A SHORT SURVEY OF THE IDEAL STRUCTURE OF
PRIVALOV SPACES ON THE UNIT DISK

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Summary. For $1 < p < \infty$, the Privalov class $N^p$ consists of all holomorphic functions $f$ on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ such that

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \left( \log^+ |f(re^{i\theta})| \right)^p \frac{d\theta}{2\pi} < +\infty.$$ 

M. Stoll [32] showed that the space $N^p$ with the topology given by the metric $d_p$ defined as

$$d_p(f, g) = \left( \int_0^{2\pi} \left( \log(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad f, g \in N^p,$$

becomes an $F$-algebra.

In this overview paper we give a survey of some known results related to the ideal structure of Privalov classes $N^p$ ($1 < p < \infty$). In Section 2 we point out that every space $N^p$ ($1 < p < \infty$) is a ring of Nevanlinna–Smirnov type in the sense of Mortini [27]. Consequently, in the next section we establish the facts that $N^p$ is a coherent ring and that $N^p$ has the Corona Property. In Section 4 we present a result of N. Mochizuki [26] which gives a complete characterization of the closed ideals in $N^p$. Consequently, if $\mathcal{M}$ is a closed ideal in $N^p$ which is not identically 0, then there is a unique modulo constants

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inner function $\varphi$ such that $\mathcal{M} = \varphi N^p$. Using this result, it can be proved that a closed subspace $E$ of $N^p$ is invariant if and only if it has the form $\varphi N^p$ for some inner function $\varphi$. This result is in fact the $N^p$-analogue of the famous Beurling’s theorem for the Hardy spaces $H^q \ (0 < q < \infty)$.

1 INTRODUCTION

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let $\mathbb{T}$ denote the boundary of $\mathbb{D}$. Let $L^q(\mathbb{T}) \ (0 < q \leq \infty)$ be the familiar Lebesgue spaces on $\mathbb{T}$. The Nevanlinna class $N$ is the set of all functions $f$ holomorphic on $\mathbb{D}$ such that

$$\sup_{0 \leq r < 1} 2\pi \int_0^{2\pi} \log^+ |f(re^{i\theta})|^q \frac{d\theta}{2\pi} < \infty,$$

where $\log^+ |x| = \max(\log |x|, 0)$ for $x \neq 0$ and $\log^+ 0 = 0$.

It is well known that for each $f \in N$, the radial limit (the boundary value) of $f$ defined as

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

exists for almost every $e^{i\theta} \in \mathbb{T}$ (e.g., see [7, p. 97]).

The Smirnov class $N^+$ consists of those functions $f \in N$ for which

$$\lim_{r \to 1} 2\pi \int_0^{2\pi} \log^+ |f(re^{i\theta})|^q \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f^*(e^{i\theta})|^q \frac{d\theta}{2\pi} < \infty.$$

Recall that we denote by $H^q \ (0 < q \leq \infty)$ the classical Hardy space on $\mathbb{D}$, defined as the set of all holomorphic functions $f$ on $\mathbb{D}$ for which

$$\|f\|_q^\max(1,q) := \sup_{0 \leq r < 1} 2\pi \int_0^{2\pi} |f(re^{i\theta})|^q \frac{d\theta}{2\pi} < +\infty.$$

Further, $H^\infty$ is the space of all bounded holomorphic functions on $\mathbb{D}$ with the supremum norm $\| \cdot \|_\infty$ defined as

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|, \quad f \in H^\infty.$$

We refer [4] for a good reference on the spaces $H^q$ and $N^+$.

For $1 < p < \infty$ the Privalov class $N^p$ consists of all holomorphic functions $f$ on $\mathbb{D}$ for which

$$\sup_{0 \leq r < 1} 2\pi \int_0^{2\pi} \left( \log^+ |f(re^{i\theta})|^p \right)^\frac{1}{p} \frac{d\theta}{2\pi} < +\infty.$$

These classes were introduced in the first edition of Privalov’s book [28, p. 93], where $N^p$ is denoted as $A_p$. It is known [26] (also see [19, Section 3]) that

$$N^q \subset N^p \ (q > p), \quad \bigcup_{p>0} H^p \subset \bigcap_{p>1} N^p, \quad \text{and} \quad \bigcup_{p>1} N^p \subset N^+, \quad$$

where the above containment relations are proper.
The study of the spaces $N^p$ ($1 < p < \infty$) was continued in 1977 by M. Stoll [32] (with the notation $(\log^+ H)^p$ in [32]). Further, the topological and functional properties of these spaces were studied by C.M. Eoff ([5] and [6]), N. Mochizuki [26], Y. Iida and N. Mochizuki [10], Y. Matsugu [12], J.S. Choa [2], J.S. Choa and H.O. Kim [3], A.K. Sharma and S.-I. Ueki [30] and in works [19]–[25] of authors of this paper; typically, the notation of these spaces varied. Linear topological structure of the spaces $N^p$ and their Fréchet envelopes was investigated in [16], [17], [21] and [22]. In particular, it was proved in [16, Theorem] that the space $N^p$ ($1 < p < \infty$) does not have the Hahn-Banach approximation property, and hence, it does not have the Hahn-Banach separation property. Furthermore, the spaces $N^p$ are neither locally convex [16, Corollary] nor locally bounded [23, Theorem 1.1]. Furthermore, the ideal structure of the algebras $N^p$ was investigated in [14], [18], [22] and [26].

We refer the recent monograph [8, Chapters 2, 3 and 9] by V.I. Gavrilov, A.V. Subbotin and D.A. Efimov for a good reference on the spaces $N^p$.

In 1977 Stoll [32] proved the following result.

**Theorem A** ([32, Theorem 4.2]). The Privalov space $N^p$ ($1 < p < \infty$) (with the notation $(\log^+ H)^p$ in [32]) with the topology given by the metric $\rho_p$ defined as

$$
\rho_p(f, g) = \left( \int_0^{2\pi} \left( \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|) \right)^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad f, g \in N^p,
$$

is an $F$-algebra, i.e., an $F$-space (a complete metrizable topological vector space with the invariant metric) in which multiplication is continuous.

Notice that (1) with $p = 1$ defines the metric $d_1$ on the Smirnov class $N^+$. N. Yanagihara proved [33] that the metric $d_1$ induces the topology on $N^+$ under which $N^+$ is an $F$-algebra.

It is well known [4, p. 26, Theorem 2.10] that every non-zero function $f \in N^+$ admits a unique factorization of the form

$$
f(z) = B(z)S_\mu(z)F(z), \quad z \in \mathbb{D},
$$

where $B$ is the Blaschke product with respect to zeros $\{z_n\} \subset \mathbb{D}$ of $f$ (the set $\{z_n\}$ may be finite), $S_\mu$ is a singular inner function, $F$ is an outer function for $N^+$, i.e.,

$$
B(z) = z^m \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z},
$$

with $\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$, $m$ a nonnegative integer,

$$
S_\mu(z) = \exp \left( -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right)
$$

with $\mu$ a measure on $[0, 2\pi]$.

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with a positive singular measure \( d\mu \), and
\[
F(z) = \lambda \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |f^*(e^{it})| \, dt \right),
\]
(5)
where \( |\lambda| = 1 \) and
\[
\log |f^*(e^{i\theta})| \in L^1(\mathbb{T}).
\]
(6)
A function \( F \) with the factorization (5) and for which \( \log |F^*(e^{i\theta})| \in L^1(\mathbb{T}) \) is called an outer function. Furthermore, a function \( \varphi \) of the form
\[
\varphi(z) = B(z)S_\mu(z), \quad z \in \mathbb{D},
\]
(7)
where the functions \( B \) and \( S_\mu \) are given by (3) and (4), respectively, is called an inner function or the inner factor of a function \( f \) factorized by (2). Notice that the function \( \varphi \) defined by (7) is a bounded holomorphic function on \( \mathbb{D} \) such that \( |\varphi^*(e^{i\theta})| = 1 \) for almost every \( e^{i\theta} \in \mathbb{T} \), and hence, \( |f^*(e^{i\theta})| = |F^*(e^{i\theta})| \) for almost every \( e^{i\theta} \in \mathbb{T} \).

The inner-outer factorization theorem for the classes \( N^p \) is given by Privalov [28] as follows.

**Theorem B** ([28, pp. 98-100]; also see [6]). A function \( f \in N^+ \) factorized by (2) with (3) -- (6) belongs to the Privalov class \( N^p \) if and only if \( \log^+ |F^*(e^{i\theta})| \in L^p(\mathbb{T}) \).

**Remark 1.** If we exclude only the condition \( (\log^+ |F^*|)^p \in L^1(T) \) from Theorem B, we obtain the well known canonical factorization theorem for the class \( N^+ \) (e.g., see [4, p. 26] or [28, p. 89]).

In this paper, we give a survey of known results related the ideal structure of the Privalov classes \( N^p \) for all \( 1 < p < \infty \).

In Section 2 of [14], the ideal structure of subrings \( N^p \) of \( N \) with \( p > 1 \) is described as consequences of the results in [27, Sections 1 and 3] given for an arbitrary ring of Nevanlinna–Smirnov type in the sense of Mortini. In particular, \( N^p \) is a ring of Nevanlinna–Smirnov type (Theorem 1). We also give a necessary and sufficient condition for an ideal \( I \) in \( H^\infty \) to be the trace of an ideal \( J \) in \( N^p \) (Theorem 2). As an application, we give another sufficient condition for an ideal \( I \) in \( H^\infty \) to be trace of an ideal \( J \) in \( N^p \), and in this case there holds \( J = I N^p \) (Theorem 3). Theorem 4 gives a necessary and a sufficient condition for a prime ideal \( P \) in \( H^\infty \) to be the trace of some prime ideal \( Q \) in \( N^p \).

In Section 3 we notice that \( N^p \) is a coherent ring for all \( p > 1 \), that is, the intersection of two finitely generated ideals in \( N^p \) is finitely generated (Theorem 5). Furthermore, the algebra \( N^p \) has the Corona Property (Theorem 6). We also give a sufficient condition for an ideal \( I \) of \( N^p \), generated by a finite number of inner functions and which contains an interpolating Blaschke product \( B \), to be equal to the whole space \( N^p \) (Theorem 7).

The basic result in Section 4 is a result of N. Mochizuki [26] which gives a complete characterization of the closed ideals of \( N^p \) (Theorem 8). A closed subspace \( E \) of \( N^p \) is invariant under multiplication by \( z \) if and only if it is an ideal (Theorem 9). Applying this result and a result of Mochizuki [26, Theorem 4], it can be proved that a closed subspace \( E \) of \( N^p \) is invariant if and only if it has the form \( \varphi N^p \) for some inner function.
φ (Theorem 10). This result is in fact the $N^p$-analogue of the famous Beurling’s theorem for the Hardy spaces $H^q$ $(0 < q < \infty)$.

2 THE IDEALS IN $N^p$ AND $H^\infty$

Following R. Mortini [27], we have the following definition.

**Definition 1.** A ring $R$ satisfying $H^\infty \subset R \subset N$ is said to be of Nevanlinna-Smirnov type if every function $f \in R$ can be written in the form $g/h$, where $g$ and $h$ belong to the space $H^\infty$ and $h$ is an invertible element in $R$.

In particular, the Nevanlinna class $N$ and the Smirnov class $N^+$ are rings of Nevanlinna-Smirnov type; hence the name (see [4, Chapter 2]). Further, Mortini noticed that by a result of M. Stoll [31], the ring $F^+ \cap N$ is of Nevanlinna-Smirnov type, where the space $F^+$ is the containing Fréchet envelope for $N^+$, consisting of those functions $f$ holomorphic in $D$ satisfying

$$\limsup_{r \to 1} (1 - r) \log M(r, f) = 0$$

with $M(r, f) = \max_{|z| = r} |f(z)|$ (see [34]).

By Theorem A, it is easy to show the following result (see [6], where $N^p$ is denoted as $N^+_{\alpha}$).

**Theorem C** ([6]). A function $f \in N$ belongs to the Privalov class $N^p$ if and only if it can be expressed as the ratio $g/h$, where $g$ and $h$ are in $H^\infty$, and $h$ is an outer function such that $\log |h^*| \in L^p(T)$.

Clearly, by Theorem B, every function $h$ described in Theorem C is an invertible element of $N^p$. Therefore, we have the following result.

**Theorem 1** ([14, Theorem B]). $N^p$ $(1 < p < \infty)$ is a ring of Nevanlinna–Smirnov type.

As an application of Theorems A and B and the results of Mortini in [27], in Section 2 of [14] were obtained some facts about the ideal structure of the algebra $N^p$.

**Definition 2.** We say that an ideal $I$ in $H^\infty$ is the trace of an ideal $J$ in $N^p$ if $I = J \cap H^\infty$.

The following result is an immediate consequence of Theorems A, B and [27, Satz 1, Satz 2].

**Theorem 2** ([14, Theorem 1]). An ideal $I$ in $H^\infty$ is the trace of an ideal $J$ in $N^p$ if and only if the following condition is satisfied: If $f \in I$, $F$ is an outer function with $\log |F^*| \in L^p(T)$, and if $fF \in H^\infty$, then $fF \in I$. In this case, $J$ is a unique ideal in $N^p$ with $I = J \cap H^\infty$, and there holds $J = IN^p$.

Further, the above theorem immediately yields the following result.

**Theorem 3** ([14, Theorem 2]). Suppose that $I$ is an ideal in $H^\infty$ such that $f \in I$ implies that the inner factor of $f$ also belongs to $I$. Then $I$ is the trace of an ideal $J$ in $N^p$, and there holds $J = IN^p$. 

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Remark 2. As noticed in [14, p. 130, Remark], it remains an open question is it true the converse of Theorem 3. While this is true for the Nevanlinna class and the Smirnov class [27, Korrolar 1 and Korrolar 2, resp.], the corresponding problem is here complicated by the fact that there exist outer functions which are not invertible in $N^p$.

Definition 3. An ideal $P$ in a ring $R$ is prime if whenever $fg \in P$, $f, g \in R$, then either $f$ or $g$ is in $P$.

A characterization of the invertible elements in $N^p$ and a result in [27, Satz 3] yield the following result established in [14].

Theorem 4 ([14, Theorem 3]). A prime ideal $P$ in $H^\infty$ is the trace of some prime ideal $Q$ in $N^p$ if and only if $P$ contains no outer functions $F$ for which $\log |F^*| \in L^p(T)$. When this is the case, $Q$ is a unique prime ideal in $N^p$ with this property, and there holds $Q = PN^p$.

Remark 3. By a result of Mochizuki [26, Theorem 3] (see [14, p. 131, Remark]), every prime ideal of $N^p$ which is not dense in $N^p$ is equal to the set of functions in $N^p$ vanishing at a specific point of $\mathbb{D}$. The analogous result for the class $N^+$ was proved in [29, Theorem 1].

3 FINITELY GENERATED IDEALS IN $N^p$

Definition 4. An ideal $J$ in the ring $R$ such that $H^\infty \subset R \subset N^p$, is called finitely generated if there exist elements $f_1, \ldots, f_n \in R$ such that

$$J = (f_1, \ldots, f_n) = \left\{ \sum_{i=1}^n g_i f_i : g_i \in R \right\}.$$

If $n$ can be chosen to be one, then $J$ is a principal ideal. A ring $R$ is said to be coherent if the intersection of two finitely generated ideals in $R$ is finitely generated.

Using the result in [13] that $H^\infty$ is a coherent ring, it was shown in [27, Satz 7] that this is true for all rings of Nevanlinna–Smirnov type. In particular, by Theorem 1, we have the following result.

Theorem 5 ([14, Theorem 4]). $N^p$ is a coherent ring for all $p > 1$.

Definition 5. We say that a commutative ring $R$ with unit of holomorphic functions on the disk $\mathbb{D}$ has the Corona Property if the ideal generated by $f_1, \ldots, f_n \in R$ is equal to $R$ if and only if there is an invertible element $f$ of $R$ such that

$$|f(z)| \leqslant \sum_{i=1}^n |f_i(z)| \quad \text{for all} \quad z \in \mathbb{D}.$$

Definition 5 is motivated by the famous Corona Theorem of Carleson (for example, see [7, p. 324] or [4, p. 202]), which states that the algebra $H^\infty$ of all bounded holomorphic functions on $\mathbb{D}$ has the Corona Property. Mortini noticed [27, Satz 4] that by a result of
Wolff [7, p. 329], it is easy to show that every ring of Nevanlinna–Smirnov type has the Corona Property. In particular, by Theorem 1 we have the following result.

**Theorem 6** ([14, Theorem 5]). The algebra $N^p$ has the Corona Property for all $p > 1$.

**Remark 4.** It was proved in [11, Theorem 7] that there exists a subalgebra of the Nevanlinna class $N$ containing the Smirnov class $N^+$ without the Corona Property.

**Definition 6.** A sequence $\{z_k\}_{k=1}^{\infty} \subset \mathbb{D}$ is called an interpolating sequence (for $H^\infty$) if for every bounded sequence $\{\omega_k\}_{k=1}^{\infty}$ of complex numbers there exists a function $f$ in $H^\infty$ such that $f(z_k) = \omega_k$ for every $k = 1, 2, \ldots$. An interpolating Blaschke product is a Blaschke product given by (3) whose (simple) zeros form an interpolating sequence.

The following theorem given in [14] generalizes Theorem 6 in [27].

**Theorem 7** ([14, Theorem 7]). Assume that $I$ is an ideal in $N^p$ generated by inner functions $\varphi_1, \ldots, \varphi_n$, and suppose that $I$ contains an interpolating Blaschke product $B$ with zeros $\{z_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} \left(1 - |z_k|^2\right)^p \left|\log \left(|\varphi_1(z_k)| + \cdots + |\varphi_n(z_k)|\right)\right|^p < \infty.$$  

Then $I = N^p$.

### 4 IDEALS IN THE SPACES $N^p$ GENERATED BY INNER FUNCTIONS

Let $U$ denote the operator of “multiplication by $z$” on the space $N^p$, that is,

$$(Uf)(z) = zf(z) \quad (f \in N^p, z \in \mathbb{D}).$$

$U$ is called the right shift or unilateral shift because the Taylor coefficients of $f$ one unit to the right.

**Definition 7.** An invariant subspace of the space $N^p$ is defined as a closed subspace $E$ of $N^p$ such that $(Uf)(z) \in E$ whenever $f \in E$.

A characterization of the closed ideals of $N^p$ is completely given by N. Mochizuki [26] as follows.

**Theorem 8** ([26, Theorem 4]; cf. also see [22, Theorem 2.1]). Let $\mathcal{M}$ be a closed ideal in $N^p$ which is not identically 0. Then there is a unique modulo constants inner function $\varphi$ defined by (7) such that $\mathcal{M} = \varphi N^p$, where

$$\varphi N^p = \{\varphi f : f \in N^p\}.$$  

The following result was attributed in [22].

**Theorem 9** ([22, Lemma 2.2]). A closed subspace $E$ of $N^p$ is invariant if and only if it is an ideal.
As an immediate consequence of Theorems 8 and 9, it is obtained in [22] the following $N^p$-analogue of the famous Beurling’s theorem for the Hardy spaces $H^q$ ([1]; also see [9, Ch. 7, p. 99]).

**Theorem 10** ([22, Theorem 2.3]; cf. also [20, the assertion 2.3 on p. 99]). *A closed subspace* $E$ *of* $N^p$ *is invariant if and only if it has the form* $\varphi N^p$ *for some inner function* $\varphi$.

**Remark 5.** Theorem 10 shows that there is a one-to-one correspondence between inner functions and invariant subspaces of $N^p$; so each invariant subspace of $N^p$ being of the form of an ideal $\varphi N^p$, where $\varphi$ is an inner function.

**Remark 6.** By [29, Theorem 2], it follows that Theorem 8 is also true for the Smirnov class $N^+$. 

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