ON SOME METRIC TOPOLOGIES ON PRIVALOV SPACES ON THE UNIT DISK

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Summary. Let N^p $(1 < p < \infty)$ be the Privalov class N^p of holomorphic functions on the open unit disk $\mathbb{D}$ in the complex plane. In 1977 M. Stoll proved that the class N^p equipped with the topology given by the metric λ_p defined by

$$\lambda_p(f,g) = \left(\int_0^{2\pi} \left(\log(1+|f^*(e^{i\theta}) - g^*(e^{i\theta})|)\right)^p \frac{d\theta}{2\pi}\right)^{1/p}, \quad f,g \in N^p,$$

becomes an F-algebra. In the recent overview paper by Meštrović and Pavićević (2017) a survey of some known results on the topological structures of the Privalov spaces N^p $(1 < p < \infty)$ and their Fréchet envelopes F^p are presented.

In this article we continue a survey of results concerning the topological structures of the spaces N^p $(1(p < \infty)$. In particular, for each $p > 1$, we consider the class N^p as the space M^p equipped with the topology induced by the metric ρ_p defined as

$$\rho_p(f,g) = \left(\int_0^{2\pi} \log^p(1+M(f-g)(\theta)) \frac{d\theta}{2\pi}\right)^{1/p}, \quad f,g \in M^p, \text{ where } Mf(\theta) = \sup_{0\leqslant r<1} \left|f\left(re^{i\theta}\right)\right|.$$

On the other hand, we consider the class N^p with the metric topology introduced by Meštrović, Pavićević and Labudović (1999) which generalizes the Gamelin-Lumer's metric which is generally defined on a measure space (Ω, Σ, μ) with a positive finite measure μ. The space N^p with the associated modular in the sense of Musielak and Orlicz becomes the Hardy-Orlicz class. It is noticed that the all considered metrics induce the same topology on the space N^p.

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1 INTRODUCTION AND PRELIMINARY RESULTS

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let $\mathbb{T}$ denote the boundary of $\mathbb{D}$. Let $L^q(\mathbb{T})$ $(0 < q \leq \infty)$ be the familiar Lebesgue spaces on the unit circle $\mathbb{T}$. The *Privalov class* N^p $(1 < p < \infty)$ consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\sup_{0\leq r<1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p \frac{d\theta}{2\pi} < +\infty, \tag{1}$$

where for $z \in \mathbb{C}$, $\log^+ |z| = \max(\log |z|, 0)$ if $z \neq 0$ and $\log^+ 0 = 0$. These classes were firstly considered by I.I. Privalov in [40, p. 93], where N^p is denoted as A_q.

Notice that for $p = 1$ the condition (1) defines the *Nevanlinna class* N of holomorphic functions on $\mathbb{D}$. Recall that the *Smirnov class* N^+ is the set of all functions $f \in N$ such that

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi} < +\infty,$$

where f^* is the boundary function of f on $\mathbb{T}$, i.e.,

$$f^*(e^{i\theta}) = \lim_{r\to 1-} f(re^{i\theta})$$

is the *radial limit* of f which exists for almost every $e^{i\theta} \in \mathbb{T}$. Recall that the classical *Hardy space* H^q $(0 < q \leq \infty)$ consists of all functions f holomorphic on $\mathbb{D}$ such that

$$\sup_{0\leq r<1} \int_0^{2\pi} \left|f\left(r\,\mathrm{e}^{i\theta}\right)\right|^q \frac{d\theta}{2\pi} < \infty$$

if $0 < q < \infty$, and which are bounded when $q = \infty$:

$$\sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

It is known that (see [36] and [25])

$$N^r \subset N^p \ (r > p), \quad \bigcup_{q>0} H^q \subset \bigcap_{p>1} N^p, \quad \text{and} \quad \bigcup_{p>1} N^p \subset M \subset N^+ \subset N,$$

where the above containment relations are proper.

It is well known (see, e.g., [4, p. 26]) that a function $f \in N^+$ has a unique factorization of the form

$$f(z) = B(z)S_\mu(z)F(z), \quad z \in \mathbb{D},$$

where B is the *Blaschke product* with respect to zeros $\{z_k\} \subset \mathbb{D}$ of f, S_μ is a *singular inner function* and F is an *outer function* in N^+, i.e.,

$$B(z) = z^m \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \cdot \frac{z_k - z}{1 - \bar{z}_k z}, \quad z \in \mathbb{D},$$

with $\sum_{k=1}^{\infty}\big(1-|z_k|\big)<\infty$, m a nonnegative integer,

$$S_\mu(z)=\exp\Big(-\int\limits_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\Big)$$

with positive singular measure $d\mu$ and

$$F(z)=\lambda\exp\Big(\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\big|F^*\big(e^{it}\big)\big|\,dt\Big),\tag{2}$$

where $\lambda\in\mathbb{C}$ with $|\lambda|=1$ and $\log|F^*|\in L^1(\mathbb{T})$.

Recall that a function I of the form

$$I(z)=B(z)S_\mu(z),\quad z\in\mathbb{D},$$

is called an *inner function*. Furthermore, it is well known that $|I^*(e^{it})|=1$ for almost every $e^{it}\in\mathbb{T}$ and hence, $|f^*(e^{it})|=|F^*(e^{it})|$ for almost every $e^{it}\in\mathbb{T}$.

I.I. Privalov [40, p. 98] (alo see [25, Theorem 5.3]) proved that a function f holomorphic on $\mathbb{D}$ belongs to the class N^p if and only if $f=IF$, where I is an inner function on $\mathbb{D}$ and F is an outer function given by (2) such that $\log^+|f^*|\in L^p(\mathbb{T})$ (or equivalently, $\log^+|F^*|\in L^p(\mathbb{T})$).

M. Stoll [44, Theorem 4.2] showed that the space N^p (with the notation $(\log^+ H)^\alpha$ in [44]) equipped with the topology given by the metric λ_p defined by

$$\lambda_p(f,g)=\left(\int\limits_0^{2\pi}\big(\log(1+|f^*(e^{i\theta})-g^*(e^{i\theta})|)\big)^p\,\frac{d\theta}{2\pi}\right)^{1/p},\quad f,g\in N^p,\tag{3}$$

becomes an F-algebra. Recall that the function $\lambda_1=\lambda$ defined on the Smirnov class N^+ by (3) with $p=1$ induces the metric topology on N^+. N. Yanagihara [45] proved that under this topology, N^+ is an F-space.

Furthermore, in connection with the spaces N^p $(1<p<\infty)$, Stoll [44] (also see [5] and [29, Section 3]) also studied the spaces F^q $(0<q<\infty)$ (with the notation $F_{1/q}$ in [44]), consisting of those functions f holomorphic on $\mathbb{D}$ for which

$$\lim_{r\to 1}(1-r)^{1/q}\log^+ M_\infty(r,f)=0,$$

where

$$M_\infty(r,f)=\max_{|z|\le r}|f(z)|.$$

Stoll [44, Theorem 3.2] also proved that the space F^q with the topology given by the family of seminorms $\{|||\cdot|||_{q,c}\}_{c>0}$ defined for $f\in F^q$ as

$$|||f|||_{q,c}=\sum_{n=0}^{\infty}|\hat{f}(n)|e^{-cn^{1/(q+1)}}<\infty$$

for each $c>0$, where $\hat{f}(n)$ is the n-th Taylor coefficient of f, is a countably normed *Fréchet algebra*. By a result of C.M. Eoff [5, Theorem 4.2], F^p is the *Fréchet envelope* of N^p and hence, F^p and N^p have the same topological duals.

Following H.O. Kim ([13] and [14]), the class M consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\int_0^{2\pi} \log^+ Mf(\theta)\frac{d\theta}{2\pi} < \infty,$$

where

$$Mf(\theta) = \sup_{0\leqslant r<1} \left|f\left(re^{i\theta}\right)\right|$$

is the *maximal radial function* of f.

The study on the class M on the disk $\mathbb{D}$ has been exstensively investigated by H.O. Kim in [13] and [14], V.I. Gavrilov and V.S. Zaharyan [9] and M. Nawrocky [39]. Kim [14, Theorems 3.1 and 6.1] showed that the space M with the topology given by the metric ρ defined by

$$\rho(f,g) = \int_0^{2\pi} \log(1 + M(f-g)(\theta))\,\frac{d\theta}{2\pi}, \quad f,g \in M, \tag{4}$$

becomes an F-algebra. Furthermore, Kim [14, Theorems 5.2 and 5.3] gave an incomplete characterization of multipliers of M into H^∞. Consequently, the topological dual of M is not exactly determined in [14], but as an application, it was proved in [14, Theorem 5.4] (also cf. [39, Corollary 4]) that M is not locally convex space. Furthermore, the space M is not locally bounded ([14, Theorem 4.5] and [39, Corollary 5]).

Nevertheless that as noticed above, the class M is essentially smaller than the class N^+, M. Nawrocky [39] showed that the class M and the Smirnov class N^+ have the same corresponding locally convex structure which was already established by N. Yanagihara for the Smirnov class in [45] and [46]. More precisely, it was proved in [39, Theorems 1] that the Fréchet envelope of the class M can be identified with the space F^+ of holomorphic functions on the open unit disk $\mathbb{D}$ such that

$$|||f|||_c := \sum_{n=0}^{\infty} |\hat{f}(n)|e^{-c\sqrt{n}} < \infty$$

for each $c > 0$, where $\hat{f}(n)$ is the n-th Taylor coefficient of f. Notice that F^+ coincides with the space F^1 defined above. It was shown in [46] (also see [45]) that F^+ is actually the containing Fréchet space for N^+ (also see [43]). Moreover, Nawrocky [39, Theorem 1] characterized the set of all continuous linear functionals on M which by a result of Yanagihara [45] coincides with those on the Smirnov class N^+.

Motivated by the mentioned investigations of the classes M and N^+, and the fact that the classes N^p $(1 < p < \infty)$ are generalizations of the Smirnov class N^+, in [20, Chapter 6] and [22] the first author of this paper investigated the classes M^p $(1 < p < \infty)$ as generalizations of the class M. Accordingly, the *class* M^p $(1 < p < \infty)$ consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\int_0^{2\pi} \left(\log^+ Mf(\theta)\right)^p\,\frac{d\theta}{2\pi} < \infty.$$

Obviously,

$$\bigcup_{p>1} M^p \subset M.$$

By analogy with the topology defined on the space M ([13] and [14]), the space M^p can be equipped with the topology induced by the metric ρ_p defined as

$$\rho_p(f,g) = \Big(\int_0^{2\pi} \log^p(1 + M(f-g)(\theta)) \,\frac{d\theta}{2\pi}\Big)^{1/p},$$

with $f, g \in M^p$.

After Privalov, the study of the spaces N^p $(1 < p < \infty)$ was continued in 1977 by M. Stoll [44] (with the notation $(\log^+ H)^\alpha$ instead of N^p in [44]). Further, the linear topological and functional properties of these spaces were extensively investigated by C.M. Eoff in [5] and [6], N. Mochizuki [36], Y. Iida and N. Mochizuki [12], Y. Matsugu [17], J.S. Choa [2], J.S. Choa and H.O. Kim [3], A.K. Sharma and S.-I. Ueki [42] and in works [19]-[35] of authors of this paper; typically, the notation varied and Privalov was mentioned in [17], [21]-[24], [29]-[32], [34], [35] and [42]. In particular, it was proved in [21, Corollary] that N^p is not locally convex space and in [30, Theorem 1.1] that N^p is not locally bounded space. We refer the recent monograph [10, Chapters 2, 3 and 9] by V.I. Gavrilov, A.V. Subbotin and D.A. Efimov for a good reference on the spaces N^p $(1 < p < \infty)$.

Let us recall that in our recent overview paper [32] it was given a survey of some known results on different topologies on the Privalov classes N^p $(1 < p < \infty)$ and their Fréchet envelopes F^p $(1 < p < \infty)$ on the open unit disk. Here we give a survey on related extended results involving some other metrics and the induced topologies on the classes N^p.

The remainder of this overview paper is organized in three sections. For any fixed $p > 1$, in Section 2 we present some results concerning the topological and functional structures on the classes M^p $(1 < p < \infty)$. Section 3 is devoted to the consideration of the Privalov class N^p as a closed subspace of some Orlicz space. In this setting N^p with the associated modular in the sense of Musielak and Orlicz becomes the Hardy-Orlicz class whose topology coincides with both metric topologies λ_p and ρ_p. Concluding remarks are presented in the last section.

2 THE ρ_p-METRIC TOPOLOGY ON PRIVALOV SPACE N^p

Here we focus our attention to certain results from [20, Chapter 6] and [22] concerning the classes M^p $(1 < p < +\infty)$. In [22] it is proved the following basic result.

Theorem 1 ([22, Theorem 2]). *The function ρ_p defined on M^p as*

$$\rho_p(f,g) = \Big(\int_0^{2\pi} \log^p(1 + M(f-g)(\theta)) \,\frac{d\theta}{2\pi}\Big)^{1/p}, \quad f, g \in M^p, \tag{5}$$

is a translation invariant metric on M^p. Further, the space M^p is a complete metric space with respect to the metric ρ_p.

Remark 1. Notice that the expression (5) with $p = 1$ defines the metric $\rho_1 = \rho$ on the class M (given by (4)) introduced by H.O. Kim in [13] and [14]. As noticed above, it was proved in [14] that the metric ρ induces the topology on M under which M is also an F-algebra.

Moreover, the following two statements are also proved in [20, Chapter 6].

Theorem 2 ([22, Theorem 11]). *$M^p = N^p$ for each $p > 1$; that is, the spaces M^p and N^p coincide.*

Theorem 3 ([22, Theorem 15]). *M^p with the topology given by the metric ρ_p defined by* (5) *becomes an F-space.*

Using Theorem 3 and the open mapping theorem (see, e.g., [41, Corollary 2.12 (b)]), the following result was also proved in [22].

Theorem 4 ([22, Theorem 16]). *For each $p > 1$ the classes M^p and N^p coincide, and the metric spaces (M^p, ρ_p) and (N^p, λ_p) have the same topological structure, where the metrics ρ_p and λ_p are given on M^p and N^p by* (5) *and* (3), *respectively.*

As an immediate consequence of Theorem 4 and [22, Lemma 8], we obtain the following assertion.

Proposition 1. *The convergence with respect to the metric ρ_p given by* (5) *on the space M^p is stronger than the metric of uniform convergence on compact subsets of the disk $\mathbb{D}$.*

Remark 2. For an outer function h let $H^2(|h^*|^2)$ denote the closure of the (analytic) polynomials in the space $L^2(|h^*|^2 d\,\theta)$. By using the famous Beurling's theorem for the Hardy space H^2 ([1]; also see [11, Ch. 7, p. 99]), it was proved in [6] (also see [27, Section 1]) that the class N^p can be represented as a union of certain weighted Hardy classes. Using this representation, the following two topologies are defined on N^p in [27]: the usual locally convex inductive limit topology, which we shall call the *Helson topology* and denote by $\mathcal{H}_p$, in which a neighborhood base for 0 is given by those balanced convex sets whose intersection with each $H^2\,(|h^*|^2)$ is a neighborhood of zero in $H^2\,(|h^*|^2)$, and a not locally convex topology, denoted by I_p, in which a neighborhood base for zero is given by all sets whose intersection with each space $H^2\,(|h^*|^2)$ is a neighborhood of zero. It was proved in [27, Theorem E] (cf. [20, Chapter 3]) that the topology $\mathcal{H}_p$ coincides with the metric topology induced on N^p by the Stoll's metric topology λ_p given by (3). Moreover, it was proved in [6] that the topology I_p coincides with the metric topology λ_p and by Theorem 4, I_p also coincides with the metric topology ρ_p, which are not locally convex. Hence, I_p is strictly stronger than $\mathcal{H}_p$. The analogous results for the space N^+ are proved by J.E. McCarthy in [18].

3 THE SPACE N^p AS THE HARDY-ORLICZ CLASS

In this section we we give a short survey about Privalov classes N^p $(1 < p < +\infty)$ as the Hardy-Orlicz classes. Related results are mainly obtained in [33].

Let (Ω, Σ, μ) be a measure space, i.e., Ω is a nonempty set, Σ is a σ-algebra of subsets of Ω and μ is a nonnegative finite complete measure not vanishing identically. Denote by $L^p(\mu) = L^p$ $(0 < p \le \infty)$ the familiar Lebesgue spaces on Ω. For each real number $p > 0$ in [33] it was considered the class $L_p^+(\mu) = L_p^+$ of all (equivalence classes of) Σ-measurable complex-valued functions f defined on Ω such that the function $\log^+ |f|$ belongs to the space L^p, i.e.,

$$\int\limits_{\Omega} \left(\log^+ |f(x)|\right)^p \, d\mu < +\infty,$$

where $\log^+ |a| = \max(\log |a|, 0)$. Clearly, $L_q^+ \subset L_p^+$ for $q > p$ and $\bigcup_{p>0} L^p \subset \bigcap_{p>0} L_p^+$ [33, Section 2]. For each $p > 0$ the space L_p^+ is an algebra with respect to the pointwise addition and multiplication. For each $p > 0$ we define the metric d_p on L_p^+ by

$$\begin{aligned} d_p(f,g) \;&=\; \inf_{t>0} \left[t + \mu\left(\{x \in \Omega : \, |f(x) - g(x)| \ge t\}\right)\right] \\ &\quad + \int\limits_{\Omega} \left| \left(\log^+ |f(x)|\right)^p - \left(\log^+ |g(x)|\right)^p \right| \, d\mu. \end{aligned} \tag{6}$$

Recall that the space L_1^+ was introduced by T. Gamelin and G. Lumer in [8, p. 122] (also see [7, p. 122], where L_1^+ is denoted as $L(\mu)$). Note that the metric d_p given by (6) with $p = 1$ coincides with the Gamelin-Lumer's metric d defined on L_1^+. It was proved in [8, Theorem 1.3, p. 122] (also see [7, Theorem 2.3, p. 122]) that the space L_1^+ with the topology given by the metric d_1 becomes a topological algebra. The following result is a generalization of the corresponding result for the case $p = 1$ given in [8, p. 122] (also see [7, p. 122]).

Theorem 5 ([33, Theorem 2.1]). *The space L_p^+ with the metric d_p given by (6) is a topological algebra, i.e., a topological vector space with a complete metric in which multiplication is continuous.*

By the inequality

$$\left(\log(1 + |z|)\right)^p \le 2^{\max(p-1,0)} \left((\log 2)^p + (\log^+ |z|)^p\right), \quad z \in \mathbb{C},$$

it follows that a function f belongs to the space L_p^+ if and only if

$$\|f\|_p := \left(\int\limits_{\Omega} \left(\log(1 + |f(x)|)\right)^p \, d\mu \right)^{1/p} < \infty. \tag{7}$$

Furthermore [33, Section 2], the function σ_p defined as

$$\sigma_p(f,g) = \left(\|f - g\|_p\right)^{\min(1,p)}, \quad f, g \in L_p^+, 0 < p \le 1, \tag{8}$$

is a translation invariant metric on L_p^+ for all $p > 0$. Notice that in the case of Privalov space N^p $(1 < p < \infty)$, the metric σ_p given by (8) coincides with Stoll's metric λ_p defined by (3).

Recall that two metrics (or norms) defined on the same space will be called equivalent if they induce the same topology on this space.

Theorem 6 ([33, Theorem 2.3]). *The metric d_p given by* (6) *defines the topology for L_p^+ which is equivalent to the topology defined by the metric σ_p given by* (8).

Remark 4. It was pointed out in [33, Remark, Section 2] that using the same argument applied in the proof of Theorem 2.3 of [33], it is easy to show that the metrics σ_p and d_p are equivalent with the metric δ_p given on L_p^+ by

$$\begin{aligned}\delta_p(f,g) &= \inf_{t>0}\left[t+\mu\left(\{x\in\Omega:\ |f(x)-g(x)|\geq t\}\right)\right]\\ &\quad +\left(\int_{\Omega}\left|\log^+|f(x)|-\log^+|g(x)|\right|^p\,d\mu\right)^{1/\max(p,1)},\ f,g\in L_p^+. \qquad (9)\end{aligned}$$

Remark 5. In [45, Remark 5, p. 460] M. Hasumi pointed out that the Yanagihara's metric $\lambda = \lambda_1$ on the Smirnov class (given by (3) with $p = 1$) defines the topology on the space $L_1^+ = L(\mu)$ which is equivalent to the metric topology $d_1 = d$ (given by (6) with $p = 1$).

As a consquence of Theorems 5 and 6, it can be obtained the following result.

Theorem 7 ([33, Corollary 2.4]). *For each $p > 0$ the space L_p^+ with the topology given by the metric σ_p is an F-algebra, i.e., a topological algebra with a complete translation invariant metric σ_p.*

Remark 6. In view of Theorem 7, note that L_p^+ may be considered as the generalized Orlicz space L_p^w with the constant function $w(t) \equiv 1$ on $[0, 2\pi)$ defined in [33, Section 6].

The real-valued function $\psi : [0,\infty) \mapsto [0,\infty)$ defined as $\psi(t) = \big(\log(1+t)\big)^p$, is continuous and nondecreasing in $[0,\infty)$, equals zero only at 0, and hence it is a *φ-function* (see, e.g., [37, p. 4, Examples 1.9]). Moreover, ψ is a log-*convex* function since it can be represented in the form $\psi(x) = \Psi(\log x)$ for $x > 0$, where $\Psi(u) := \max(u^p, 0)$ $(u \in [0,\infty))$ is a *convex* function on the whole real axis, satisfying the condition $\lim_{u\to+\infty}\frac{\Psi(u)}{u} = +\infty$. Notice that convex φ-functions are a particular case of log-convex φ-functions.

Further, observe that [33, Section 4] the space $L_p^+(dt/(2\pi) = L_p^+$ $(p > 0)$, consisting of all complex-valued functions f, defined and measurable on $[0, 2\pi)$, for which

$$\|f\|_p := \left(\int_0^{2\pi}(\log(1+|f(t)|))^p\,\frac{dt}{2\pi}\right)^{1/p} < +\infty \qquad (10)$$

is the *Orlicz class* (see [37, p. 5]; cf. [33, Section 4]), whose generalization was given in [33, Section 6]. It follows by the dominated convergence theorem that the class L_p^+

coincides with the associated *Orlicz space* (see [37, Definition 1.4, p. 2]) consisting of those functions $f \in L_p^+$ such that

$$\int_0^{2\pi} (\log(1+c|f(t)|))^p \, \frac{dt}{2\pi} \to 0 \quad \text{as} \quad c \to 0+ \, .$$

Since $\sigma_p(f,g) = \big(\|f-g\|_p\big)^{\min(p,1)} (f,g \in L_p^+)$ is an invariant metric on L_p^+, the function $\|\cdot\|_p$ given by (10) is a *modular* in the sense of Definition 1.1 in [37, p. 1], where σ_p is the metric defined by (8). For any function $f \in L_p^+$, by the monotone convergence theorem, it follows that $\lim_{c\to 0} \|cf\| = 0$ and thus (L_p^+, σ_p) is a *modular space* in the sense of Definition 1.4 in [37, p. 2]. In other words, the function $\|\cdot\|_p$ is an F-norm. It is known (see [37, Theorem 1.5, p. 2 and Theorem 7.7, p. 35]) that the functional $|\cdot|_p$ defined as

$$|f|_p = \inf\Big\{\varepsilon > 0 : \int_0^{2\pi} \left(\log\left(1+\frac{|f(t)|}{\varepsilon}\right)\right)^p \frac{dt}{2\pi} \le \varepsilon\Big\}, \quad f \in L_p^+, \tag{11}$$

is a complete F-*norm* on L_p^+. Furthermore (see [16, p. 54]), if we denote by L_p^0 the class of all functions f such that $\alpha f \in L_p^+$ for every $\alpha > 0$, then L_p^0 is the closure of the space of all continuous functions on $[0, 2\pi)$ in the space $(L_p^+, |\cdot|_p)$.

We also give the following two results.

Theorem 8 ([33, Theorem 4.1]). *The F-norms $\|\cdot\|_p$ and $|\cdot|_p$ (given by* (10) *and* (11)*, respectively), induce the same topology on the space L_p^+. In other words, the norm and modular convergences are equivalent.*

Proposition 2 ([33, Corollary 4.2]). *There does not exist a nontrivial continuous linear functional on the space $(L_p^+, \|\cdot\|_p)$.*

Note that [33, Section 5] the algebra N^p may be considered as the *Hardy-Orlicz space* with the Orlicz function $\psi : [0,\infty) \mapsto [0,\infty)$ defined as $\psi(t) = \big(\log(1+t)\big)^p$. These spaces were firstly studied in 1971 by R. Leśniewicz [15]. For more information on the Hardy-Orlicz spaces, see [37, Ch. IV, Sec. 20]. Identifying a function $f \in N$ with its *boundary function* f^*, by [16, 3.4, p. 57], the space N^p is identical with the closure of the space of all functions holomorphic on the open unit disk $\mathbb{D}$ and continuous on $\overline{\mathbb{D}} : |z| \le 1$ in the space $(L_p^+(dt/2\pi) \cap N, |\cdot|_p)$, where $dt/2\pi$ is the usual normalized Lebesgue measure on the unit circle $\mathbb{T}$. Using this fact, Theorem 4 and Theorem 6, the main surveyed results of this paper can be summarized as follows.

Theorem 9 ([33, Theorem]). *For each $p > 1$ Privalov class is the Hardy-Orlicz space with the Orlicz function $\psi(t) = \big(\log(1+t)\big)^p$ $(t \in [0, 2\pi))$. Moreover, the metrics λ_p, ρ_p, d_p, δ_p and the functional $|\cdot|_p$ (defined respectively by* (3), (5), (6), (9) *and* (11)*) induce the same topology on N^p under which N^p becomes an F-algebra.*

4 CONSLUSION

This paper continues an overview of topologies on the Privalov spaces N^p $(1 < p < +\infty)$ induced by different metrics. Notice that the class N^p equipped with the topology given by the metric λ_p introduced by M. Stoll becomes an F-algebra. The same statement is also true for the class M^p with respect to the ρ_p-metric topology. These facts are used in [20] to prove that for each $p > 1$ the classes M^p and N^p coincide and the metric spaces (M^p, ρ_p) and (N^p, λ_p) have the same topological structure In Section 3 we give a short survey about Privalov spaces N^p $(1 < p < +\infty)$ whose topology is induced by the generalized Gamelin-Lumer's metric d_p defined on the space $L_p^+(dt/(2\pi)$. Notice that the space $L_p^+(dt/(2\pi)$ coincides with the Orlicz class associated to the log-convex φ-function $\psi(t) = \big(\log(1+t)\big)^p$ $(t \in [0, +\infty)$. Accordingly, it follows that for each $p > 1$ Privalov space is the Hardy-Orlicz space with the Orlicz function $\psi(t) = \big(\log(1+t)\big)^p$ $(t \in [0, 2\pi))$. Moreover, the metrics λ_p, ρ_p and d_p induce the same topology on N^p under which N^p becomes an F-algebra. We believe that presented results would be useful for future research on related topics, as well as for some applications in Functional and Complex Analysis.

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