STOCHASTIC-THERMODYNAMIC MODELING OF THE DEVELOPED STRUCTURED TURBULENCE

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Summary. The paper deals with a phenomenological model of the developed turbulence in a compressible homogeneous medium with an account for nonlinear cooperative processes. The bottom line is representation of turbulized fluid motion as a thermodynamic system consisting of two continua, the subsystem of averaged motion and the subsystem of turbulent chaos, the latter addressing as conglomerate of vortex structures of various spatiotemporal scales. We develop the ideas of a stationary non-equilibrium state of the dissipative active subsystem of turbulent chaos that emerges due to influx of negentropy from external medium (the subsystem of averaged motion) and the appearance of relatively stable coherent vortex structures in the system when the flow control parameters are varied. This allows us to consider some of the turbulent field rearrangements as self-organization processes in an open system. Methods of the stochastic theory of irreversible processes and extended irreversible thermodynamics are used to derive the defining relations for the turbulent fluxes and forces to close the system of averaged hydrodynamic equations and describe the transport and self-organization processes in the stationary non-equilibrium case sufficiently to be used in some practical applications.

1 INTRODUCTION

Turbulence is regarded as the most common type of fluid motion in nature. Traditionally, it was mainly represented as a fine-grained fluctuating continuum in a state of total stochastic chaos. However, a different viewpoint on turbulence that was probably first put forward explicitly by Prigogine[1] is also admissible. According to this approach, a real turbulent fluid flow is a less-random and macroscopically more organized process than it seems at the first glance: transition from a laminar flow to a turbulent one is regarded as the process of self-organization whereby part of the energy of turbulent chaos corresponding to small-scale fluctuations of thermo-hydrodynamic fluid parameters passes into a macroscopically organized motion of vortex coherent structures (CSs). This increases the internal order of the turbulized hydrodynamic system compared to molecular chaos (laminar fluid motion). In particular, the cascade fragmentation of vortices in fully developed turbulence can also be interpreted as an unbounded sequence of self-organization processes. In this case, the set of spatiotemporal scales on which such a process unfolds corresponds to a coherent behavior of an enormous

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number of particles exhibited in the formation of relatively stable mesoscale super-molecular structures (when molecules are involved in collective, coordinated, and interrelated motions corresponding to different-scale vortices continuously distributed in a real fluid flow). This change of the view on turbulence is clearly expressed in Prigogine's utterance[2], who "could predict thirty years ago that non-equilibrium leads to self-organization in the form in which we observe it in hydrodynamic instabilities like Bernard cells."

As has now become clear, the presence of relatively large coherent vortex structures (turbulent filaments, rings, vortex spirals, etc.) randomly distributed in space and time is a characteristic feature of many, if not all, developed turbulent flows[3-6]. According to the latest viewpoints[7], hydrodynamic turbulence, which is among the most complex dynamical phenomena, is related, in particular, to the formation and development of an enormous number of organized dissipative vortex structures with various spatiotemporal scales under certain fluid flow conditions in an essentially non-equilibrium open system. For instance, the self-organization processes against the background of a chaotic fluctuating motion of cosmic matter appear to be the most important mechanism that form the peculiar features in astro- and geophysical objects at various stages of their evolution, including formation of galaxies and galaxy clusters, birth of stars from the diffuse medium of gas-dust molecular clouds, formation of protoplanetary disks and subsequent accumulation of planetary systems, formation of planetary and cometary gaseous envelopes (atmospheres), and the different-scale flows in atmospheres and circumplanetary plasma environment, just to mention a few.

Based on the currently available experimental data[8] where a comprehensive overview of the relevant publications is contained), a coherent vortex structure can be defined as a connected, liquid mass with phase-correlated (i.e., coherent) vortices in the entire region of coordinate space occupied by the structure. In the last decade, a large number of various CSs have been discovered and their topological properties firmly established owing to progress achieved in developing the technique for a visual observation of turbulized fluid flows. As examples, we could mention such CSs as vortex filaments, Taylor vortices, turbulent spots, vortex balls, hairpin-like vortices, burstings, vortex spirals, streaks, Brawn–Thomas structures, and mushroom-shaped vortices. The frequency of occurrence of a particular structure depends on the type of flow (a boundary layer, a mixing layer, a jet, etc.) and on the geometry and regime of turbulized fluid motion. As a rule, such vortex structures are localized in space and do not overlap and therefore, they can often be considered as lumped objects–clusters. Their mean free paths is much larger than their own sizes. By definition, the characteristic CS size is the largest spatial scale l on which coherent vorticity exists; the latest results show that l can be much smaller than the characteristic hydrodynamic flow scale L but larger than the Kolmogorov scale η, i.e., it can lie in the inertial range of scales, $\eta < l << L$. Due to the interaction of individual CSs (e.g., according to the Biot–Savart law in the case of two simple vortices), which generally is essentially nonlinear, they combine or break up, i.e., a new single structure (e.g., a spiral vortex) or adjacent similar structures (rings, balls, etc.) set up, which can be connected by ropes – the regions of low coherent vorticity. Below, we will proceed from the idea that the mesoscale spatiotemporal CSs are random in all range of the parameters used for their description.

Unfortunately, a direct numerical simulation of the developed turbulent motion based on exact (instantaneous) hydrodynamic equations encounters great mathematical difficulties, while constructing a full-blown hydromechanics theory of structured turbulence is hardly possible because mechanisms of the formation, interaction, and evolution of different-scale vortex structures are extremely complex. For this reason, the development of new model approaches (including phenomenological ones) to describing fully developed turbulence, the introduction of additional internal parameters of a medium (including those characterizing small-scale flow structures), the introduction of universal laws and special relations supplementing the known mass, energy, and momentum conservation laws equations are required. Basically, while semiempirical modeling of the structured turbulence is idealized description of a real fluid flow, such an approach is inevitable in order to preserve the main features of the most important hydrodynamic effects and minimize computational efforts.

Hence, to construct an adequate model of “highly organized” turbulent flow, it is generally necessary to incorporate relatively large dissipative coherent structures, along with fine-grained patterns of fluctuating field of thermo-hydrodynamic flow parameters. They should be taken into account when modeling both the process of yet non-turbulized surrounding fluid transfer into turbulized motion and the fully developed turbulent mass, momentum, and heat transport processes. In other words, any efficient continuum model of turbulence cannot be constructed without including explicitly enough spatiotemporal coherent structures and without describing them by some internal state parameters of fluctuating flow. According to Frisch[9], such regular vortex structures are, in a sense, the “sinews” of turbulence. Consequently, the true possibilities to overcome efficiently various mechanical and mathematical problems when dealing with their formulations and numerical implementations become open up only in turbulence modeling with an account of its internal structurization[10-13].

The phenomenon of structured turbulence is rooted in the order–chaos relationship in turbulized fluid flows. Different complex patterns in the behavior of open fluctuating hydrodynamic systems form spatiotemporal vortex structures with no specific impact from outside through self-organization, i.e., establish “order through fluctuations” far from local thermodynamic equilibrium. Unfortunately, although more than 30 years has elapsed since the understanding of the synergetic nature of turbulence as a self-organization process, the views of dissipative coherent structures emerging in a flow have not yet been implemented into development of model approaches aimed at creating practical (engineering) methods of turbulence calculation based, as a rule, on hydrodynamic equations. At the same time, extending the formalism of non-equilibrium thermodynamics to media with excited internal degrees of freedom of macromolecules (which can be described by additional parameters characterizing the internal microstructure of the medium) probably allows us to extend this approach to modeling the cascade transport of kinetic energy by vortices of various sizes formed by their successive fragmentation in the developed turbulent flow.

Here we consider a synergetic approach to the phenomenological modeling the developed turbulence in a compressible homogeneous fluid with an account for the nonlinear cooperative processes. The goal of this study is an attempt to construct a phenomenological hydrodynamic model of stationary non-equilibrium turbulence using the methods of extended irreversible thermodynamics with internal variables[14,15] and statistical thermodynamics of irreversible processes[16,17]. It turned out very efficient in the study of different physical problems, by taking into account the nonlinear cooperative processes in them giving rise to various non-equilibrium dissipative vortex structures far from thermodynamic equilibrium[18]. In other words, we will attempt to obtain in phenomenological way a closed system of averaged hydrodynamic equations which would described self-consistently the various turbulent transport and self-organization processes. In this approach, the vortex CSs belonging to the strongly localized regions of small-scale motion of the subsystem of turbulent chaos should be taken into account both at the stage of laminar flow turbulization due to the development of a hierarchy of some type instabilities (e.g., the Kelvin–Helmholtz instability in free shear flows — in mixing layers, jets, wakes, etc., or the Taylor–Gertler instability in the sublayer of a turbulent boundary layer where longitudinal vortices can appear in near-wall layers) and at the stage of developed turbulence given all possible resonant situations associated, for example, with the synchronization of small-scale vorticity in a strongly turbulized fluid flow. Herewith, by the synchronization of regular and chaotic (stochastic) oscillations we mean the establishment of some relations between the characteristic frequencies and phases of self-oscillating systems as a result of their mutual influence.

We will also discuss the synergetic approach to constructing a phenomenological model of the developed turbulence in a compressible homogeneous fluid by taking into account the nonlinear cooperative processes. Including a set of random variables in the model as the internal parameters of the subsystem of turbulent chaos related to its microstructure allows us in this case to derive the Fokker–Planck–Kolmogorov (FPK) kinetic equations in configuration space by thermodynamic methods. These equations are designed to determine the temporal evolution of the conditional probability density for the random structural parameters of chaos (referring, in particular, to the cascade fragmentation of large-scale vortices or temperature inhomogeneities). They will be also used to analyze the Markovian stochastic processes of the transition from one quasi-non-equilibrium state of turbulent chaos to another through a successive loss of fluid flow stability when varying the corresponding control parameters. As it will be shown, stabilization of the subsystem of chaos near a subsequent stationary-non-equilibrium state in configuration space corresponds to the transition of turbulized system to a new state appropriate to the formation of complex spatiotemporal CSs in a turbulized flow.

2 SYSTEM OF HYDRODYNAMIC EQUATIONS OF THE MEAN MOTION FOR A SINGLE-COMPONENT COMPRESSIBLE FLUID

Developed turbulence is known to be realized at the finite but fairly large Reynolds number and is characterized by continuous Fourier spectra (both temporal and spatial ones) suggesting an existence of multiscale structure in the field of hydro-thermodynamic parameters. Basically, multiscale structure of a fluid flow when an enormous number of degrees of freedom are excited is a key signature of the developed turbulence. Therefore, any model approach to describing fully developed turbulence is regarded as one or another way of limiting the degrees of freedom.

In this study, when phenomenologically constructing the model of structured turbulence (aimed at describing "regular" fields of hydro-thermodynamic parameters), we will represent a moving fluctuating fluid as a hydro-thermodynamic system consisting of two interrelated continua (subsystems) that simultaneously fill the same volume of coordinate space continuously − the subsystem of averaged motion and the subsystem of turbulent spatiotemporal chaos[19-21]. The continuum of averaged motion obtained by the probability-theoretical averaging of the instantaneous hydrodynamic equations for a fluid is designed mainly to investigate the evolution of the averaged fields of hydro-thermodynamic parameters (including the description of large vortex structures). In turn, the subsystem of turbulent chaos (turbulent "superstructure") consists of two components (phases): proper turbulent chaos (the so-called incoherent turbulence) associated with the stochastic small-scale fluctuating motion of a turbulized fluid, and a coherent component associated with mesoscale vortex structures in a turbulized fluid flow embedded in this almost uniform (fine-grained) fluctuating field of t hydro-thermodynamic parameters and generally having the topology of a fractal set. In the thermodynamic description of the subsystem of turbulent chaos, we will include the set of internal coordinates that ultimately correspond to excited macroscopic degrees of freedom of a turbulized fluid. This makes it possible to use the generalized Onsager formalism when modeling the processes of turbulent transport and kinetics in the total continuum. In the case under consideration, this theory describes not only the linear relaxation of the averaged extensive thermodynamic parameters to their stationary values but also the behavior of the turbulent fluctuations in the neighborhood of stationary non-equilibrium states of chaos[17]. By applying the well-known extension of the formalism of non-equilibrium thermodynamics to systems with internal degrees of freedom[22], we can obtain the FPK kinetic equations for the distribution functions of the characteristic parameters of small-scale turbulence and on their basis to model the Richardson–Kolmogorov cascade process.

It should be noted that our separation of a real fluctuating fluid flow into imaginary (averaged and turbulent) ones, in general, depends on the choice of the spatiotemporal region for which the mean values of the local physical variables that are continuous functions of the coordinates $\mathbf{r}$ and time t have been established, i.e., to some extent, it is arbitrary. Below, we will assume the hydrodynamic scale of averaged motion L (the Obukhov[23,24] scale of observation or the resolution step of the finite-difference grid) lying in the inertial range $\eta < \Lambda << L$ and determined by the size $d\mathbf{r} \sim \Lambda^3$ of the averaging region G to be such that the subsystem of turbulent chaos contains the entire set of mesoscale CSs whose size is smaller than the averaging region, $l < \Lambda$. Here, $\eta = (\nu^3 / \varepsilon_b)^{1/4}$ is the Kolmogorov length scale that characterizes the effect of viscous dissipation on the structure of small-scale turbulence; L is the external or integral scale of turbulence characterizing its generation mechanism; ε_b is a key parameter of the Richardson–Kolmogorov cascade vortex fragmentation process that is the mean dissipation rate of turbulent energy per unit fluid mass per unit time and simultaneously (under quasi-equilibrium conditions) the transfer rate of the kinetic energy of fluctuating motion over the hierarchy of vortices. According to the existing estimates, for the averaged flow to contain the bulk (80 or 90 %) of the total energy of a turbulized flow, the averaging scale Λ must be smaller than the integral turbulence scale L by a factor of 10–20. Note that the specificity of the two-phase structure of turbulent chaos manifests itself in the additional turbulent momentum and energy transport by vortex coherent structures. This slightly modifies the known closure models (defining relations) and necessitates refining the effective (given the presence of CSs in the flow) turbulent viscosity and thermal conductivity coefficients.

We will follow the classical approach to the phenomenology of fully developed turbulence. It is based on Reynolds' idea of averaging the instantaneous equations of fluid motion for fluctuating hydro-thermodynamic parameters over space and/or time or through a different equivalent procedure, for example, through the probability-theoretical averaging over an ensemble of statistically similar hydrodynamic systems under identical external conditions adopted in statistical hydrodynamics[25]. Under the standard in statistical physics assumption about the system's ergodicity, when the time (space) average of any physical variable can be identified with its probability-theoretic average, these averaging's filter out the modes of motion whose scale is smaller than the spatiotemporal averaging interval. These small-scale fluctuating motions excluded in the averaging process are assumed to contribute to the turbulent fluid motion determined by the fluctuations of hydrodynamic parameters with respect to the corresponding average values. Actually, it is this small-scale turbulent motion that is modeled below by the subsystem of turbulent chaos.

It is important to notice that the averaging problem is a central one in the mechanics of natural media. In the case of such a complex system as structured turbulence, the construction of its macroscopic model itself often depends precisely on the averaging method. Bearing in mind the various applications of the turbulence model being developed, in particular, to astrophysical phenomena in which the ratio of the characteristic fluid velocity to the average speed of sound (a measure of significance of the fluid density fluctuations) is much greater than unity, we will assume the system's mass density $\rho(\mathbf{r},t)$ to be variable. As is known, the averaging's in classical theories of turbulence with a constant mass density usually for all physical parameters of the medium without exception are introduced in some identical way, as a rule, without any weight coefficients. At the same time, such an identical averaging for all physical parameters in the case of a fluid with a variable mass density leads not only to cumbersome hydrodynamic equations of mean motion but also to difficulties in the physical interpretation of some individual terms in them. Therefore, when constructing a phenomenological model of structured turbulence in a compressible medium, in addition to the "ordinary" means $f(\mathbf{r},t)$ of some hydro-thermodynamic variables $\overline{f(\mathbf{r},t)}$ (e.g., the density or pressure), we will also use the Favre means[26] for some other parameters $g(\mathbf{r},t)$ (e.g., the temperature or hydrodynamic velocity) specified by the relation

$$\langle g(\mathbf{r},t)\rangle = \overline{\rho g(\mathbf{r},t)}/\overline{\rho} = \lim_{M\to\infty}\frac{1}{M}\sum_{p=1}^{M}\rho^{[p]}g^{[p]} \Big/ \lim_{M\to\infty}\frac{1}{M}\sum_{p=1}^{M}\rho^{[p]} \ . \tag{1}$$

Here, the summation is over the set of possible realizations $[p]$ $(1\le[p]\le M)$ [or a stochastic hydrodynamic system; $f(\mathbf{r},t)=\overline{f}+f'$, $g(\mathbf{r},t)=\langle g\rangle+g''$, where g'' and f' are the corresponding turbulent fluctuation, with $\overline{g''}\neq 0$ and $\overline{f'}=0$. Below, unless stated otherwise, the letter in angle brackets will denote the weighted-mean averaging of the corresponding physical quantity. Let us write out here the properties of the weighted-mean averaging of physical quantities that are often used in this paper:

$$\overline{\rho g f}=\overline{\rho}\langle g\rangle\langle f\rangle+\overline{\rho g'' f''}, \quad \overline{\rho g''}=0, \quad \overline{g''}=-\overline{\rho' g''}/\overline{\rho},$$

$$\overline{\rho\left\{\frac{\partial f}{\partial t}+\mathbf{u}\cdot\frac{\partial f}{\partial \mathbf{r}}\right\}}=\overline{\rho}\left\{\frac{\partial\langle f\rangle}{\partial t}+\langle\mathbf{u}\rangle\cdot\frac{\partial\langle f\rangle}{\partial \mathbf{r}}\right\}+\frac{\partial}{\partial\mathbf{r}}\left(\overline{\rho f''\mathbf{u}''}\right)=\overline{\rho}\frac{D\langle f\rangle}{Dt}+\frac{\partial}{\partial\mathbf{r}}\mathbf{J}_f^{turb},$$

$$\frac{D}{Dt}(..)=\frac{\partial}{\partial t}(..)+\langle\mathbf{u}\rangle\cdot\frac{\partial}{\partial\mathbf{r}}(..).$$

Here, the turbulent flux of an attribute $f(\mathbf{r},t)$ is denoted by

$$\mathbf{J}_f^{turb}(\mathbf{r},t) \equiv \overline{\rho f''\mathbf{u}''} = \overline{\rho}\langle f''\mathbf{u}''\rangle .$$

These relations can be easily derived from definition (1) and the Reynolds averaging postulates.

The system of exact hydrodynamic equations of mean motion for a single component compressible fluid obtained by the probability-theoretical averaging of the corresponding instantaneous hydrodynamic equations which is valid on the microscale is as follows:

$$\overline{\rho}\frac{D\langle v\rangle}{Dt} = div\langle\mathbf{u}\rangle\,,\quad (\langle v\rangle = 1/\overline{\rho}), \tag{2}$$

$$\overline{\rho}\frac{D\langle\mathbf{u}\rangle}{Dt} = -\frac{\partial\overline{p}}{\partial\mathbf{r}} + \left(\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{P}^{\Sigma}\right) + \overline{\rho}\mathbf{F}\,, \tag{3}$$

$$\overline{\rho}\frac{D\langle E\rangle}{Dt} = -div\left(\overline{\mathbf{q}} + \mathbf{q}^{turb} - \overline{p'\mathbf{u}''}\right) - \overline{p}div\langle\mathbf{u}\rangle + \left(\overline{\mathbf{P}} : \frac{\partial\langle\mathbf{u}\rangle}{\partial\mathbf{r}}\right) -$$

$$- \overline{p'div\,\mathbf{u}''} + \left(\mathbf{J}_v^{turb}\cdot\frac{\partial\overline{p}}{\partial\mathbf{r}}\right) + \overline{\rho}\langle\varepsilon_b\rangle\,, \tag{4}$$

$$\overline{p} = \overline{\rho}\mathrm{R}\langle T\rangle. \tag{5}$$

Here, $\mathbf{u}(\mathbf{r},t)$; $p(\mathbf{r},t)$, $v(\mathbf{r},t)$; $E(\mathbf{r},t)$ are, respectively, the instantaneous values of the hydrodynamic velocity, pressure, specific volume ($v = 1/\rho$), and specific internal energy for a fluid particle; $\langle\mathbf{u}\rangle = \overline{\rho\mathbf{u}}/\overline{\rho}$ is the Favre-averaged hydrodynamic velocity of the medium; $D(..)/Dt = \partial(..)/\partial t + \langle\mathbf{u}\rangle\cdot\partial(..)/\partial\mathbf{r}$ is the total time derivative relative to the averaged velocity field; $\overline{p}(\mathbf{r},t)$ is the average pressure; R is the gas constant; $\mathbf{F}(\mathbf{r},t)$ is the external force acting on a unit mass (in this section, we neglect the fluctuations in mass force); $\mathbf{P}^{\Sigma}(\mathbf{r},t) = \overline{\mathbf{P}}(\mathbf{r},t) + \mathbf{R}(\mathbf{r},t)$ is the total stress tensor in a turbulent flow; $\overline{\mathbf{P}}(\mathbf{r},t)$ is the average molecular viscous stress tensor; $\mathbf{R}(\mathbf{r},t) \equiv -\overline{\rho\mathbf{u}''\mathbf{u}''}$ is the turbulent (Reynolds) stress tensor; $\overline{\mathbf{q}}(\mathbf{r},t)$ is the average molecular heat flux; $\mathbf{q}^{turb}(\mathbf{r},t) = \overline{\rho H''\mathbf{u}''}$ and $\mathbf{J}_v^{turb}(\mathbf{r},t) = -\overline{\rho'\mathbf{u}''}/\overline{\rho}$ are, respectively, the turbulent heat and specific volume fluxes (here, we use the relation $\rho v'' = -\rho'/\overline{\rho}$; $H(\mathbf{r},t) = E + p/\rho$ is the instantaneous value of the specific fluid enthalpy; $\langle\varepsilon_b(\mathbf{r},t)\rangle \equiv \overline{\rho\varepsilon_b}/\overline{\rho} = \overline{(\mathbf{P} : \partial\mathbf{u}''/\partial\mathbf{r})}/\overline{\rho}$ is the weighted-mean value of the specific dissipation rate of turbulent kinetic energy into heat due to molecular viscosity ν.

It can be seen from the system of equations (2)–(5) that the averaged motion of a turbulized homogeneous fluid is characterized by: 1) the average molecular thermodynamic fluxes $\overline{\mathbf{q}}(\mathbf{r},t)$ and $\overline{\mathbf{P}}(\mathbf{r},t)$ for which the corresponding defining relations are needed (they are derived for a turbulized medium using the thermodynamic approach, for example[27]); and 2) the as yet indefinite mixed one-point one-time correlations (second-order moments) $\mathbf{R}(\mathbf{r},t)$,

$\mathbf{q}^{turb}(\mathbf{r},t)$, and $\mathbf{J}_v^{turb}(\mathbf{r},t)$, which represent the so-called turbulent fluxes of the fluid characteristics related to the fluctuations of hydro-thermodynamic parameters. The correlation terms including pressure fluctuations $\overline{p' div\, \mathbf{u}''}$ and $\overline{div(p'\mathbf{u}'')}$ as well as the weighted-mean viscous dissipation rate of turbulent energy $\langle \varepsilon_b \rangle$ should also be determined. When the model of developed turbulence is constructed phenomenologically, the defining (constitutive) relations closing the system (2)–(5) can be established by the same method as in the laminar case, i.e., in accordance with the thermodynamic rules of continuum mechanics by the Onsager method. However, the additional basic idea that the corresponding thermodynamic forces are also responsible for the linear relaxation of the fluctuating characteristics of turbulized chaos to its stable stationary non-equilibrium state[17] will be further used as well.

3 THERMODYNAMICS OF THE STRUCTURED TURBULENCE. INTERNAL FLUCTUATING COORDINATES OF THE SUBSYSTEM OF TURBULENT CHAOS

An important task we face is to suggest a concept that would allow us to go beyond the classical formalism of irreversible thermodynamics. This goal can be achieved by expanding the space of independent basic variables by introducing the internal coordinates defining the microstructure of the subsystem of turbulent chaos. The subsequent step involves finding the evolutionary equations for these additional state parameters.

Within the framework of the complete model of structured turbulence, the system of equations (2)–(5) obtained by the probability-theoretical averaging of the instantaneous hydrodynamic equations for a single-component fluid is designed to study the spatiotemporal evolution of the averaged fields of hydrodynamic quantities, including various large vortex structures. Following Prigogine's viewpoint on hydrodynamic turbulence as a macroscopically highly organized flow we will address the subsystem of turbulent (vortex) chaos as a continuum with a certain internal microstructure. Moreover, recalling the above said, we will assume that the vortex continuum consists of two components: proper turbulent chaos (the so called incoherent turbulence) associated with the stochastic small-scale fluctuating motion of a turbulized fluid, and a coherent component embedded in this almost uniform fluctuating field of hydrodynamic parameters. The latter component is an ensemble of mesoscale vortex structures (multi-molecular structures) whose images in the phase space of an equivalent dynamical system are classical attractors (e.g., limit cycles) or strange attractors (having a fractal structure). Each of these two subsystems is thermodynamically open, i.e., capable to exchange energy and entropy (but not mass) with the adjacent subsystem. We also assume the hydrodynamic velocity fields for the subsystems of averaged motion and turbulent chaos to be coincident, because no separation of the corresponding Lagrangian volumes (diffusion effect) occurs in the process of a real turbulent fluid motion, i.e. the subsystem of turbulent chaos has no macroscopic hydrodynamic velocity relative to the subsystem of averaged motion. Let us note that a different approach to modeling structured turbulence associated, in particular, with the triple decomposition of the instantaneous motion of a fluid into an averaged motion and coherent and incoherent fluctuations is also known. The hydrodynamic equations with double averaging over time and over a specially chosen ensemble determined by some features characteristic of CSs, serve as a basis for such models. However, this procedure encounters some internal inconsistencies[8].

It should be emphasized once again that our artificial separation of a real turbulized fluid flow into imaginary averaged and turbulent (vortex, fluctuating) ones is just a way of pictorially describing the phenomenon to be convenient for modeling. For any elementary volume $d\mathbf{r}$ of the medium in each of these subsystems, we determine local "coarse-grained" thermodynamic parameters (which are continuous functions of the coordinates $\mathbf{r}$ and time t), such as density, pressure, temperature, internal energy, and entropy[i]. Furthermore, we additionally characterize the subsystem of turbulent chaos by a number of internal coordinates that are ultimately related to its microstructure. Note that such primary concepts as the generalized temperature and entropy of the subsystem of turbulent chaos have no precise physical interpretation and they are introduced only to ensure coherence of the theory[15].

It is also assumed that the generalized thermodynamic state parameters characterizing the stationary non-equilibrium vortex structure of turbulent chaos are related by ordinary relations in the local equilibrium thermodynamics such as the Gibbs and Gibbs–Dugham relations. They serve exclusively as a constraint on the form of the derived defining (constitutive) relations. In other words, such identities also remain valid far from local thermodynamic equilibrium of the subsystem of chaos provided that chaos is in a stable stationary non-equilibrium state choosing as a reference one. This fundamental assumption is a kind of new postulate[22] underlying the thermodynamic approach to description of the fully developed turbulence. One should be aware that since the energy of turbulent motions continuously dissipates due to molecular viscosity, it is impossible to reach the situation when the subsystem of turbulent chaos comes to a local statistically equilibrium state in principle. At the same time, for a stationary turbulized fluid flow, where the viscous energy dissipation is, on average, compensated for by the energy from the external source on a long time scale, the stationary non-equilibrium transport processes in the subsystem of turbulent chaos are affordable and do not differ much in physical sense from the local equilibrium processes in a dissipative system. Let us note that in the case under consideration, the *H*-theorem is valid since any initial state of turbulent chaos approaches a stationary non-equilibrium state in sufficiently long time.

In constructing our turbulence model, key concept of the Kolmogorov[28] theory was used. In the limit of very large $\mathbf{Re} = Lu_0 / \nu$ and Peclet $\mathbf{Pe} = L_T u_{T0} / \chi$ numbers corresponding to large-scale motions in anisotropic, inhomogeneity, and non-stationary averaged flow, the randomness of the fragmentation of vortices (or macrostructural temperature inhomogeneities) and the chaoticity of their energy transfer over the cascade downward are assumed to cause the statistical regime of turbulent fluctuations within the small spatiotemporal averaging region G of the instantaneous values of hydrodynamic parameters to be almost locally isotropic−homogeneous, isotropic, and quasi-stationary. This allows us to varying only with control parameters and primarily with the Reynolds number $\mathbf{Re}$ which eventually determines the number of cascades in the hierarchy of vortices of various order. Basically, there can be no

[i] In this connection, it is pertinent to recall the following: according to Onsager[29], the methods of statistical mechanics can be used to describe turbulent chaos in which different-scale vortices are well mixed and, hence, the methods of statistical thermodynamics of irreversible processes are also applicable.

complete local isotropy due to the presence of mesoscale vortex structures. Here, u_0 and u_{T0} are typical changes in the mean velocity at distances $\sim L$ and L_T, respectively; χ and ν are the molecular thermal diffusivity and kinematic viscosity, respectively; L_T is the distance at which the mean temperature changes noticeably. It is also assumed, for simplicity, that the Prandtl number $\mathbf{Pr} = \nu / \chi$ is of the order of unity and $L \approx L_T$; in this case, boundaries of the inertial and convective ranges within which the molecular viscosity and molecular heat conduction effects are significant may be considered to be coincident.

Let us now use the methods of extended irreversible thermodynamics[15] and non-equilibrium statistical thermodynamics[17] to obtain the defining (closing) relations for the thermodynamic fluxes and forces that describe the various turbulent transport processes in coordinate space and self-organization processes in phase space with an efficiency sufficient for practical purposes.

3.1 Thermodynamics of the Subsystem of Averaged Motion

We will begin by analyzing the balance equations for the average entropy $\langle S\rangle(\mathbf{r},t)$ of a turbulized homogeneous fluid. The probability-theoretical averaging of the Gibbs identity, which is assumed to be valid for micromotions

$$T\delta S = \delta E - p\delta v ,$$

where $T(\mathbf{r},t)$, $S(\mathbf{r},t)$ are the instantaneous values of the absolute temperature and specific entropy in a fluid particle, respectively, leads to the fundamental Gibbs identity for the subsystem of averaged motion. This identity written along the averaged trajectory of the center of mass of a physically elementary volume $d\mathbf{r}$ takes the form

$$\frac{D\langle S\rangle}{Dt} = \frac{1}{\langle T\rangle}\frac{D\langle E\rangle}{Dt} + \frac{p}{\langle T\rangle}\frac{D\langle v\rangle}{Dt} . \tag{6}$$

Identity (6) can be rewritten in the form of a local balance equation for the system's average entropy $\langle S\rangle(\mathbf{r},t)$ if we eliminate the substantial time derivatives of the parameters $\langle E\rangle(\mathbf{r},t)$ and $\langle v\rangle(\mathbf{r},t)$ from it using the averaged hydrodynamic equations (2) and (4). This results in

$$\bar{\rho}\frac{D\langle S\rangle}{Dt} + div\left(\frac{\mathbf{q}^{\Sigma}}{\langle T\rangle}\right) = \sigma_{\langle S\rangle} \equiv \sigma^{(i)}_{\langle S\rangle} + \sigma^{(e)}_{\langle S\rangle} , \tag{7}$$

where

$$0 \le \sigma^{(i)}_{\langle S\rangle}(\mathbf{r},t) \equiv \frac{1}{\langle T\rangle}\left\{-\left(\mathbf{q}^{\Sigma}\cdot\frac{\partial \ln\langle T\rangle}{\partial \mathbf{r}}\right) + \bar{\pi}\, div\langle\mathbf{u}\rangle + \left(\overset{\circ}{\bar{\mathbf{P}}}{}^{s} : \overset{\circ}{\bar{\mathbf{D}}}\right)\right\}, \tag{8}$$

$$\sigma^{(e)}_{\langle S\rangle}(\mathbf{r},t) \equiv \frac{1}{\langle T\rangle}\left\{-\overline{p' div\mathbf{u}''} + \left(\mathbf{J}^{turb}_{(v)} \cdot \frac{\partial \overline{p}}{\partial \mathbf{r}}\right) + \overline{\rho}\langle \varepsilon_b\rangle\right\} \equiv \frac{\Im}{\langle T\rangle}. \tag{9}$$

Here, $\overset{\circ}{\overline{\mathbf{P}}}{}^{s}(\mathbf{r},t)$ is the (average) shear rate tensor and $\overline{\pi}(\mathbf{r},t) \equiv \frac{1}{3}\overline{\mathbf{P}} : \mathbf{I}$ is the average viscous pressure. The positive quantity $\sigma^{(i)}_{\langle S\rangle}(\mathbf{r},t)$ defines the local production rate of the system's average entropy $\langle S\rangle(\mathbf{r},t)$ due to dissipative transport processes inside the subsystem of averaged fluid motion; as will be seen from the subsequent analysis, the quantity $\sigma^{(e)}_{\langle S\rangle}(\mathbf{r},t) \equiv \Im / \langle T\rangle$ (the entropy sink or source) reflects the entropy exchange between the subsystems of turbulent chaos and averaged motion. It is important to note that the quantity $\sigma^{(e)}_{\langle S\rangle}(\mathbf{r},t)$ can be different in sign depending on the specific regime of fluid motion. Indeed, the turbulent energy dissipation rate $\langle \varepsilon_b\rangle(\mathbf{r},t)$ is always positive quantity. However, the energy transition rate $\overline{p' div\mathbf{u}''}$ (the work done on turbulent vortices by the environment per unit time per unit volume due to pressure fluctuations and the expansion ($div\mathbf{u}'' > 0$) or compression ($div\mathbf{u}'' < 0$) of turbulent vortices) can be different in sign. The $\mathbf{J}^{turb}_{(v)} \cdot (\partial \overline{p} / \partial \mathbf{r})$ is positive for small-scale turbulence, but it can be both positive and negative for large-scale and mesoscale vortices[11]. Thus, it follows from (7) that, generally, the average entropy of a turbulized medium $\langle S\rangle(\mathbf{r},t)$ can both increase and decrease. This is a characteristic feature of any thermodynamically open systems.

This implies that there is some internal openness in an externally closed turbulized system modeled by two continua. It stems from the fact that the averaged motion of a turbulized fluid is described only by one of the two continua. At the same time, each physically infinitesimal volume element $d\mathbf{r} \sim \Lambda^3$ (where Λ is the averaging scale) of the medium is still assumed to be so large that the additional information about the pattern of the fluctuating motion (turbulent superstructure) on scales smaller than or equal to the size of a "mathematical point" can be taken into account in the model. Hence it follows, in particular, that the average entropy alone is clearly not enough for an adequate description of all features of structured turbulence because this quantity is not related to any parameters characterizing the internal structure and thermodynamics of the subsystem of turbulent chaos and, in particular, to such a key parameter of the theory as the turbulence energy (the average fluctuation kinetic energy per unit mass of the medium)

$$\langle b\rangle(\mathbf{r},t) = \overline{\rho(\mathbf{u}'')^2} / 2\overline{\rho}\,. \tag{10}$$

For this reason, to macroscopically describe structured turbulence and, in particular, the cascade transport of turbulent energy by vortices of various scales (downward over the range of sizes), we will use a well-known generalization of the formalism of irreversible thermodynamics to media with an internal structure by introducing for this purpose generalized extensive thermodynamic parameters (internal energy $E_{turb}(\mathbf{r},t)$, generalized chemical potentials $\mu(\mathbf{q},\mathbf{r},t)$, etc.) of the subsystem of turbulent chaos related to the fluctuating fluid motion[30] and the so-called internal coordinates corresponding to the system's microstructure. In other words, we will proceed just as has long been done in non-equilibrium thermodynamics, for example, in order to take into account various transformations in the internal degrees of freedom of molecules, in particular, to take into account the orientation of polar molecules relative to the external electric field[ii] when the formalism of a generalized chemical potential equal to the standard chemical potential and the field term dependent on the internal coordinate ϑ is used[22].

3.2 Internal Coordinates of the Subsystem of Turbulent Chaos

Thus, when modeling the stochastic system corresponding to the subsystem of turbulent chaos, we use the formalism of generalized statistical thermodynamics[16,17]. It suggests the study of a statistical ensemble of macroscopically identical subsystems of chaos with the same generalized extensive state parameters such as the average specific volume $v(\mathbf{r},t)$, internal energy $E_{turb}(\mathbf{r},t)$, and entropy $S_{turb}(\mathbf{r},t)$ of chaos and some infinite sequence of internal variables $n(\mathbf{q},\mathbf{r},t)$. The internal variables $n(\mathbf{q},\mathbf{r},t)$ can be the number densities of small-scale vortices or temperature inhomogeneities in states characterized by specified values of the parameters $\mathbf{q}$ − the internal coordinates defining the system's microstructure. We assume that the vortex structures of chaos are somehow localized in both coordinate space $\mathbf{r}$ and configuration space $\mathbf{q}$ .The internal coordinates $q_k(\mathbf{r},t)$ ($k=1,...n$), which are some characteristics of the ensemble of vortex (or temperature) chaos corresponding to small-scale turbulent fluctuations, are generally random (stochastic) variables fluctuating relative to their mean (stationary) values q_k^{st}. They serve as a measure of the differences in any set of thermodynamically identical subsystems of the vortex ensemble. Allowance for the fluctuations in the internal coordinates describing the state of chaos in purely dynamical modeling refines its statistical description and leads to a more adequate reflection of reality.

The continuously changing local random parameters that adequately characterize the evolution of a turbulized fluid, including the spatiotemporal evolution of various mesoscale coherent structures, can be attributed to the stochastic internal coordinates describing the macroscopic state of turbulent chaos. Thus, some of the internal coordinates q_k can refer to the incoherent component of the subsystem of turbulent chaos, while others can characterize individual CSs. In particular, the following positive definite quantities (or their logarithms) can be chosen as the stochastic coordinates $q_k(\mathbf{r},t)$: the kinetic energy of vortices, $b=|\mathbf{u}''|^2/2$; the dissipation rate of turbulent energy into heat, $\varepsilon(\mathbf{r},t)=\frac{1}{2}\nu\sum_{i,j}(\partial u_i''/\partial r_j+\partial u_j''/\partial r_i)^2$; the scalar dissipation of temperature inhomogeneities, $\varepsilon_T(\mathbf{r},t)=\chi\sum_j(\partial T''/\partial r_j)^2$; the mixing rate of a substance with a concentration $\theta(\mathbf{r},t)$ to the molecular level (which does not affect the flow

[ii] As it is well known, this can be done by introducing an internal coordinate ϑ, the angle between the field and dipole directions.

dynamics), $\varepsilon_\theta(\mathbf{r},t)=\chi\sum_j(\partial\theta''/\partial r_j)^2$ [iii]; the system's enstrophy (in the case of 2D turbulence); the mean vorticities of the field of velocity fluctuations referring to k-th type mesoscale vortex structures (the fundamental quantities to characterize CSs), etc.

As will be further shown, using the internal coordinates as additional macroscopic parameters of turbulent chaos allows us to obtain thermodynamically the evolutionary Fokker–Planck–Kolmogorov (FPK) equations in the space of configurations $\mathbf{q}$ given in this approach Prigogine's central postulate concerning the direction of irreversible processes in any local volume of the space of internal coordinates[22] (see *Chap. 3, Sect. 11*). These kinetic equations are designed to determine the temporal evolution of the probability density function for various stochastic small-scale turbulence characteristics. They also allow one to analyze the conditions for the transition from one stable stationary non-equilibrium state of turbulent chaos to another that are eventually caused by a successive loss of fluid flow stability when changing the parameters controlling the regime of turbulent motion as a whole.

The purely classical (i.e., without any introduction of internal stochastic coordinates) thermodynamic approach to modeling turbulence seems not quite adequate in the case of a structured vortex continuum, because any two realizations of the ensemble (the set of subsystems of chaos with the same set of extensive thermodynamic state parameters) are identical in all respects when applying it, which does not correspond to the real situation. This is attributable to turbulent fluctuations in the internal coordinates of the state of chaos, which ultimately serve as a measure of the differences in any ensemble of thermodynamically identical subsystems of chaos. Actually, it is these turbulent fluctuations which are not suppressed under strongly non-equilibrium conditions but, on the contrary, are enhanced in certain situations by internal irreversible processes within the vortex subsystem at the so called bifurcation points (at which the subsystem "can choose" between various states), that lead to various mechanical manifestations of a real turbulized flow. In particular, the subsystem of turbulent chaos in some stable stationary state (far from the local thermodynamic equilibrium) at certain values of control parameters can shift to a new stationary state with a neutral stability (associated with the so-called critical point of stability loss) and, subsequently, abruptly pass to another asymptotically stable stationary state corresponding to one or another form of the supramolecular coherent behavior of a large number of particles (e.g., the oscillations of different scale vortices). Here it worth to recall that, according to Prigogine[2], this ability to bring order via fluctuations is a fundamental property of any open strongly non-equilibrium thermodynamic systems. Thus, because of the continuous turbulent fluctuations, it is convenient to imagine any quasi-stationary state of turbulent chaos as a state of not one individual subsystem but the whole physical ensemble of subsystems. That is why it is necessary to refine the thermodynamic description of a turbulized flow so as to be able to take into account in modeling the effects of stochasticity of the vortex continuum[20].

Thus, in accordance with the stochastic-thermodynamic approach we assume that for a complete statistical description of the random vector process $\mathbf{q}(\mathbf{r},t)$ in a turbulized continuum (the set of structural small-scale characteristics of chaos $q_k(\mathbf{r},t)$, where $k=1,...n$, which

[iii] The quantity ε_θ is a measure of the concentration field inhomogeneity disappearing per unit time through molecular diffusion $D\approx\chi$.

is often convenient to gather into one column vector $\mathbf{q}$ in n-dimensional configuration space), it is sufficient to keep the one-point probability density $W_1(\mathbf{q},t)$ and the joint two-point probability density $W_2(\mathbf{q}_0,t_0;\mathbf{q},t)$. The random processes that are completely described only by these two distribution functions are known to be the Markovian ones. This key assumption determines the class of random processes (turbulent fluctuations) to which the analyzed stochastic-thermodynamic model of developed turbulence is applicable. We will also use below the two-point conditional probability density, $P_2(\mathbf{q}_0,t_0;\mathbf{q},t)$, which allows the probable value of the parameter $\mathbf{q}$ at time t to be found if $\mathbf{q}=\mathbf{q}_0$ at time t_0 with a probability equal to unity. These probability densities can be used to obtain the mean values of various functions $f(\mathbf{q}(t))$ of the random state vector $\mathbf{q}(t)$; in particular, the formula $\overline{f(\mathbf{q}(t))}=\int f(\mathbf{q}(t))W_1(\mathbf{q},t)d\mathbf{q}$ defines the unconditional mean of $f(\mathbf{q}(t))$ and the formula $\overline{f(\mathbf{q}(t))}^0=\int f(\mathbf{q})P_2(\mathbf{q}_0|\mathbf{q},t)d\mathbf{q}$ introduces the mean of $f(\mathbf{q}(t))$ at time t provided that $f(\mathbf{q}(t_0))=f(\mathbf{q}_0)$ (conditional mean). The relationship between the means over the conditional sub-ensemble $\overline{f(\mathbf{q}(t))}^0$ and over the entire physical ensemble $\overline{f(\mathbf{q}(t))}$ is implicitly contained in the relation

$$P_2(\mathbf{q}_0|\mathbf{q},t)=W_2(\mathbf{q}_0,t_0;\mathbf{q},t)\,/\,W_1(\mathbf{q}_0,t_0),$$

which actually defines the so-called transition probability function P_2.

Our analysis is restricted to the so-called stationary physical ensemble of turbulent chaos consisting of adequate (in the above mentioned sense) sub-systems maintained by continuously acting external sources of turbulence in a state in which the random variables $\mathbf{q}(t)$ are invariant with respect to a shift along the time axis, i.e., $\mathbf{q}(t_p+\tau)=\mathbf{q}(t_p)$ for all p and τ. Clearly, in this case, the one-time probability density $W_1(\mathbf{q})$ will not depend on time, while the joint probability densities will be determined only by the difference $t-t_0$; for example, $P_2(\mathbf{q}_0,t_0|\mathbf{q},t)=P_2(\mathbf{q}_0,0|\mathbf{q},t-t_0)$. Bearing this in mind, we omit the initial time in the expressions for W_2 and P_2 and write them in an abridged form: $P_2(\mathbf{q}_0|\mathbf{q},t)$ etc. Also, in the case under consideration, the positive function P_2 has the following properties[17]:

$$\int P_2(\mathbf{q}_0|\mathbf{q},t)d\mathbf{q}=1, \quad \int W_1(\mathbf{q}_0)P_2(\mathbf{q}_0|\mathbf{q},t)d\mathbf{q}_0=W_1(\mathbf{q}),$$

$$\lim_{t\to\infty}P_2(\mathbf{q}_0|\mathbf{q},t)=W_1(\mathbf{q}), \qquad (11)$$

the latter relation implying that the conditional probability density for stationary processes ceases to depend on the initial condition asymptotically with time.

3.3 Basic Kinetic Equation

The key proposition of the Kolmogorov[28] theory for the generation of small-scale turbulence is that the excitation of vortex structures, their non-linear interactions, and the viscous dissipation of turbulent energy are strictly separated in the space of wave numbers, when the energy influx into a turbulized flow occurs near the wave number k_L corresponding to the turbulence macroscale L, while the energy dissipation becomes efficient near the wave number k_η where η is the turbulence microscale often called the Kolmogorov's scale. In other words, the existence of an inertial range of scales ($k_L << k << k_\eta$) is characteristic signature of the developed turbulence. The energy transfer from large-scale turbulent vortices to small-scale ones can be imagined as a cascade process of their fragmentation.[iv]

For our analysis, we need the so-called basic kinetic equation for the rate of change in the number of vortex moles $n(\mathbf{q})$ in the cascade interaction between turbulent motions of different scales or for the function $P_2(\mathbf{q},t) \equiv n(\mathbf{q})/n_\Sigma$ .The latter is the (conditional) probability density to detect the system in the interval $(\mathbf{q},\mathbf{q}+d\mathbf{q})$ at time t if it was in a state $\mathbf{q}^{st}$ at the initial time $t=0$ with a probability equal to unity. Here,

$$n_\Sigma(\mathbf{r},t) = \int n(\mathbf{q},\mathbf{r},t)d\mathbf{q} \tag{12}$$

is the total number of vortex structures with an attribute $\mathbf{q}$ in the volume $d\mathbf{r}$.

When deriving the equation for the rate of change in the number of vortex moles $n(\mathbf{q})$ in the cascade interaction between turbulent vortices of different scales, we will assume that the cascade vortex fragmentation mechanism associated with the transition of kinetic energy from large vortices to increasingly small ones is such that the medium "retains memory" only about the last transition (Markovian process). If the number density of vortices $n(\mathbf{q}_i)$ in a state $\mathbf{q}_i$ can be changed only through their transition from the neighboring states $\mathbf{q}_i$ (the decay of large vortices with attributes $\mathbf{q}_{k-1}$ into smaller vortices with attributes $\mathbf{q}_{i-1}$ or to the neighboring states $\mathbf{q}_{i+1}$ (the interaction locality principle), then we have $\partial n/\partial t + (J_i - J_{i-1}) = 0$, where J_i is the rate of the transition $i \to i+1$ and J_{i-1} is the rate of the transition $(i-1) \to i$. Since the distribution of vorticity in a real turbulized fluid flow is continuous[v] we also assume that this transfer of the kinetic energy of averaged motion over the Richardson–Kolmogorov cascade downward can be adequately described in terms of the change in random parameters $\mathbf{q}$ taking on continuous values. In particular, this corresponds to the cascade destruction of large vortices and the formation of small ones whereby only infinitesimal changes in quantities $\mathbf{q}$ occur in a single interaction, while finite changes arise from the cumulative action of a large number of vortex interactions. In terms of chemistry, this process can be considered as a

[iv] The idea of an energy cascade was first put forward by L. Richardson in 1922.

[v] As it is well known, there is a continuum of excited degrees of freedom in the case of developed turbulence[31].

series of consecutive chemical reactions expressed by the scheme $\ldots \to (k-1) \to k \to (k+1) \to \ldots$. Note that the principle of detailed balance can generally break down for thermodynamic systems far from complete thermal equilibrium; this is also true, in particular, for the stationary non-equilibrium cascade formation of vortex structures. Consequently, for a real cascade process, the equation for the rate of change in the number density of turbulent vortices $n(\mathbf{q},t)$ should be rewritten as the standard continuity equation

$$\partial n(\mathbf{q},t)/\partial t = -(\partial/\partial\mathbf{q})\cdot\mathbf{J}(\mathbf{q},t)$$

in the configurations space. Here, $\mathbf{J}(\mathbf{q},t)$ is the vortex flux in the inertial range corresponding to the number of "vortex moles" that pass from a state $\mathbf{q}$ to a state $\mathbf{q}+d\mathbf{q}$ in a unit time. In the more general case where the number density of turbulent vortices also depends on the spatial coordinate $\mathbf{r}$, the kinetic equation for the rate of change in $n(\mathbf{q},\mathbf{r},t)$ can be written as the continuity equation, both in the coordinate space $\mathbf{r}$ and in the space of internal coordinates $\mathbf{q}$,

$$\frac{\partial n(\mathbf{q},\mathbf{r},t)}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\cdot\left\{n(\mathbf{q},\mathbf{r},t)\langle\mathbf{u}\rangle\right\} = -\frac{\partial}{\partial\mathbf{q}}\cdot\mathbf{J}(\mathbf{q},\mathbf{r},t), \tag{13}$$

or, in view of (2), as

$$\bar{\rho}\frac{D}{Dt}\left(\frac{n(\mathbf{q},\mathbf{r},t)}{\bar{\rho}}\right) = -\frac{\partial}{\partial\mathbf{q}}\cdot\mathbf{J}(\mathbf{q},\mathbf{r},t).. \tag{13*}$$

Here, $\mathbf{J}(\mathbf{q},\mathbf{r},t)$ is the thermodynamic flux in the space of internal coordinates $\mathbf{q}$ to be determined.

3.4 Thermodynamics of the Subsystem of Structured Chaos

Thus, the transfer of turbulent energy over the cascade of vortices in the case of developed turbulence can be considered as a kind of chemical transformations with the corresponding chemical potential $\mu_{turb}(\mathbf{q},\mathbf{r},t)$ for the internal degrees of freedom $\mathbf{q}$ and the de Donder chemical affinity

$$\mathbf{A}_{turb}(\mathbf{q}) = -(\partial\mu_{turb}/\partial\mathbf{q}).$$

The latter can be treated as the driving force of the cascade process corresponding to one equivalent $n(\mathbf{q}) \to n(\mathbf{q}+\delta\mathbf{q})$ of the vortex fragmentation process. Note that the concept of chemical potential is distinguished by great generality: it is applicable to almost any model of a continuous medium if the concept of thermodynamic temperature (which is generally not the absolute temperature of the medium) can be introduced for it in one or another way.

Basically, extending the formalism of the generous chemical potential to stationary non-equilibrium turbulent chaos, we determine intensive thermodynamic parameters, such as the generalized turbulization temperature $T_{turb}(\mathbf{r},t)$ and pressure $p_{turb}(\mathbf{r},t)$ (which are not related in any way to the molecular temperature and pressure of the underlying flow). Similarly, the turbulent chemical potential $\mu_{turb}(\mathbf{q},\mathbf{r},t)$ for the internal degrees of freedom $\Phi_0(\mathbf{q})$ follows from the fundamental Gibbs relation for the generalized entropy (specified a priori as the characteristic function)

$$S_{turb}(\mathbf{r},t) = S_{turb}\left(E_{turb}(\mathbf{r},t),\ \langle v\rangle(\mathbf{r},t),\ n(\mathbf{q})/\bar{\rho}\right) \tag{14}$$

using the relations[32]

$$\frac{1}{T_{turb}} = \left(\frac{\partial S_{turb}}{\partial E_{turb}}\right)_{\langle v\rangle, n/\bar{\rho}}; \quad \frac{p_{turb}}{T_{turb}} = \left(\frac{\partial S_{turb}}{\partial(1/\bar{\rho})}\right)_{E_{turb}, n/\bar{\rho}};$$

$$\frac{\mu_{turb}(\mathbf{q})}{T_{turb}} = -\left(\frac{\partial S_{turb}}{\partial(n(\mathbf{q})/\bar{\rho})}\right)_{E_{turb}, 1/\bar{\rho}}. \tag{15}$$

The chemical potential $\mu_{turb}(\mathbf{q})$ for the internal degrees of freedom is generally defined as a functional derivative. The turbulization entropy $S_{turb}(\mathbf{r},t)$ is assumed to contain all thermodynamic information about the subsystem of stationary non-equilibrium turbulent chaos, i.e., it is related to the stability, fluctuations, and dynamical changes in a quasi-stationary state in exactly the same way as the equilibrium entropy of some thermodynamic system in a local equilibrium state[17]. A remarkable feature of the entropy $S_{turb}(\mathbf{r},t)$ is that many statistical properties of turbulent chaos in quasi-stationary states can be deduced from this quantity. In particular, it is admissible to interpret the various algebraic relations for the intensive variables $E_{turb}(\mathbf{r},t)$, $T_{turb}(\mathbf{r},t)$, $p_{turb}(\mathbf{r},t)$, and $\mu_{turb}(\mathbf{q},\mathbf{r},t)$ that can be derived in a standard thermodynamic way from (15) as specific "equations of state" for the subsystem of turbulent chaos.

It is worth to emphasize that the concepts of turbulization temperature and entropy are pertinent not only to the state near local thermodynamic equilibrium of the subsystem of turbulent chaos. On the contrary, these quantities are introduced with the goal of describing thermodynamically chaos in stationary non-equilibrium states. Indeed, in this approach, in contrast to the local equilibrium entropy of the subsystem of averaged motion $\langle S\rangle(\mathbf{r},t)$, the generalized turbulence entropy $S_{turb}(\mathbf{r},t)$ generally remains an indefinite quantity. There are no experimental or physical methods to establish its true functional dependence on state parameters. Hence, this quantity is introduced into the theory exclusively with the goal of ensuring its coherence, while the explicit form of the functional equation (14) is postulated depending on the goals of modeling approach.

The differential form of the fundamental Gibbs relation for the turbulization entropy $S_{turb}(\mathbf{r},t)$ written along the trajectory of the center of mass of a physically elementary volume $d\mathbf{r}$ takes the following form[33]

$$\frac{DS_{turb}}{Dt} = \frac{1}{T_{turb}}\frac{DE_{turb}}{Dt} + \frac{p_{turb}}{T_{turb}}\frac{D\langle v\rangle}{Dt} - \frac{1}{T_{turb}}\int_{\mathbf{q}} \mu_{turb}(\mathbf{q})\frac{D}{Dt}\left(n(\mathbf{q})/\bar{\rho}\right)d\mathbf{q}. \tag{16}$$

Here the internal energy of chaos $E_{turb}(\mathbf{r},t)$ is identified with the turbulence energy[20]

$$E_{turb}(\mathbf{r},t)=\langle b\rangle+const=\overline{\rho(\mathbf{u}'')^2}/2\overline{\rho}+const\,, \qquad (17)$$

and it is assumed that the subsystem of turbulent chaos is represented thermodynamically as an ideal gas with three degrees of freedom in which the energy is distributed uniformly. Then the following equations of state can be written in the form

$$\overline{\rho}\langle b\rangle(\mathbf{r},t)=\tfrac{3}{2}\mathsf{R}\overline{\rho}T_{turb}\,,\quad p_{turb}(\mathbf{r},t)=\mathsf{R}\overline{\rho}T_{turb}\,,$$

$$\mu_{turb}(\mathbf{q},T_{turb})=\mathrm{k_B}T_{turb}\ln n(\mathbf{q})+\Phi(\mathbf{q},T_{turb})\,, \qquad (18)$$

where $\mathsf{R}(=n_\Sigma\,\mathrm{k_B}/\overline{\rho})$ is the "gas constant" for the vortex continuum, $\mathrm{k_B}$ is the Boltzmann constant, and $\Phi(\mathbf{q},T_{turb})$ is the so-called potential energy in internal coordinate $\mathbf{q}$, which also generally depends on the turbulization temperature $T_{turb}(\mathbf{r},t)$.

An important caveat is that the potential energy $\Phi(\mathbf{q},T_{turb})$ can be eliminated from (18) using the equilibrium distribution $P_2^{st}(\mathbf{q})$ of internal coordinates $\mathbf{q}$ corresponding to some asymptotically stable stationary state of turbulent chaos specified *a priori*. As it was said above, since the energy of turbulent motions dissipates continuously due to viscosity, it is impossible to accomplish a statistically equilibrium state while, in contrast, the stationary state is usually dissipative. Indeed, as it is well known[33], as some chemically active molecular continuum passes to a stable stationary (but fairly close to equilibrium) state characterized by minimal entropy production, the entropy itself also decreases. By analogy with molecular chemically reacting systems, for a stationary state of the subsystem of turbulent chaos when the internal coordinates $\mathbf{q}$ can fluctuate near some stable stationary value $\mathbf{q}^{st}$ at certain (for a given elementary volume $d\mathbf{r}$) internal energy and specific volume, the turbulization entropy $S_{turb}(\mathbf{r},t)$ must also be minimal among all of the other states with the same values of $E_{turb}(\mathbf{r},t)$ and $\langle v\rangle(\mathbf{r},t)$. Thus, the following condition is valid:

$$\delta S_{turb}=-\frac{1}{\overline{\rho}T_{turb}}\int\limits_{\mathbf{q}}\mu_{turb}(\mathbf{q})\delta n(\mathbf{q})d\mathbf{q}=0\,,$$

where $\delta n(\mathbf{q})=n(\mathbf{q})-n(\mathbf{q}^{st})$. Since the total number of vortex moles n_Σ is constant, we also have $\int\delta n(\mathbf{q})d\mathbf{q}=0$. It follows from these two conditions that the chemical potential $\mu_{turb}(\mathbf{q})$ in internal coordinates for a stable stationary state of turbulent chaos does not depend on the state vector $\mathbf{q}$ ($\mu_{turb}^{st}=const$). Using this fact, we can obtain the expression

$$\mu_{turb}(\mathbf{q},t)=\mathrm{k_B}\,T_{turb}\ln\left\{n(\mathbf{q})/n(\mathbf{q}^{st})\right\}+\mu_{turb}^{st}\,, \qquad (19)$$

which allows the chemical potential to be determined from the stationary distribution $n(\mathbf{q}^{st})$ of an attribute $\mathbf{q}$ known in advance. Relation (19) can be rewritten as

$$\mu_{turb}(\mathbf{q},t) = \mathrm{k_B}\,T_{turb}\ln\left\{\frac{P_2(\mathbf{q}_0|\mathbf{q},t)}{W_1^{st}(\mathbf{q})}\right\} + \mu_{turb}^{st}\,. \qquad (19^*)$$

The latter is a generalization of the well-known Einstein formula (for a statistically equilibrium state) to quasi-stationary states in configuration space $\mathbf{q}$; in writing(19*), we used the expression

$$\mathbf{f}(\mathbf{q},T_{turb}) = -\frac{\partial}{\partial\mathbf{q}}\Phi(\mathbf{q},T_{turb}) = \mathrm{k_B}T_{turb}\frac{\partial}{\partial\mathbf{q}}\ln W_1^{st}(\mathbf{q}) \qquad (19^{**})$$

for the friction force (in configuration space $\mathbf{q}$) generated by the potential field $\Phi(\mathbf{q},T_{turb})$. The function

$$W_1^{st}(\mathbf{q}) = const\exp\left\{-\frac{\Phi(\mathbf{q},T_{turb})}{\mathrm{k_B}\,T_{turb}}\right\}$$

defines the (maximum) probability of a stable stationary state $\mathbf{q}^{st}$ when the fluctuating internal coordinates $\mathbf{q}$ remain constant, while the function $\Phi(\mathbf{q},T_{turb})$ acts as the thermodynamic potential for a stationary state.

Let us recall that no distinction is made between the two concepts of equilibrium in the equilibrium thermodynamics − the equilibrium state corresponding to maximum entropy and the equilibrium distribution in possible states that are physically almost equivalent[14]. A similar situation also holds for stationary states in the thermodynamics of non-equilibrium processes[17]. This is because the asymptotic probability densities of the states are concentrated in an extremely narrow region and these Gaussian quantities transform into delta functions concentrated in $\mathbf{q}^{st}$ in the thermodynamic limit.

Now, using (13*), let us transform the Gibbs identity (16) by integration by parts and assuming that the flux $\mathbf{J}(\mathbf{q},\mathbf{r},t)$ becomes zero at both boundaries $\mathbf{q}_1$ and $\mathbf{q}_2$ of the domain of definition of the variable $\mathbf{q}$ (a corollary of the condition $\int_{\mathbf{q}_1}^{\mathbf{q}_2}\delta n(\mathbf{q})d\mathbf{q} = 0$). This results in

$$\frac{DS_{turb}}{Dt} = \frac{1}{T_{turb}}\frac{DE_{turb}}{Dt} + \frac{p_{turb}}{T_{turb}}\frac{D\langle v\rangle}{Dt} - \frac{1}{T_{turb}}\int_{\mathbf{q}}\mathbf{J}(\mathbf{q})\cdot\frac{\partial\mu_{turb}(\mathbf{q},\mathbf{r},t)}{\partial\mathbf{q}}d\mathbf{q}\,. \qquad (20)$$

The last term in this relation

$$\frac{DS_{turb}}{Dt} = \frac{1}{T_{turb}}\frac{DE_{turb}}{Dt} + \frac{p_{turb}}{T_{turb}}\frac{D\langle v\rangle}{Dt} - \frac{1}{T_{turb}}\int_{\mathbf{q}}\mathbf{J}(\mathbf{q})\cdot\frac{\partial\mu_{turb}(\mathbf{q},\mathbf{r},t)}{\partial\mathbf{q}}d\mathbf{q} \qquad (21)$$

describes the total growth of the turbulization entropy $S_{turb}(\mathbf{r},t)$ due to irreversible processes of the formation of various vortex structures characterized by the complete set of internal coordinates $\mathbf{q}$. It can be seen from (21) that the local fluctuation entropy production $\sigma_{\mathbf{q}}(S_{turb})$ corresponding to each part of the space of internal coordinates $\mathbf{q}$ has an ordinary thermodynamic form:

$$\sigma_{\mathbf{q}}(S_{turb}) = -\frac{1}{T_{turb}}\mathbf{J}\cdot\frac{\partial\mu_{turb}}{\partial\mathbf{q}} = \frac{1}{T_{turb}}\mathbf{J}(\mathbf{q},\mathbf{r},t)\cdot\mathbf{A}_{turb}(\mathbf{q},\mathbf{r},t),$$

where

$$\mathbf{A}_{turb} = -\frac{\partial\mu_{turb}}{\partial\mathbf{q}} = \frac{\mathrm{k_B}\,T_{turb}}{n}\frac{\partial n}{\partial\mathbf{q}} - \frac{\partial\Phi}{\partial\mathbf{q}} = \frac{\mathrm{k_B}\,T_{turb}}{n}\exp\left(\frac{\Phi}{\mathrm{k_B}\,T_{turb}}\right)\frac{\partial}{\partial\mathbf{q}}\exp\left(\frac{\mu_{turb}}{\mathrm{k_B}\,T_{turb}}\right) \quad (22)$$

is the generalized de Donder chemical affinity for configuration $\mathbf{q}$ (the state function of the subsystem of turbulent chaos) written here by taking into account (15) for the generalized chemical potential $\mu_{turb}(\mathbf{q},\mathbf{r},t)$.

3.5 Transfer Equation for the Turbulization Entropy

Let us now derive the transfer equation for the turbulization entropy $S_{turb}(\mathbf{r},t)$ by applying the same procedure that led to (7). For this purpose, we will eliminate from (20) the substantial derivatives of the specific volume $\langle v\rangle(\mathbf{r},t)$ and turbulent energy $\langle b\rangle(\mathbf{r},t)$ $(\equiv E_{turb}(\mathbf{r},t))$ for which the differential equation[21] is

$$\bar{\rho}\frac{D}{Dt}\langle b\rangle = -div\mathbf{J}^{turb}_{\langle b\rangle} + \mathbf{R}:\frac{\partial\langle\mathbf{u}\rangle}{\partial\mathbf{r}} + \overline{p' div\mathbf{u}''} - \left(\mathbf{J}^{turb}_{v}\cdot\frac{\partial\bar{p}}{\partial\mathbf{r}}\right) - \bar{\rho}\langle\varepsilon_b\rangle.$$

As a result, we will have

$$\frac{\partial}{\partial t}(\bar{\rho}S_{turb}) + div\left(\bar{\rho}S_{turb}\langle\mathbf{u}\rangle + \mathbf{J}_{(S_{turb})}\right) = \sigma_{(S_{turb})} \equiv \sigma^{(i)}_{(S_{turb})} + \sigma^{(e)}_{(S_{turb})}, \quad (23)$$

where

$$\sigma^{(e)}_{(S_{turb})}(\mathbf{r},t) \equiv \frac{1}{T_{turb}}\left\{\overline{p' div\mathbf{u}''} - \left(\mathbf{J}^{turb}_{v}\cdot\frac{\partial\bar{p}}{\partial\mathbf{r}}\right) - \bar{\rho}\langle\varepsilon_b\rangle\right\} \equiv -\frac{\mathfrak{I}}{T_{turb}}, \quad (24)$$

$$0 \le \sigma^{(i)}_{(S_{turb})}(\mathbf{r},t) = \frac{1}{T_{turb}}\left\{-\left(\mathbf{J}^{turb}_{\langle b\rangle}\cdot\frac{\partial\ln T_{turb}}{\partial\mathbf{r}}\right) - p_{turb}div\langle\mathbf{u}\rangle + \right.$$

$$\left. + \overset{0}{\mathbf{R}} : \overset{0}{\bar{\mathbf{D}}} + \bar{\rho}\int_{\mathbf{q}}\mathbf{J}(\mathbf{q})\cdot\mathbf{A}_{turb}(\mathbf{q})d\mathbf{q}\right\}. \quad (25)$$

Here, $\nu \to 0$ is the so-called turbulent energy diffusion flux; $\overline{\varepsilon}$ is the part with a zero trace of the Reynolds stress tensor;

$$p_{turb}(\mathbf{r},t) = \tfrac{1}{3}(\mathbf{R}:\mathbf{I}), \tag{26}$$

is the turbulization pressure. The quantities $\sigma^{(i)}_{(S_{turb})}(\mathbf{r},t)$ and $\sigma^{(e)}_{(S_{turb})}(\mathbf{r},t)$ are the local production and sink of the entropy $S_{turb}(\mathbf{r},t)$ for the subsystem of turbulent chaos, respectively. Note that the work of turbulent stresses $\overset{0}{\mathbf{R}}:\overset{0}{\overline{\mathbf{D}}}$ causes the entropy of chaos to grow, while the viscous dissipation $\langle\varepsilon_b\rangle > 0$ reduces the turbulization entropy S_{turb}.

4 BALANCE EQUATION FOR THE TOTAL ENTROPY OF THE SUBSYSTEMS OF AVERAGED MOTION AND STRUCTURED TURBULENT CHAOS

Adding (7) and (23) yields the balance equation for the total entropy $S_\Sigma = \langle S\rangle + S_{turb}$ of a turbulized fluid system

$$\overline{\rho}\frac{DS_Y}{Dt} + div\left\{\frac{\mathbf{J}^{turb}_{\langle b\rangle}}{T_{turb}} + \frac{\left(\mathbf{q}^Y - \sum_{\alpha=1}^{N}\langle\mu_\alpha\rangle\mathbf{J}^Y_\alpha\right)}{\langle T\rangle}\right\} = \sigma_Y, \tag{27}$$

where

$$0 \le \sigma_Y \equiv \sigma^{(i)}_{\langle S\rangle} + \sigma^{(i)}_{S_{turb}} + \sigma_{\langle S\rangle, S_{turb}} = \sigma^{(i)}_{\langle S\rangle} + \sigma^{(i)}_{S_{turb}} + \frac{T_{turb} - \langle T\rangle}{\langle T\rangle T_{turb}}\Im_{E,\langle b\rangle} \tag{28}$$

is the local entropy production related to irreversible processes inside the total turbulized continuum. The quantity σ_Y written by taking into account (8), (9) and (24), (25) has the structure of a bilinear form, $\sigma_\Sigma = \sum_\alpha \Im_\alpha(\mathbf{r},t)\mathbf{X}_\alpha(\mathbf{r},t)$,

$$0 \le \sigma_\Sigma(\mathbf{r},t) \equiv \frac{1}{\langle T\rangle}\left\{-\left(\tilde{\mathbf{q}}^\Sigma\cdot\frac{\partial\ln\langle T\rangle}{\partial\mathbf{r}}\right) + \overline{\pi}\, div\langle\mathbf{u}\rangle + \overset{0}{\overline{\mathbf{P}}}:\overset{0}{\overline{\mathbf{D}}} + \right.$$

$$\left. + \frac{1}{T_{turb}}\left\{-\left(\mathbf{J}^{turb}_{\langle b\rangle}\cdot\frac{\partial\ln T_{turb}}{\partial\mathbf{r}}\right) + \overset{0}{\mathbf{R}}:\overset{0}{\overline{\mathbf{D}}} + \overline{\rho}\int_{\mathbf{q}}\mathbf{J}(\mathbf{q})\cdot\mathbf{A}_{turb}(\mathbf{q})d\mathbf{q}\right\} + \Im\left(\frac{T_{turb} - \langle T\rangle}{T_{turb}\langle T\rangle}\right). \tag{29}$$

According to the basic postulate of the nonlinear thermodynamics of non-equilibrium processes, if the system is near a relatively stable quasi-stationary state, then the thermodynamic fluxes can be represented as linear functions of the conjugate macroscopic forces[17]:

$$\Im_{\alpha i}(\mathbf{r},t) = \sum_{\beta} \Lambda^{ij}_{\alpha\beta} X_{\beta j}(\mathbf{r},t) \quad (\alpha,\beta = 1,2,...) \tag{30}$$

It is important to emphasize that the matrix of phenomenological coefficients $\Lambda^{ij}_{\alpha\beta}$ for a turbulized fluid continuum depends not only on the system's average state parameters (i.e., on its average temperature $\langle T\rangle(\mathbf{r},t)$, density $\overline{\rho}(\mathbf{r},t)$, etc.) but also on turbulent superstructure characteristics, i.e., on parameters such as $\langle\varepsilon_b\rangle(\mathbf{r},t)$ $T_{turb}(\mathbf{r},t)$. Such a situation, where there is a functional dependence of the coefficients $\Lambda^{ij}_{\alpha\beta}$ on the thermodynamic fluxes $\Im_{\alpha i}$ themselves (e.g., on the dissipation rate $\langle\varepsilon_b\rangle$, which is also the energy flux over the cascade of vortices in the stationary case), is known to be typical of self-organizing systems[34,35]. Generally, this can lead to the individual terms $\Im_{\alpha}(\mathbf{r},t)\mathbf{X}_{\alpha}(\mathbf{r},t)$ in the sum σ_Σ being not positive definite, although the entire sum is always greater than or equal to zero, $\sigma_\Sigma \geq 0$. The superposition of various fluxes can then, in principle, lead to negative individual diagonal elements of the matrix $\Lambda_{\alpha\beta}$, which probably explains the negative viscosity effect in some turbulent flows.

As can be seen from (29), the spectrum of possible cross effects for a turbulent flow generally widens compared to a laminar one. Thus, for example, the total heat flux $\mathbf{q}^{\Sigma}(\mathbf{r},t)$ in a turbulized continuum can emerge under the influence of not only its conjugate thermodynamic force $\partial(1/\langle T\rangle)/\partial\mathbf{r}$ but also the force $\partial(1/T_{turb})/\partial\mathbf{r}$ conjugate to the flux $\mathbf{J}^{turb}_b(\mathbf{r},t)$ describing the "diffusive" transport of turbulent kinetic energy. However, at present there are no reliable experimental data which would quantitatively described such cross effects. Besides, the contribution from any cross effects to the total rate of some process is usually an order of magnitude smaller than that from direct ones[14]. Taking this circumstance into account, we use the requirement that the intensities $\sigma_\Sigma(\mathbf{r},t)$, $\sigma^i_{\langle S\rangle}(\mathbf{r},t)$, $\sigma^i_{S_{turb}}(\mathbf{r},t)$ are positive independently of one another and omit a number of cross effects in the linear constitutive relations (30) without special stipulations.

In view of the second law of thermodynamics, the last term on the right-hand side of (29) describing the entropy production within the total continuum through irreversible entropy exchange between the subsystems of turbulent chaos and averaged motion is always positive,

$$\sigma_{\langle S\rangle,S_{turb}} = \Im\left(\frac{T_{turb}-\langle T\rangle}{T_{turb}\langle T\rangle}\right) \geq 0\,. \tag{31}$$

Therefore, the "direction" of the thermodynamic flux $\Im(\mathbf{r},t)$ is determined by the sign of the state function $X_{\Im}(\mathbf{r},t) = \left(1/\langle T\rangle - 1/T_{turb}\right)$, which should be considered as the conjugate thermodynamic force (a macroscopic factor) producing the entropy flux $\Im(\mathbf{r},t)$. Such entropy exchange between two mutually open subsystems is known[2] to be an indispensable condition for a structured collective behavior, i.e., it can be a source of self-organization in one of them.

In turn, the dissipative activity of the subsystem of turbulent chaos in the case of stationary non-equilibrium turbulence is determined by the influx of negative entropy ($\sigma^{e}_{S_{turb}} \equiv -\Im / T_{turb} < 0$) from the subsystem of averaged motion.

5 STATIONARY NON-EQUILIBRIUM STATE OF THE TURBULENT FIELD: DEFINING RELATIONS FOR THE STRUCTURED TURBULENCE

Since turbulence is accompanied by the dissipation of kinetic energy, a continuously acting external (relative to the medium under consideration) source is needed to maintain its quasi-stationary regime when the energy input and dissipation are nearly balanced. An energy source could be, e.g., a turbulence-producing wire grid placed perpendicular to the forced fluid flow, stationary boundary conditions causing a large-scale flow velocity shear, a thermal-convective large-scale instability, etc. Such a source should be powerful enough to compensate for the expenditure of turbulent energy dissipated through molecular viscosity. For quasi-stationary turbulence, almost the entire energy being expended will be transferred without any significant (but, in general, existing) losses through the inertial range from the energy range to the viscous one, where it dissipates into heat. The energy transfer from large-scale vortices to small-scale ones can be visualized as a random Richardson–Kolmogorov cascade turbulent vortex fragmentation process.

In the model approach we assume that a continuous process of energy transfer from the subsystem of averaged fluid motion to the subsystem of turbulent chaos corresponds to such quasi-stationary turbulence. Obviously, a stationary non-equilibrium regime between the influx of energy from the "external source" (attributable to the averaged fluid flow) and its dissipation (due to irreversible processes within the subsystem of turbulent chaos itself) is then established in the vortex continuum associated with small-scale turbulence in which $dS_{turb} / dt \cong 0$. Incidentally, the quasi-stationary state in which the entropy production is minimal is an attractor for the open subsystem of turbulent chaos, while the state corresponding to the total entropy maximum serves as an attractor for the turbulized system as a whole. The condition $dS_{turb} / dt \cong 0$ implies that the production $\sigma^{(i)}_{(S_{turb})}(\mathbf{r},t)$ of turbulization entropy S_{turb} is compensated for by its efflux $\sigma^{(e)}_{(S_{turb})}(\mathbf{r},t)$ to such extent that the total generation of entropy S_{turb} is almost absent,

$$\sigma_{(S_{turb})}(\mathbf{r},t) = \sigma^{(e)}_{(S_{turb})}(\mathbf{r},t) + \sigma^{(i)}_{(S_{turb})}(\mathbf{r},t) \simeq 0$$

It should also be kept in mind that the turbulization entropy flux in the stationary case is constant, $\mathbf{J}_{(S_{turb})} = \mathbf{J}^{turb}_{\langle b\rangle} / T_{turb} = const$ ($div\mathbf{J}_{S_{turb}} \approx 0$). Since $\sigma^{(i)}_{(S_{turb})} > 0$, the inequality $0 > \sigma^{e}_{S_{turb}} \cong -\sigma^{i}_{S_{turb}}$ holds, i.e., the subsystem of turbulent chaos must export entropy into the "external medium" in order to compensate for the entropy production through irreversible internal processes within itself. In other words, an influx of negative entropy (negentropy) from the "external medium" is needed to maintain the stationary nonequilibrium state within the subsystem of turbulent chaos,

$$\sigma^{e}_{S_{turb}}(\mathbf{r},t) \equiv -\Im / T_{turb} = -\langle T\rangle \sigma^{e}_{\langle S\rangle} / T_{turb} < 0 .$$

Such a condition is known to be sufficient for the formation of dissipative coherent structures in vortex continuum[1,2]. Indeed, since the entropy efflux from the subsystem of averaged motion in the stationary non-equilibrium state of chaos is positive $(0<\sigma^{e}_{S} \equiv -\Im/\langle T\rangle)$, the rate $\Im(\mathbf{r},t)$ of entropy (heat) exchange between the averaged and turbulent motions is also positive, $\Im \ge 0$. It then follows from inequality (31) that the turbulization temperature $T_{turb}(\mathbf{r},t)$ is higher than the average turbulized fluid temperature ($T_{turb} > \langle T\rangle$), which is in the full agreement with the basic synergetic principle about self-organization of dissipative system. According to this principle, the formation of coherent structures (in our case, the formation of different-scale coherent vortex structures in the subsystem of turbulent chaos) when heat is removed from the system, i.e., when passing to lower temperatures, is a universal property of matter [18].

5.1 Defining Relations for Structured Turbulence

Thus, the negentropy entering the subsystem of turbulent chaos is expended to maintain and improve its internal structure. The relation $0 \le \sigma^{e}_{\langle S\rangle} = -T_{turb}\sigma^{e}_{S_{turb}}/\langle T\rangle \cong T_{turb}\sigma^{i}_{S_{turb}}/\langle T\rangle$ is then valid and the balance equation (7) for the averaged entropy $\langle S\rangle(\mathbf{r},t)$ of a turbulized fluid system takes the form

$$\bar{\rho}\frac{D\langle S\rangle}{Dt} + div\left(\frac{\mathbf{q}^{\Sigma})}{\langle T\rangle}\right) = \sigma^{i}_{\langle S\rangle} + \sigma^{e}_{\langle S\rangle} \cong \sigma^{i}_{\langle S\rangle} + \frac{T_{turb}}{\langle T\rangle}\sigma^{i}_{S_{turb}} \cong \sigma_{\Sigma}, \tag{32}$$

where the following expression holds for the local energy dissipation $\langle T\rangle\sigma_{\Sigma}$:

$$\langle T\rangle\sigma_{\Sigma}(\mathbf{r},t) \equiv -\left(\mathbf{q}^{\Sigma}\cdot\frac{\partial \ln\langle T\rangle}{\partial \mathbf{r}}\right) + \overset{\circ}{\mathbf{R}} : \overset{\circ}{\bar{\mathbf{D}}} + \bar{\rho}\int_{\mathbf{q}} \mathbf{J}(\mathbf{q})\cdot\mathbf{A}_{turb}(\mathbf{q})d\mathbf{q} \ge 0 . \tag{33}$$

Here, $\mathbf{q}^{\Sigma}(\mathbf{r},t) \equiv \mathbf{q}^{turb} - \overline{p'\mathbf{u}''}$ is the total heat flux in the subsystem of averaged motion for the developed turbulence. Based on (33), we can write the following defining (gradient) relations for the turbulent fluxes and their conjugate thermodynamic forces in the linear approximation and using the Curie–Prigogine principle (according to which no relation is possible between tensors of different ranks in locally isotropic medium[14]):

$$\mathbf{q}^{\Sigma}(\mathbf{r},t) = -\lambda^{turb}\frac{\partial \ln\langle T\rangle}{\partial \mathbf{r}}, \tag{34}$$

$$\mathbf{R}(\mathbf{r},t) = -\tfrac{2}{3}\bar{\rho}\langle b\rangle\,\mathbf{I} + \bar{\rho}\nu^{turb}\left\{\tfrac{1}{2}\left(\frac{\partial\langle\mathbf{u}\rangle}{\partial\mathbf{r}} + \left(\frac{\partial\langle\mathbf{u}\rangle}{\partial\mathbf{r}}\right)^{transp}\right) - \tfrac{1}{3}\,div\langle\mathbf{u}\rangle\,\mathbf{I}\right\}, \qquad (35)$$

$$\mathbf{J}(\mathbf{q},\mathbf{r},t) = \int_{\tilde{\mathbf{q}}} \mathbf{L}(\mathbf{q},\tilde{\mathbf{q}},\mathbf{r},t)\cdot\mathbf{A}(\tilde{\mathbf{q}},\mathbf{r},t)\,d\tilde{\mathbf{q}}\,. \qquad (36)$$

They correspond to the stationary state of the turbulent field. Note that the linearity condition is not strong enough to deprive the case under consideration of practical significance.

Assessing the status of the problem of closing the averaged hydrodynamic equations as a whole, it should be recognized that currently almost all semi-empirical turbulence models are mainly based on the gradient relations. The phenomenological coefficients (turbulent exchange coefficients) $\lambda^{turb}(\mathbf{r},t)$, $\nu^{turb}(\mathbf{r},t)$ in these relations are scalar quantities, because, as has been emphasized above, strong turbulence is locally homogeneous and isotropic. In contrast to the molecular exchange coefficients, these quantities are not material constants. This is because in turbulized continuum the processes of mass, momentum, and energy transfer from one region of the system to another are determined by the collective motions of molecules (vortex structures) and, hence, must depend on turbulence intensity parameters, in particular, on the parameters ε_b and L_1 (or $\langle b\rangle$). For example, in the inertial range of vortex scales ($\eta < k < L_1$), the turbulent viscosity $\nu^{turb}(\mathbf{r},t)$ corresponding to the Richardson–Obukhov empirical "law of four thirds"[vi] is $\nu^{turb} \sim \langle\varepsilon_b\rangle^{1/3} L_1^{4/3} \sim \langle b\rangle^2/\langle\varepsilon_b\rangle$.

Thus, when stationary-inhomogeneous turbulence is modeled in the system where the energy dissipation processes are important, a heat transfer equation for averaged motion in form (32) should be invoked; this equation should be supplemented by the linear defining relations (34)–(36).

6 PRIGOGINE'S PRINCIPLE: THERMODYNAMIC DERIVATION OF THE FOKKER–PLANCK–KOLMOGOROV EQUATIONS

According to (36), the phenomenological relation for the thermodynamic flux $\mathbf{J}(\mathbf{q},\mathbf{r},t)$ in the space of internal coordinates $\mathbf{q}$ and the corresponding "instantaneous" affinity $\mathbf{A}(\mathbf{q},\mathbf{r},t)$ generally has an integral form. Following Prigogine[22] (see Chap. 3, Sect. 11), we will now assume that the irreversible processes in each part of the internal coordinate space $\mathbf{q}$ proceed in such a way that only positive entropy increment occurs. This implies that not only integral (19) but also the quantity

$$T_{turb}\sigma_{\mathbf{q}}(S_{turb}) = \mathbf{J}(\mathbf{q},\mathbf{r},t)\cdot\mathbf{A}_{turb}(\mathbf{q},\mathbf{r},t) \geq 0\,, \qquad (37)$$

which is the energy dissipation per unit volume of configuration space $\mathbf{q}$ will be positive. The defining relation between the flux $\mathbf{J}(\mathbf{q},\mathbf{r},t)$ and affinity $\mathbf{A}_{turb}(\mathbf{q},\mathbf{r},t)$ of a state $\mathbf{q}$ (corresponding to one equivalent of the vortex decay process) is then

[vi] This law follows, in particular, from dimensional and similarity considerations.

$$\mathbf{J}(\mathbf{q},t) = \mathbf{L_q} \cdot \mathbf{A}_{turb}(\mathbf{q},t) = -\mathbf{L_q} \cdot \frac{\partial \mu_{turb}(\mathbf{q},t)}{\partial \mathbf{q}} = \mathbf{L_q} \cdot \left(\frac{\mathrm{k_B}\, T_{turb}}{n(\mathbf{q},t)} \frac{\partial n(\mathbf{q},t)}{\partial \mathbf{r}} + \frac{\partial \Phi(\mathbf{q},t)}{\partial \mathbf{r}} \right). \quad (38)$$

Here, $\mathbf{L_q}$ is the positive definite local matrix of transport coefficients satisfying the Onsager–Casimir reciprocity relation $\mathbf{L}^{transp} = \mathbf{L}$; $n(\mathbf{q},t)$ are the numbers of vortex moles in the cascade interaction between turbulent motions of different scales.

Relation (38) taken together with (13) allows us to derive the following evolutionary Fokker–Planck–Kolmogorov equation in the so-called kinetic form[36] in the space of stochastic variable $\mathbf{q}$ for the distribution functions of various statistical small-scale turbulence characteristics:

$$\frac{\partial P_2(\mathbf{q},\mathbf{r},t)}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \left(\langle \mathbf{u} \rangle P_2(\mathbf{q},\mathbf{r},t) \right) =$$

$$= \frac{\partial}{\partial \mathbf{q}} \cdot \left\{ -\mathbf{K}(\mathbf{q}) P_2(\mathbf{q},\mathbf{r},t) + \frac{\varepsilon^2}{2} \mathbf{Q}(\mathbf{q}) \cdot \frac{\partial P_2(\mathbf{q},\mathbf{r},t)}{\partial \mathbf{q}} \right\}. \quad (39)$$

Here, the probability flux $\mathbf{J} = \mathbf{J}_{dr} + \mathbf{J}_{dif}$ is the sum of the drift $\mathbf{J}_{dr}$ and diffusion $\mathbf{J}_{dif}$ components: $\mathbf{J}_{dr} = \mathbf{K} P_2$, $\mathbf{J}_{dif} = -\frac{\varepsilon^2}{2} \mathbf{Q} \cdot \frac{\partial P_2}{\partial \mathbf{q}}$, where the following notation is used for the drift vector $\mathbf{K}$ and the matrix of generalized diffusion coefficients $\mathbf{D}$ in the space of stochastic variable $\mathbf{q}$:

$$\mathbf{K}(\mathbf{q}) \equiv \text{€}(\mathbf{q}) \cdot \mathbf{f}(\mathbf{q}), \quad \mathbf{D} = \varepsilon^2\, \text{€}(\mathbf{q}) = \frac{1}{2} \varepsilon^2 \mathbf{Q}, \quad \mathbf{Q}(\mathbf{q}) \equiv 2\text{€}(\mathbf{q}). \quad (40)$$

The function $P_2(\mathbf{q},t) = n / n_\Sigma$ is the (conditional) probability density to detect the system in an interval $(\mathbf{q}, \mathbf{q} + d\mathbf{q})$ at a time t if it was in a state $\mathbf{q}^{st}$ at an initial time (at $t = 0$) with probability equal to unity. The parameter $\varepsilon \equiv \sqrt{\mathrm{k_B}\, T_{turb}} = \sqrt{\rho (\mathbf{u}'')^2 / 3\bar{\rho}}$ characterizes the total intensity of the action of internal noise in the subsystem of turbulent chaos (generated by its "thermal" structure) on the random process $\mathbf{q}(t)$. In writing (39), we assumed in the first approximation the mobility parameter $\text{€}(\mathbf{q}) \equiv \mathbf{L}(\mathbf{q}) / n(\mathbf{q})$ in internal coordinate $\mathbf{q}$ to be independent on the density $n(\mathbf{q})$. Generally, the matrix K does not form a vector unless the modeling is confined by only linear transformations of coordinates.

Note that, apart from the kinetic form of the FPK equation (39), other representations of this equation can be also used, Ito's and Stratonovich's representations being the most common. They are based on a different treatment of the so-called stochastic integrals[37] that emerge from the solution of the corresponding nonlinear stochastic Langevin's equations. Although these representations are based on stochastic nearly identical equations [see (86)], they lead to the different form of FPK equations:

$$\frac{\partial P_2(\mathbf{q},\mathbf{r},t)}{\partial t} + div\big(\langle \mathbf{u}\rangle P_2(\mathbf{q},\mathbf{r},t)\big) =$$

$$= \frac{\partial}{\partial \mathbf{q}} \cdot \left\{ -\big(\mathbf{K}(\mathbf{q}) + \mathbf{H}(\mathbf{q})\big) P_2(\mathbf{q},\mathbf{r},t) + \tfrac{1}{2}\varepsilon^2 \frac{\partial}{\partial \mathbf{q}} \cdot \big(\mathbf{Q}(\mathbf{q}) P_2(\mathbf{q},\mathbf{r},t)\big) \right\}, \qquad (39^*)$$

where $\mathbf{H}(\mathbf{q}) \equiv \lambda \dfrac{\varepsilon^2}{2} \dfrac{\partial \mathbf{Q}(\mathbf{q})}{\partial \mathbf{q}}$ is a "fictitious force" dependent on the choice of calculus. In the case $\lambda = 0,\ 1/2,\ 1$ Ito's - Stratonovich's, and Klimontovich's representations of the FPK equation, respectively, are the most appropriate.

If response of the subsystem of turbulent chaos to impact of the external medium (the subsystem of averaged motion) does not depend on its internal state specified by stochastic variable $\mathbf{q}$, then the diffusion coefficient does not change with coordinate $\mathbf{q}$ either (i.e., $\mathbf{Q}(\mathbf{q}) = const$). The stochastic system can then be assumed to possess additive noise that, in our case, is just reduced to the intensity $\varepsilon^2 = \mathrm{k_B} T_{turb}$ of the internal noise in the subsystem of turbulent chaos. The fictitious force $\mathbf{H}(\mathbf{q})$ for such systems is identically equal to zero, i.e., they both are completely insensitive to the choice of calculus. In other words, all three forms of the FPK equation take the same simplest form (39). However, a feedback is generally possible between the internal states $\mathbf{q}$ of the stochastic subsystem of chaos and the subsystem of averaged motion. It turns out that not only the external fluctuations (associated, for example, with the degrees of freedom of the turbulent field that are not described by the selected coordinates $\mathbf{q}$) affect the stochastic subsystem of turbulent chaos but also the latter has the reverse effect on their intensity. As applied to the case under consideration, this implies that the diffusion coefficient becomes dependent on the random coordinate $\mathbf{q}$, i.e., the external fluctuations are multiplicative. In the presence of multiplicative noise, when $\mathbf{Q}(\mathbf{q}) \neq const$, the simplest form of the FPK equation (39) holds only in Klimontovich's representation ($\lambda = 1$).

Thus, the problem of choosing calculus arises for systems with multiplicative noise because the force entering into the FPK equation is determined ambiguously. In Ito's calculus, it is reduced to a real force acting on the selected degree of freedom. When using Stratonovich's calculus, an addition proportional to the derivative of the effective diffusion coefficient emerges. The value of this addition doubles in Klimontovich's kinetic representation. It is important to notice that it affects significantly the behavior of the stochastic system. As a result, the question about the physical nature of this addition and the choice of calculus seems topical.

7 EXAMPLES OF THE FOKKER–PLANCK–KOLMOGOROV EQUATIONS DESCRIBING EVOLUTION OF THE FLUCTUATING CHARACTERISTICS OF TURBULENT CHAOS

Now proceeding from simple examples let us show that Prigogine's principle (37) can serve as a basis for deriving the evolutionary partial differential Fokker–Planck–Kolmogorov equations in the space of stochastic variable $\mathbf{q}$ for the distribution functions of various stochastic small-scale turbulence characteristics. This is valid if appropriate hypotheses about the distribution in a stationary non-equilibrium state of the latter are adopted in advance. It should be kept in mind, however, that such hypotheses are almost always not quite rigorous and are a strong idealization related to the simplification of a real turbulent motion under natural conditions [38].

7.1 Evolution of Vortices in the Space of Fluctuating Velocities

Let us first address assumption (37) to derive the kinetic equation describing the change in the probability density function of vortex velocities $P_2(u''\mathbf{r},t)$ $(\equiv n(u'',\mathbf{r},t)/n_\Sigma)$, $n(u'',\mathbf{r},t)$ being the number density of turbulent vortices and n_Σ is the total number of vortex moles, see (12). We consider this function as the internal variable of the subsystem of turbulent chaos and the fluctuating velocity u'' as the internal coordinate $\mathbf{q}$. The probability distribution function for the fluctuating velocity is known to be not universal in the case of developed turbulence because it depends on the mechanism generating the turbulent field. Nonetheless, following Millionshchikov[40], we may use the hypothesis about normal distribution of fluctuating velocities (for a locally isotropic turbulent field) in the stationary case

$$W_1(u'') \equiv n^{stat}(u'')/n_\Sigma = \left(\beta/\sqrt{\pi}\right)\cdot\exp\left(-\beta^2 u''^2\right). \tag{41}$$

Millionshchikov applied such a distribution in the turbulence theory for special purposes that we will not discuss in more detail. Note only that there are many examples where the velocity distribution function is approximately Gaussian, for instance, in turbulence generated by grids in wind tunnels or turbulence in an atmospheric boundary layer [9,41]. For the subsystem of turbulent chaos we accept the constant β in (41) as being related to the turbulization temperature T_{turb} in exactly the same way as it is done in the gas kinetic theory[42]. Then, using (17) and (41), it is easy to find that $\beta^2 = (2\mathsf{R}T_{turb})^{-1}$, whence we obtain a different equivalent expression for the function $n^{stat}(u'')$:

$$n^{stat}(u'') = n_\Sigma\left(\frac{1}{2\pi\mathsf{R}T_{turb}}\right)^{1/2}\exp\left(-\frac{u''^2}{2\mathsf{R}T_{turb}}\right) = const\cdot\exp\left(-\frac{u''^2}{2\mathsf{R}T_{turb}}\right). \tag{41*}$$

Substituting this distribution into (16) yields the following representation of the chemical potential $\mu_{turb}(u'',\mathbf{x},t)$ for the configuration u'':

$$\mu_{turb}(u'',\mathbf{r},t) = \frac{\mathrm{k_B}}{\mathsf{R}}(u''^2/2) + \mathrm{k_B}\,T_{turb}(\mathbf{r},t)\ln\left[n(u'',\mathbf{r},t)\right] + const. \tag{42}$$

Given this formula, the phenomenological relation (36) for the probability flux $J(u'',\mathbf{x},t)$ takes the form

$$J(u'') = -\frac{\bar{\rho}}{n_\Sigma} L_{u''}\left(u'' + \frac{\mathsf{R}T_{turb}}{n(u'')}\frac{\partial n(u'')}{\partial u''}\right) = -\alpha\left(u'' n(u'') + \mathsf{R}T_{turb}\frac{\partial n(u'')}{\partial u''}\right), \tag{43}$$

where $\mathsf{R}(= n_\Sigma \mathrm{k_B}/\bar{\rho})$. Here, we introduce the coefficient $\alpha \equiv \mathrm{k_B}\, L_{u''} / \mathsf{R}n(u'')$ that can be interpreted as the "mobility" in the space of internal coordinate u'' per unit volume; in the first approximation, this coefficient does not depend on $n(u'')$ and is assumed to be independent of u''. Substituting (43) into (19) yields the sought for kinetic FPK equation for the conditional probability density function of the fluctuating velocity u'':

$$\frac{\partial P_2}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\cdot\left(P_2\langle\mathbf{u}(\mathbf{r},t)\rangle\right) - \alpha\frac{\partial}{\partial u''}\left(u'' P_2 + \mathsf{R}T_{turb}(\mathbf{r},t)\frac{\partial P_2}{\partial u''}\right) = 0\,. \tag{44}$$

This dynamical equation supplemented by the initial condition $P_2 = \delta(u''-0)$ (the $\delta-$function concentrated at point «0» appears on the right-hand side) describes the temporal evolution of the probability density function P for the fluctuating velocity u'', in particular, for decaying (so-called degenerating) turbulence. It should be noted that the quantity $K \equiv -\alpha u''$ (the friction coefficient in the corresponding Langevin's equation) acts as the drift coefficient in the FPK equation written in standard form and the quantity

$$D \equiv 2L_{u''}\,\mathrm{k_B}\,T_{turb} / n(u'') = 2\alpha\,\mathsf{R}T_{turb} = \alpha\beta^{-2}$$

is the diffusion coefficient.

The normal distribution (41), which is a stationary solution of the one-dimensional (in parameter u'') FPK equation (44) can be taken as the initial statistical state of the fluctuating velocity field for the whole class of various motions of degenerating turbulence. A non-stationary solution of this equation can then be obtained in an analytically closed form:

$$P_2(u'',\mathbf{r},t) = \left\{\pi\, a(\mathbf{r},t)\right\}^{-1/2} \exp\left\{-\left[u'' - b(\mathbf{r},t)\right]^2 / a(\mathbf{r},t)\right\}, \tag{45}$$

where

$$a(\mathbf{r},t) = \frac{D}{\alpha}\left\{1 - \exp(-2\alpha t)\right\} + a_0\exp(-2\alpha t), \quad b(\mathbf{r},t) = b_0(\mathbf{r})\exp(-\alpha t)\,; \tag{46}$$

$a_0(\mathbf{r})$ and $b_0(\mathbf{r})$ are the initial conditions. This solution allows us to calculate the various n-point moments (correlation functions) of the m-th order describing the statistical relation between the random velocities at various points of space-time. In particular, for the quantities $\overline{u''}^{0}(\mathbf{r},t)$ (the conditional mean velocity of the ensemble of vortices at a time t) and $\overline{u''(\mathbf{r},t)u''(\mathbf{r},t_1)}$ (the two-time one-point correlation function), we have

$$\overline{u''}^{0}(\mathbf{r},t)=\int u''P_2(0|u'',t)du''=b_0(\mathbf{r})\exp(-\alpha t), \qquad (47)$$

whence $b(\mathbf{r},t)\equiv\overline{u''}^{0}(\mathbf{r},t)$;

$$\overline{u''(\mathbf{r},t)u''_1(\mathbf{r},t_1)}=\int u''du''\int u''_1du''W_2(u'',t;u''_1,t_1)=$$

$$=\overline{u''^2(\mathbf{r},t)}\exp\{-\alpha|t-t_1|\}=\frac{1}{2}\frac{D}{\alpha}\exp\{-\alpha|t-t_1|\}, \qquad (48)$$

where the average is taken over the stochastic process[34]. Here, $W_2(u'',t;u''_1,t_1)$ is the joint probability density. Because generation of new modes of fluctuating motion (the fragmentation of vortex structures) is a Markovian process, it is represented as the product of the probability density at time t_1, $W_1(u''_1,t_1)$, and the conditional probability $P_2(u'',t|u''_1,t_1)$ (which is reduced to the δ-function at $t=t_1$):

$$\delta(u''-u''_1)):\ W_2(u'',t;u''_1,t_1)=W_1(u''_1,t_1)P_2(u'',t|u''_1,t_1).$$

Since $D=2\alpha\,\mathrm{R}T_{turb}=\tfrac{4}{3}\alpha\langle b\rangle$, then we have: -first, the correct relation $\overline{u''^2}=\tfrac{2}{3}\langle b\rangle\cong\tfrac{1}{3}\overline{|\mathbf{u}''|^2}$ is consistent with the assumption about local isotropy of the vortex velocity field in the case of developed turbulence and, -and second, the effective formula for one of the most important correlation quantities in the theory of statistical turbulence

$$\overline{u''(\mathbf{r},t)u''(\mathbf{r},t_1)}=\tfrac{2}{3}\langle b\rangle\exp\left(-\alpha|t-t_1|\right), \qquad (49)$$

defining the speed with which the fluctuating velocity “forgets its past” (according to this formula, this occurs in a time $t\cong 1/\alpha$ follow from (48).

Solution (45) at zero values of the parameters a_0 and b_0 takes the form

$$P_2(u'',t)=\left\{2\pi\,\mathrm{R}T_{turb}\left[1-\exp(-2\alpha t)\right]\right\}^{-1/2}\exp\left\{-\frac{u''^2}{2\mathrm{R}T_{turb}\left[1-\exp(-2\alpha t)\right]}\right\}. \qquad (50)$$

It allows the temporal evolution of the conditional probability distribution function for the fluctuating velocity to be traced if the velocity distribution was Gaussian in the case of stationary turbulence.

It should be kept in mind that choosing the fluctuating velocity u'' as a suitable characteristic of turbulent vortices (the internal coordinate of the subsystem of turbulent chaos) generally does not justified, because the Gaussian probability distribution of the fluctuating velocity u'' has been confirmed with a sufficient degree of reliability neither experimentally (it was established, for example, that the deviation from “normal behavior” behind the grid increases considerably with increasing Reynolds number $\mathbf{Re}$) nor theoretically (the classical “two and five thirds" laws of turbulence are known to break down for this distribution). Earlier it was pointed out that the most acceptable characteristics of small-scale turbulence to serve as an internal coordinate are non-negative macroscopic variables even functions of rates such as the dissipation rate of turbulent energy[38]. According to Kolmogorov's hypothesis, such random characteristics asymptotically satisfy a log-normal probability distribution. This is because the successive fragmentation of vortex structures is similar to the coagulation of solid particles (the latter is known to lead to a log-normal particle size distribution). We note that the log-normal distribution does not accurately describe the edges of the true distribution of random variable and, hence, can be used to calculate the high order moments only with a great caution.

8 CASCADE PROCESS. THERMODYNAMIC TREATMENT CORRESPONDING TO THE KOLMOGOROV'S ORIGINAL SIMILARITY HYPOTHESES

Let us now apply Prigogine's principle (37) to the derivation of the kinetic equation describing the temporal evolution of the vortex distribution function in the space of kinetic energy. We will describe the Richardson–Kolmogorov cascade process (large vortices towards small vortices and heat) using an analogy with the process of consecutive chemical reactions. In this case, the original kinetic equation (13) for the distribution function $P(0|q;\mathbf{r},t)$ of turbulent vortices in the space of fluctuating energy $q=\rho|\mathbf{u}''|^2/2$ takes the form

$$\frac{\partial P_2(0|q;\mathbf{r},t)}{\partial t}+\frac{\partial}{\partial \mathbf{r}}\cdot\left(P_2(0|q;\mathbf{r},t)\langle\mathbf{u}\rangle\right)=\frac{\partial}{\partial q}\left(P_2(0|q;\mathbf{r},t)\,\varepsilon(q,\mathbf{r},t)\right), \tag{51}$$

where

$$\varepsilon(q,\mathbf{r},t)\equiv -J(q,\mathbf{r},t)/n(q,\mathbf{r},t) \tag{52}$$

is the transition reaction rate from state q to state $q+dq$ corresponding to the probability flux $J(q)$ in state q; n is the number density of turbulent vortices. In other words, relation (51) defines the parameter $\varepsilon(q,\mathbf{r},t)$ that can be interpreted as the transfer rate of kinetic energy $\rho|\mathbf{u}''|^2/2$ over hierarchy of turbulent vortices along the coordinate q. Concurrently, this quantity also defines the kinetic energy dissipation in vortices of type q. Indeed, the equation for the first moment

$$\int q\,P_2(0|q;t)dq=\overline{\rho|\mathbf{u}''|^2/2}^{\,0}=\overline{\rho}\langle b\rangle .$$

Derived from (51) as a result of integration by parts (and by assuming the flux $J(q,\mathbf{r},t)$ at the boundaries of the domain of integration to be zero) takes the classical form[31]

$$\overline{\rho}\frac{D\langle b\rangle}{Dt}\cong-\int\varepsilon(q,\mathbf{r},t)P_2(0|q;\mathbf{r},t)dq=-\overline{\varepsilon(\mathbf{r},t)}^{\,0}\cong-\overline{\rho}\langle\varepsilon(\mathbf{r},t)\rangle . \tag{53}$$

Here the conditional average is taken over the stochastic process q, in which parameter $\langle\varepsilon\rangle$ defines the mean turbulent energy dissipation rate at point $(\mathbf{r},t)$. For this reason the quantity $\varepsilon(q)$ can be interpreted as the energy dissipation rate in vortices of type q (at the point $q=\rho\left|\mathbf{u}''\right|^2/2$ of configuration space. Then for the part of the dissipation energy $\langle T\rangle\sigma_\Sigma$ [see (33)] attributable to the transfer of turbulent energy over the cascade, we have

$$\left(\langle T\rangle\sigma_\Sigma\right)^{Ch}\equiv-\bar{\rho}\int_{\mathrm{q}}\varepsilon(\mathrm{q},\mathbf{r},t)\,n(q,\mathbf{r},t)\,A_{turb}(q,\mathbf{r},t)dq\geq 0\,. \tag{54}$$

Hence, a local phenomenological Prigogine-type equation follows:

$$\varepsilon(\mathrm{q},\mathbf{r},t)=L_q A_{turb}(q,\mathbf{r},t)\,/\,n(q,\mathbf{r},t)=-\alpha' A_{turb}(q,\mathbf{r},t)\,, \tag{55}$$

in which

$$A_{turb}(q)=-\mathrm{k_B}\,T_{turb}\frac{\partial\ln n(q)}{\partial q}+f(q) \tag{56}$$

is the chemical affinity of the vortex fragmentation process (the state function of the subsystem of turbulent chaos); $f(\mathrm{q})=-\partial\Phi/\partial q$ is the so-called friction force; and $\alpha'=-L_q/n(q)$ is the mobility coefficient, which is assumed to be independent of q.

In the case where a stationary non-equilibrium flow is established in a turbulent medium when the energy transfer rate over the cascade is constant, $\varepsilon(q,\mathbf{r},t)\cong\langle\varepsilon(\mathbf{r},t)\rangle$ [vii] inequality (54) takes the form

$$\langle T\rangle\sigma_\Sigma=-\bar{\rho}\,\langle\varepsilon(\mathbf{r},t)\rangle\,A^{gl}_{turb}(\mathbf{r},t)\geq 0\,. \tag{57}$$

Here,

$$A^{gl}_{turb}\equiv\int_q n(q)A_{\mathrm{turb}}(q)dq=n_\Sigma\overline{A_{\mathrm{turb}}}$$

is the so-called global affinity of the formation of turbulent structures. Given (56), it can be rewritten as

$$\overline{A_{turb}}(\mathbf{r},t)=-\mathrm{k_B}\,T_{turb}\int_q\frac{\partial P_2(0|q;t)}{\partial q}dq+\int_q P_2(0|q;t)f(q)dq=$$

$$=-\mathrm{k_B}\,T_{turb}[P_2(q_{L_1})-P_2(q_\eta)]+\overline{f(\mathbf{r},t)}\cong\overline{f(\mathbf{r},t)}, \tag{58}$$

because P_2 becomes zero at the boundaries of the domain of integration (here, η is the local value of the Kolmogorov's microscale; $\overline{\tilde{\mathbf{F}}(\mathbf{q},t_1)\tilde{\mathbf{F}}^T(\mathbf{q},t)}^0=RT_{\mathrm{turb}}\mathbf{Q}(\mathbf{q})\delta(t-t_1)$ is the total

[vii] This assumption was first adopted in the original formulation of the well-known Kolmogorov[28] hypotheses.

number of vortex moles). Thus, the production σ_Σ of averaged entropy in such a stationary process is the product of total energy transfer rate over the cascade $\langle\varepsilon\rangle$ and the global affinity A^{gl}_{turb} is referring to the entire cascade fragmentation of large vortices into small ones. In this case, the linear phenomenological relation

$$\langle\varepsilon(\mathbf{r},t)\rangle = -\alpha' A^{gl}_{turb}(\mathbf{r},t) \tag{59}$$

is valid, in complete agreement with the results of irreversible thermodynamics for consecutive chain of chemical reactions.

On the other hand, we can adopt the more realistic condition

$$J(q,\mathbf{r},t) \cong J(\mathbf{r}) \equiv -n_\Sigma(\mathbf{r})\langle\varepsilon(\mathbf{r})\rangle$$

for the thermodynamic kinetic energy flux $J(q,\mathbf{r},t)$ over the cascade of vortices being quasi-stationary, i.e., for the flux J on various scales of motion being independent of the parameter $q = \rho|\mathbf{u}''|^2/2$. This assumption considered together with the linear relation (55) leads to a more general (than 59) form accounting for non-linear relation between ε and the chemical affinity

$$\tilde{A}(\mathbf{r},t) \equiv \int_q A(q)dq = \mu(q_{L_1},\mathbf{r}) - \mu(q_\eta,\mathbf{r})\,. \tag{60}$$

for the cascade process as a whole. The non-linear defining relation in the case can be easily obtained by applying the second formula in (22) for the "local affinity" $A_{\mathrm{turb}}(q,\mathbf{r},t)$ and as a result, we have for the chemical reaction rate

$$\langle\varepsilon\rangle \cong \gamma\left[1-\exp\left(-\frac{\tilde{A}}{\mathrm{k_B}\,T_{turb}}\right)\right], \tag{61}$$

where

$$\gamma = \frac{\left(\alpha'\dfrac{\mathrm{k_B}\,T_{turb}}{n_\Sigma}\right)\exp\left(\dfrac{\mu(q_\eta)}{\mathrm{k_B}\,T_{turb}}\right)}{\displaystyle\int_{q_\eta}^{q_{L_1}}\exp\left(\frac{\Phi(q)}{\mathrm{k_B}\,T_{turb}}\right)dq}\,. \tag{62}$$

Thus, the existing deep analogy between the consecutive chemical reactions ($A\to B\to C\to$ etc.) and the Richardson–Kolmogorov cascade fragmentation of vortices with the corresponding chemical potential and chemical affinity allows us to describe macroscopically structured turbulence by the methods of extended irreversible thermodynamics and represent it as self-organization process in an open system. Using the two interpretations of Kolmogorov's parameter ε as the quantity describing the dissipation rate of energy into heat and, simultaneously, as the transfer rate of turbulent energy over the cascade of vortices in the stationary-equilibrium case, we have been able to obtain the defining relations for a key characteristic of the turbulent field - the turbulent energy dissipation rate $\langle\varepsilon\rangle$ by means of thermodynamical modeling structured turbulence. We may recall that in the Kolmogorov[28] theory this quantity is constant and is called Kolmogorov's parameter. Relations (59) and (60) closing the system of hydrodynamic equations (2)–(5) make the thermodynamic approach to modeling the developed turbulence to a certain extent self-sufficient. Obviously, when addressing various problems of numerical simulations including broad class of natural phenomena synergetic approach to describing stationary non-equilibrium turbulence should be further refined.

An additional important remark is worth to mention. By analogy with a laminar fluid motion, the condition for increase in total continuum entropy (33) seems would place some constraints on the turbulent transport coefficients in the defining relations (34), (35), and (59). Positiveness of the direct molecular exchange coefficients is known to follow precisely from such conditions whereas the cross coefficients can be different in sign[31]. Substituting relations (34), (35), and (59) into (33) for the total entropy production of turbulent system, we obtain

$$0 \le \sigma_\Sigma \equiv \frac{1}{\langle T\rangle}\left\{\lambda^{turb}\left(\frac{\partial \ln\langle T\rangle}{\partial \mathbf{r}}\right)^2 + \bar{\rho}\nu^{turb}\left(\mathbf{D} - \frac{1}{3}(\mathrm{div}\langle\mathbf{u}\rangle)\mathbf{U}\right)^2 + \bar{\rho}\alpha'\left(A^{gl}_{turb}\right)^2\right\}. \tag{63}$$

Specificity of interactions associated with the functional dependence of the turbulent transport coefficients on parameters $\langle\varepsilon\rangle$ and $\langle b\rangle$ between various dissipative processes in a turbulized continuum is such that “switching off” one of the thermodynamic forces (e.g., the affinity A^{gl}_{turb}) can change (or even “switch off”) other processes (e.g., the viscous ones). This implies that the second law of thermodynamics, which requires that the entire sum (63) is to be positive, in general, cannot be applied to its individual terms. For example, it could happen that the quantity

$$\bar{\rho}\nu^{turb}\left(\mathbf{D} - \frac{1}{3}(\mathrm{div}\langle\mathbf{u}\rangle)\mathbf{U}\right)^2 < 0,$$

provided that the $\sigma_\Sigma \ge 0$. This indicates that turbulent flows with a negative turbulent viscosity, $\nu^{turb} < 0$, can appear under certain peculiar conditions. The above considerations serves as thermodynamic justification for the possibility that a negative viscosity can appear in turbulent fluid flows.

12 CONCLUSIONS

We performed a stochastic-thermodynamic analysis of developed turbulence in a homogeneous fluid and constructed a phenomenological model of structured turbulence as a self-organization process in an open system based on our previous study summarized in the works[11,27,40,43]. A turbulized continuum was represented as a thermodynamic complex consisting of two mutually open subsystems − the subsystem of averaged motion and the subsystem of turbulent chaos, which, in turn, is considered as an ensemble of vortices with various spatiotemporal scales. This representation allowed us to obtain the defining relations for the turbulent fluxes and forces in the subsystem of turbulent chaos in a non-equilibrium stationary state by the methods of thermodynamics with internal variables. By introducing a number of additional random parameters for the medium characterizing the excited macroscopic degrees of freedom of a strongly turbulized continuum, we obtained various Fokker–Planck–

Kolmogorov equations for the distribution functions of small-scale turbulence characteristics using Prigogine's postulate concerning the direction of the irreversible processes localized in the space of configurations. We also described thermodynamically Kolmogorov's cascade process. At the same time, a deeper understanding of the phenomenology of Kolmogorov's cascade is possible only by taking into account a large number of statistically correlated stochastic processes $\mathbf{q}$ that comprehensively characterize vortex spatiotemporal structures. Nonetheless, the simplified thermodynamic analysis of quasi-stationary turbulence performed here and the idealized macroscopic model constructed on its basis allow us to extend further our original views of the properties of open dissipative hydrodynamic systems. The latter is a "hot spot" in one of the most important and rapidly developing branches of nonlinear dynamics including evolution of chaotic motions and the formation of ordered dissipative structures.

The dual nature of the irreversible processes leading to disordering near equilibrium and ordering far from equilibrium clearly manifests itself when analyzing the current problems of turbulence, specifically in natural environment and outer space, in the entire variety of spatiotemporal scales. They involve origin and evolution of the Universe, stellar and planetary objects formation and evolution, the processes in the gaseous envelopes of celestial bodies, as well as different patterns of ecosystems in which the cascades of spatiotemporal configurations are created. From our viewpoint, the concept of entropy itself becomes much more substantive and deeper owing to the approach to modeling structured turbulent chaos being developed here. One of the main objectives of this study was the development of theoretical approach to the stationary non-equilibrium state of turbulent chaos using stochastic-thermodynamic methods and finding conditions of self-organization in such open systems.

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REFERENCES

[1] I. Prigogine, I. Stengers, *Order out of chaos*. Heinemann, London, (1984).

[2] I. Prigogine, I. Stengers. *Vremia. Khaos. Kvant. K resheniiu paradoksa vremeni*. Progress Publishing House, Moscow, (1994).

[3] S.C. Crow, F.H. Champagne, "Orderly structures in jet turbulence", *J. Fluid Mech*. **48**, 547-591 (1971).

[4] G.L. Brown, A. Roshko, "On density effects and large structures in turbulent mixing layers", *J. Fluid Mech*. **64**, 775-816 (1974).

[5] M.I. Rabinovich, M.M. Sushchik, "Regular and chaotic dynamics of structures in fluid flows". *Usp. Fiz. Nauk,* **160 (**1**),** 3-64 (1990).

[6] A.S. Monin, A.M. Yaglom. *Statistical Fluid Mechanics*, *Vol*. 2. Cambridge, Mass., USA: MITPress, (1975).

[7] Yu. L. Klimontovich, *Statistical Theory of Open Systems.* Kluwer academic publishers, Dordrecht, (1995).

[8] Y.I. Khlopkov, V.A. Zharov, S.L. Gorelov, *Kogerentnye Struktury v Turbulentnom Pogranichnom Sloe,* Moskwa, MFTI, (2002).

[9] U. Frisch, *The Legacy of A.N. Kolmogorov*. Cambridge University Press, Cambridge, (1995).

[10] L.I. Sedov, *O perspektivnykh napravleniiakh i zadachakh v mekhanike sploshnykh sred*. Nauka, Moskwa, (1980).

[11] M.Y. Marov, A.V. Kolesnichenko, *Mechanics of Turbulence of Multicomponent Gases*. KluwerAcademic, Dordrecht-Boston-London, (2001).

[12] A.V. Kolesnichenko, «К teorii inversnogo energeticheskogo kaskada v zerkalno-nesemetrichnoi turbulentnosti nemagnitnogo astrofizicheskogo diska» *Mathematica Montisnigrix*, **29**, 11-37 (2014).

[13] A.V. Kolesnichenko, «К MGD -modelirovaniiu struktury i evoliutsii turbulentnogo akkretsionnogo diska protozvezdy», *Mathematica Montisnigrix*, **25**, 65-93 (2012).

[14] S. de Groot, P. Mazur, *Non-equilibrium Thermodynamics*. North-Holland Publishing Company, Amsterdam, (1962).

[15] D. Jou, J. Casas-Vazquez, G. Lebon, *Extended Irreversible Thermodynamics*, Springer, Verlag Berlin Heidelberg, (2001).

[16] R.L. Stratonovich, *Nelineinoia neravnovesnaia termodinamika,* Nauka, Moskwa, (1994).

[17] J. Keizer, *Statistical Thermodynamics of Nonequilibrium Processes,* Springer, Verlag New York, (1987).

[18] W. Ebeling, A. Engel, R. Feistel, *Fizika evoliutsionnykh protsessov. Sinergeticheskie prilozheniia*, Editorial USSR, Moscow, (2001).

[19] A.V. Kolesnichenko, M.Ya. Marov, *Turbulentnost mnogokomponentnykh sred*, MAIK-Nauka, Moskwa, (1999).

[20] A.V Kolesnichenko, ”Synergetic approach to describing developed turbulence”, *Astron, Vestn*., **36(2)**, 121-139 (2002).

[21] A.V. Kolesnichenko, “Thermodynamic modeling of developed structured turbulence with energydissipation fluctuations”, *Astron. Vestn*., **38**(2), 144-170 (2004).

[22] I. Prigogine, *Introduction to Thermodynamics of Irreversible Processes,* John Wiley & Sons, New York, (1967).

[23] A. M. Obukhov, “On the energy distribution in the spectrum of turbulent flow”, *Izv. Akad. Nauk USSR, Serg. Geogr. Geophys,* **5**, 453-466 (1941)

[24] A.M. Obukhov, “Some specific features of atmospheric turbulence”, *J. Fluid Mech*., **13**, 77-81 (1962).

[25] A.S. Monin, A.M. Yaglom, *Statisticheskaia gidrodinamika, Tom* 1, Gidrometeoizdat, St. Petersburg, (1992).

[26] A. Favre, “Statistical Equations of Turbulents Gases”, *In: Problems of Hydrodynamics and Continuum Mechanics, SIAM, Philadelphia,* 231-267 (1969)

[27] A.V. Kolesnichenko, “Sootnosheniia Stefana-Maksvella I potok tepla dlia turbulentnoy mnogokomponentnoy sredy”, *V zb.: Problemy sovremennoi mekhaniki, Moskwa, Moskovskii Gosudarstvennyi Universitet,* 52-75 (1998).

[28] A.N. Kolmogorov, “Local structure of turbulence in an incompressible fluid at very large Reynolds numbers”, *Dokl. Akad. Nauk USSR,* **30**, 299-303 (1941).

[29] L. Onsager, “Statistical hydrodynamics”, *Nuovo Cimento (Suppltmtnt),* **6**, 279-287 (1949).

[30] A.K. Blackadar, “Extension of the laws of thermodynamics to turbulent system”, *J. Meteorol.*, **12**, 165 -175(1955).

[31] L.D. Landau, V.M. Lifshitz, *Gidrodinamika,* Nauka, Moskwa, (1988).

[32] A. Munster, *Khemicheskaia Termodinamika,* Editorial URSS, Moscow, (2002).

[33] D. Kondepudi, I. Prigogine, *Modern Thermodynamics, from Heat Engines to Dissipative Structures,* John Wiley and Sons, New York, Weinheim, Brisbane, Toronto, Singapore, (1999).

[34] H. Haken, *Synergetics*, Springer, Heidelberg-Berlin-New York, (1981).

[35] H. Haken, *Information and Self-Organization*. Springer, Berlin- Heidelberg- New York, (1988).

[36] Yu. L. Klimontovich, *Turbulent motion and Structure of Chaos,* Kluwer academic Publishers, Dordrecht, (1991).

[37] V.I. Tikhonov, M.A. Mironov, *Markovskie Protsessy*, Sov. Radio, Moskwa, (1977).

[38] A.S. Monin, A.M Yaglom, *Statisticheskaia gidrodinamika, tom* 2. Gidrometeoizdat, St. Petersburg, (1996).

[39] M.D. Millionshchikov, “To the theory of homogeneous and isotropic turbulence”, *Izv. Akad. Nauk USSR, Ser. Geogr. Geoph*, **5**(4/5), 433-446 (1941).

[40] M. Ya. Marov, A. V. Kolesnichenko. *Turbulence and Self-Organization: Modelling Astrophysical Objects*, Springer, ASSL, New York-Helderberg-Dordrecht-London, (2013).

[41] G.K. Batchelor, *The Theory of Homogeneous Turbulence*, Cambridge University Press, Cambridge, (1953).

[42] J. Ferziger, H. Kaper, *Mathematical Theory of Transport Processes in Gases*, North-Holland Publishing Company, Amsterdam-London, (1972).

[43] A. V. Kolesnichenko, M. Y. Marov, “Methods of nonequilibrium thermodynamics for description of multicomponent turbulent gas mixtures”, *Arch. Mech*., **37**(1/2), 3-19 (1985).

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