamma(B\bigcup S\Gamma B\bigcup B\Gamma S\bigcup S\Gamma B\Gamma S]\\&\subseteq(A\Gamma S\bigcup S\Gamma A\Gamma B]\\&\subseteq I(A\Gamma B)\\&\subseteq I(P)\subseteq (P]=P.\end{aligned}$$

Since P is a weakly prime Γ-hyperideal of S, we get $I(A)\subseteq P$ or $I(B)\subseteq P$. So, we have $A\subseteq P$ or $B\subseteq P$. Therefore, P is prime.

Lemma 2.4 Let $(S,\Gamma,\leq)$ be an ordered Γ-semihypergroup. If A and B are Γ-hyperideals of S, then $A\bigcup B$ and $A\bigcap B$ are Γ-hyperideals of S.

Proof. The proof is straightforward.

Theorem 2.5 Let P be a Γ-hyperideal of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$. Then P is weakly prime if and only if for all Γ-hyperideals A and B of S such that $(A\Gamma B]\bigcap(B\Gamma A]\subseteq P$, we have $A\subseteq P$ or $B\subseteq P$.

Proof. Suppose that P is a weakly prime Γ-hyperideal of S. Let A and B be Γ-hyperideals of S such that $(A\Gamma B]\bigcap(B\Gamma A]\subseteq P$. First, we show that $(A\Gamma B]$ is a Γ-hyperideal of S. Let $y\epsilon S$ and $x\epsilon(A\Gamma B]$. Then there exist $x\epsilon(A\Gamma B]$, $a\epsilon A$, $b\epsilon B$ and $\alpha\epsilon\Gamma$ such that $x\leq c\leq a\alpha b$. Since S is an ordered Γ-semihypergroup, we get

$$x\beta y \leq c\beta y \leq (a\alpha b)\beta y = a\alpha(b\beta y) \subseteq A\Gamma(B\Gamma S) \subseteq A\Gamma B,$$

where $\beta\epsilon\Gamma$. Hence, $x\beta y \subseteq (A\Gamma B]$. Similarly, we have $y\beta x \subseteq (A\Gamma B]$. If $y \leq x$, then $y \leq x \leq z\epsilon A\Gamma B$, and so $y\epsilon(A\Gamma B]$. Therefore $(A\Gamma B]$ is a Γ-hyperideal of S. Similarly, we can prove that $(B\Gamma A]$ is a Γ-hyperideal of S. Thus,

$$(A\Gamma B]\Gamma(B\Gamma A] \subseteq (A\Gamma B]\Gamma S \subseteq (A\Gamma B] \text{ and } (A\Gamma B]\Gamma(B\Gamma A] \subseteq S\Gamma(B\Gamma A] \subseteq (B\Gamma A].$$

So, we have

$$(A\Gamma B]\Gamma(B\Gamma A] \subseteq (A\Gamma B] \cap (B\Gamma A] \subseteq P.$$

Since P is a weakly prime Γ-hyperideal of S, we get $(A\Gamma B] \subseteq P$ or $(B\Gamma A] \subseteq P$. By Theorem 1.3(1), we have $A\Gamma B \subseteq P$ or $B\Gamma A \subseteq P$. This implies that $A \subseteq P$ or $B \subseteq P$.

Conversely, assume that A and B are Γ-hyperideals of S such that $A\Gamma B \subseteq P$. By Theorem 1.3(2), we have $(A\Gamma B] \cap (B\Gamma A] \subseteq (A\Gamma B] \subseteq (P] \subseteq P$. By hypothesis, we have $A \subseteq P$ or $B \subseteq P$. Therefore, P is a weakly prime Γ-hyperideal of S.

In the following, we define right weakly prime Γ-hyperideals in ordered Γ-semihypergroups and investigate some of their related results.

Definition 2.6 A right Γ-hyperideal I of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is said to be a right weakly prime Γ-hyperideal of S if $(A\Gamma B] \cap (B\Gamma A] \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ for all right Γ-hyperideals A,B of S.

Theorem 2.7 Let $(S,\Gamma,\leq)$ be anordered Γ-semihypergroup. If M is a maximal right Γ-hyperideal of S such that $I \cap I^2 \neq \phi$, where $I=S\backslash M$, then M is a right weakly prime Γ-hyperideal of S.

Proof. Suppose that M is a maximal right Γ-hyperideal of S such that $I \cap I^2 \neq \phi$, where $I=S\backslash M$. If M is not a right weakly prime Γ-hyperideal of S, then there exist right Γ-hyperideals A,B of S such that $(A\Gamma B] \cap (B\Gamma A] \subseteq M$, $A \not\subset M$ and $B \not\subset M$. Since $A \not\subset M$, it follows that $M \subseteq A \cup M$. By Lemma 2.4, $A \cup M$ is a right Γ-hyperideal of S. So, we get $A \cup M=S$. Similarly, we have $B \cup M=S$. Hence,

$$I=S\backslash M=(A \cup M)\backslash M=A\backslash M \subseteq A \text{ and } I=S\backslash M=(B \cup M)\backslash M=B\backslash M \subseteq B,$$

which imply that

$$I^2 \subseteq A\Gamma B \cap B\Gamma A \subseteq (A\Gamma B] \cap (B\Gamma A] \subseteq M.$$

Since $I \cap I^2 \neq \phi$, it follows that $I \cap M \neq \phi$, which is a contradiction. Therefore, M is a right weakly prime Γ-hyperideal of S.

An element a of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is called an idempotent of S if $a\epsilon a\gamma a$ for every $\gamma\epsilon\Gamma$. In view of Theorem 2.7, we have the following corollaries.

Corollary 2.8 Let $(S,\Gamma,\leq)$ be an ordered Γ-semihypergroup. If M is a maximal right Γ-hyperideal of S such that $S\backslash M$ contains an idempotent of S, then M is a right weakly prime Γ-hyperideal of S.

Proof. Suppose that M is a maximal right Γ-hyperideal of S such that $I=S\backslash M$ contains an idempotent a of S. Set $I=S\backslash M$. Since $a\epsilon a\gamma a\subseteq I\Gamma I=I^2$, it follows that $I\bigcap I^2\neq\phi$. By Theorem 2.7, M is a right weakly prime Γ-hyperideal of S.

Corollary 2.9 Let $(S,\Gamma,\leq)$ be an ordered Γ-semihypergroup. If $S=(a\gamma S]$ for some $a\epsilon S$ and $\gamma\epsilon\Gamma$, then every maximal right Γ-hyperideal of S is a right weakly prime Γ-hyperideal of S.

Proof. Suppose that $S=(a\gamma S]$ for some $a\epsilon S$ and $\gamma\epsilon\Gamma$. Let M be a maximal right Γ-hyperideal of S and $I=S\backslash M$. If a is not in I, then $a\epsilon M$. It follows that

$$S=(a\gamma S]\subseteq(M\Gamma S]\subseteq(M]=M,$$

which is a contradiction. This leads to $a\epsilon I$. So, we have $a\gamma a\subseteq I\Gamma I=I^2$. Now, let $a\gamma a\bigcap I=\phi$. Then $a\gamma a\subseteq M$. This implies that

$$S=(a\gamma S]=(a\gamma(a\gamma S]]\subseteq((a]\gamma(a\gamma S]]=(a\gamma a\gamma S]\subseteq(M\Gamma S]\subseteq(M]=M.$$

This is a contradiction. Hence, $a\gamma a\bigcap I\neq\phi$.Thus there exists $x\epsilon a\gamma a\subseteq I^2$ such that $x\epsilon I$. Therefore, $I\bigcap I^2\neq\phi$. Now, by Theorem 2.7, M is a right weakly prime Γ-hyperideal of S.

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