MATHEMATICA MONTISNIGRI
Vol XXVIII (2013) 99-106

USING “WOLFRAM MATHEMATICA 9.0” TO SIMULATE PROBABILITY PROBLEMS

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Summary. The topic and objective of this paper is to illustrate how Wolfram Mathematica 9, through her symbolic programming environment, can be used to simulate probability problems. Apart from the general discussion of its uses, several examples of Mathematica's application will be shown. The examples will clarify Markov Chains and Queues whose simbolic representation in this software makes visualisation the process, simulation process path and computation the stationary distribution much easier.

1 INTRODUCTION

Consider a family of random variables X_t defined on a common probability space (Ω, F, P) and indexed by a parameter $t \in T$. If the parameter set T is a subset of the real line (most commonly $\mathbb{Z}, \mathbb{Z}^+, \mathbb{R}$ or $\mathbb{R}^+$), we refer to the parameter t as time, and to X_t as a random process. If T is a subset of a multi-dimensional space, then X_t called a random field.

In a random process $\{X(t), t \in T\}$, the values assumed by $X(t)$ are called states, and the set of all possible values forms the state space E of the random process. If the index set T of a random process is discrete, then the process is called a discrete-parameter (or discrete-time) process. If T is continuous, then we have a continuous-parameter (or continuous-time) process. If the state space E of a random process is discrete, then the process is called a discrete-state process, often referred to as a chain. If the state space E is continuous, then we have a continuous-state process.

2 CHARACTERIZATION OF RANDOM PROCESSES

Consider a random process X(t). For a fixed time t, $\mathrm{X(t_1) = x_1}$ is a random variable, and its distribution function $\mathrm{F_x(x_1, t_1)}$ is defined as $\mathrm{F_x(x_1, t_1) = P\{X(t_1) \le x_1\}}$. $\mathrm{F_x(x_1, t_1)}$ is known as the first-order distribution of X(t). In the same way, if we consider two random variables $\mathrm{X(t_1) = x_1}$ and $\mathrm{X(t_2) = x_2}$ then their joint distribution functions is known as second-order distribution of X(t) and is defined by $\mathrm{F_x}(\mathrm{x_1}, x_2, \mathrm{t_1}, t_2) = \mathrm{P\{X(t_1) \le x_1}, X(t_2) \le x_2\}$. In general, we define the *nth-order distribution* of $X(t)$ by

$$F_x(x_1, \cdots, x_n, t_1, \cdots, t_n) = P\{X(t_1) \le x_1, \cdots, X(t_n) \le x_n\}. \tag{1}$$

If $X(t)$ is a discrete-time process, then $X(t)$ is specified by a collection of pmf s:

2010 Mathematics Subject Classification: 60J10, 60K35, 97R20.
Key words and Phrases: Random Processes, Markov Chain, Brownian bridge

$$p_x(x_1,\dots,x_n,t_1,\dots,t_n) = P\{X(t_1) = x_1,\dots,X(t_n) = x_n\}. \tag{2}$$

If $X(t)$ is a continuous-time process, then $X(t)$ is specified by a collection of pdf's:

$$f_x(x_1,\cdots x_n,t_1,\cdots,t_n) = \frac{\partial^n F_x(x_1,\cdots,x_n,t_1,\cdots,t_n)}{\partial x_1 \partial x_2 \cdots \partial x_n}. \tag{3}$$

The *mean* of $X(t)$ is defined by $\mu_x(t) = E[X(t)]$ where $X(t)$ is treated as a random variable for a fixed value of t. A measure of dependence among the r.v.'s of $X(t)$ is provided by its autocorrelation function, defined by

$$R_x(t,s) = E\big(X(t)X(s)\big). \tag{4}$$

It is clear that $R_x(t,s) = R_x(s,t)$ and $R_x(t,t) = E[X^2(t)]$.

The *autocovariance function* of $X(t)$ is defined by

$$\begin{aligned} K_x(t,s) = Cov[X(t),X(s)] &= E\{[X(t) - \mu_x(t)][X(s) - \mu_x(s)]\} = \\ &= R_x(t,s) - \mu_x(t)\mu_x(s). \end{aligned} \tag{5}$$

The *variance* of X(t) is given by

$$\sigma_x^2(t) = Var[X(t)] = E\{[X(t) - \mu_x(t)]^2\} = K_x(t,t).$$

3 CLASSIFICATION OF RANDOM PROCESSES

3.1 Stationary Processes

A random process $\{X(t),\ t \in T\}$ is said to be stationary if, for all n and for every set of time instants $(t \in T,\ i = 1,2,\dots,n)$

$$F_x(x_1,\cdots,x_n,t_1,\cdots,t_n) = F_x(x_1,\cdots,x_n,t_1+\tau,\cdots,t_n+\tau) \tag{5}$$

for any τ. Hence, the distribution of a stationary process will be unaffected by a shift in the time origin, and $X(t)$ and $X(t + \tau)$ will have the same distributions for any τ.

If stationary condition of a random process $X(t)$ does not hold for all n but holds for $n \leq k$, then we say that the process $X(t)$ is stationary to order k. If $X(t)$ is stationary to order 2, then $X(t)$ is said to be wide-sense stationary.

3.2 Independent Processes

If in a random process $X(t)$, $X(t_i)$ for $i = 1,2,\dots,n$ are independent r.v.'s, so that for $n = 2,3,\dots,n$

$$F_x(x_1,\cdots,x_n,t_1,\cdots,t_n) = \prod_{i=1}^{n} F_x(x_i,t_i) \tag{6}$$

then we call $X(t)$ an independent random process. Thus, a first-order distribution is sufficient to characterize an independent random process $X(t)$.

3.3 Processes with Stationary Independent Increments

A random process $\{X(t), t \geq 0\}$ is said to have independent increments if whenever $0 < t_1 < t_2 < \cdots < t_n$,

$$X_0, X(t_1) - X(0), X(t_2) - X(t_1), \cdots, X(t_n) - X(t_n - 1) \tag{7}$$

are independent.

If $\{X(t), t > 0$ has independent increments and $X(t) - X(s)$ has the same distribution as $X(t+h) - X(s+h)$ for all $s, t, h > 0$, $s < t$, then the process $X(t)$ is said to have stationary independent increments.

4 MARKOV PROCESSES

Markov process is a random process whose future probabilities are determined by its most recent values. A random process $(\boldsymbol{X(t), t > 0})$ is said to be a markov process if

$$\begin{aligned} P\{X(t_{n+1}) \leq x_{n+1} | X(t_1) = x_1, X(t_2) = x_2, \cdots, X(t_n) = x_n\} \\ = P\{X(t_{n+1}) \leq x_{n+1} | X(t_n) = x_n\} \end{aligned} \tag{8}$$

whenever $t_1 < t_2 < \cdots < t_n < t_{n+1}$.

A discrete-state Markov process is called a Markov chain. For a discrete-parameter Markov chain $\{X_n, n \geq 0\}$ we have for every n

$$P(X_{n+1} = j | X_0 = i_0, X_1 = i_1, \cdots, X_n = i) = P(X_{n+1} = j | X_n = i). \tag{9}$$

Let $\{X_n, n \geq 0\}$ be a homogeneous Markov chain with a discrete infinite state space $E = (0, 1, 2, \ldots)$. Then

$$p_{ij} = P\{X_{n+1} = j | X_n = i\}, \ \ i \geq 0, j \geq 0 \tag{10}$$

regardless of the value of $\boldsymbol{n}$. A transition probability matrix of $\{\boldsymbol{X_n, n \geq 0}\}$ is defined by

$$P = \{p_{ij}\} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{11}$$

where the elements satisfy

$$p_{ij} \geq 0, \ \ \sum_{j=0}^{\infty} p_{ij} = 1, \ \ \ i = 0, 1, 2, \cdots \tag{12}$$

Wolfram *Mathematica 9* provides cohesive and comprehensive random process support. Using a symbolic representation of a process makes it easy to simulate its behavior, estimate parameters from data, and compute state probabilities at different times.

There is additional functionality for special classes of random processes such as markov chains. Some of these functions will be illustrated throught the following examples.

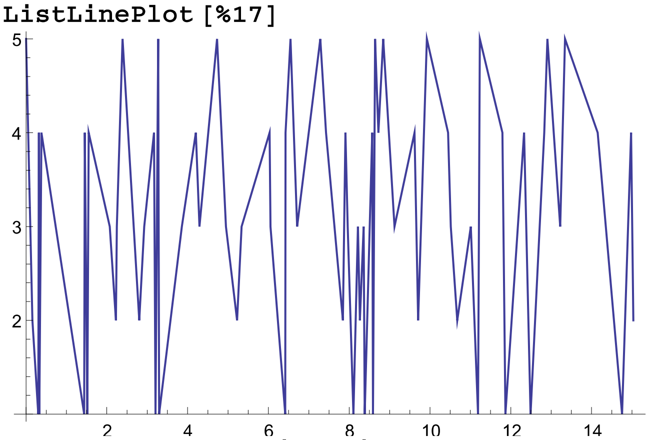
EXAMPLE 1. In order to simulate Markov chain we use Wolfram Mathematica function CountinuousMarkovProcess[p_0,q] where p_0 is initial state probability vector, and q is transition rate matrix. These function allows q to be an $\boldsymbol{n} \times \boldsymbol{n}$ matrix where $\boldsymbol{q_{ii}} \leq \boldsymbol{0}$ and $\boldsymbol{q_{ij}} \geq \boldsymbol{0}$ for $\boldsymbol{i} \neq \boldsymbol{j}$ with rows that sums to 0, p_0 is a vector of length *n* of non-negative elements that sum up to 1. We can use these function with other functions such as MarkovProcessProperties, PDF, CDF, Probability, RandomFunction ect. The example below shows simulation of countinuous-time and discrete-state Markov process.

In[16]:= mc = ContinuousMarkovProcess $\left[\left\{\frac{1}{2}, 0, 0, 0, \frac{1}{2}\right\}, \begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}\right]$

In[17]:= podaci = RandomFunction[mc, {0, 15}]

In[18]:= ListLinePlot[%17]

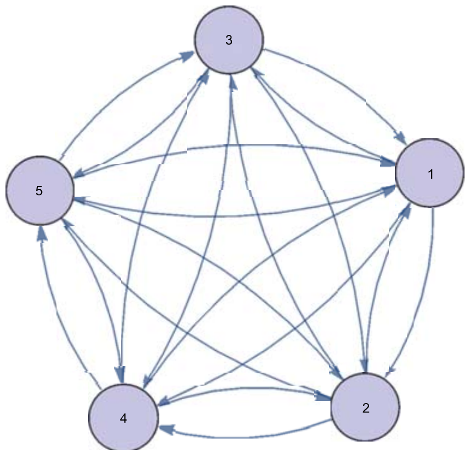
Out[18]=



In[19]:= PDF[mc[t], k] // PiecewiseExpand

Out[19]= $\begin{cases} \frac{1}{5}\, e^{-5t}\left(-1 + e^{5t}\right) & k == 2 \,||\, k == 3 \,||\, k == 4 \\ \frac{1}{10}\, e^{-5t}\left(3 + 2\, e^{5t}\right) & k == 1 \,||\, k == 5 \\ 0 & \text{True} \end{cases}$

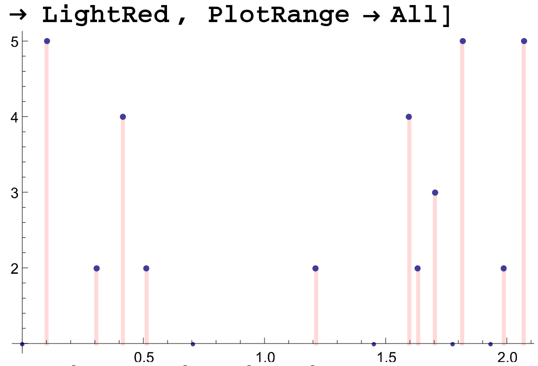
In[22]:= Graph[mc]

Out[22]= 

In[24]:= `RandomFunction [mc]`

In[28]:=
```
simplepath =
 RandomFunction[ContinuousMarkovProcess[{1/2,0,0,0,1/2},
 {{-4, 1, 1, 1, 1}, {1, -4, 1, 1, 1}, {1, 1, -4, 1, 1},
 {1, 1, 1, -4, 1}, {1, 1, 1, 1, -4}}],{0, 2}] // Normal
```

In[30]:=
```
ListPlot[simplepath, Filling → Axis, FillingStyle
 → LightRed, PlotRange → All]
```

Out[30]= 

In[31]:= `StationaryDistribution [mc]`

In[32]:=
```
CDF[ProbabilityDistribution [1/5 Boole[x == 1] + 1/5 Boole
 [x == 2] + 1/5 Boole[x == 3] + 1/5 Boole[x == 4] + 1/5 Boole[x == 5],
 {x, 1, 5, 1}], x]
```

Out[32]=
$$\begin{cases} \frac{1}{5} & 1 \le x < 2 \\ \frac{2}{5} & 2 \le x < 3 \\ \frac{3}{5} & 3 \le x < 4 \\ \frac{4}{5} & 4 \le x < 5 \\ 1 & x \ge 5 \\ 0 & \text{True} \end{cases}$$

5 COUNTING PROCESSES

A random process $\{X(t),\ t \geq\ 0$ is said to be a counting process if $X(t)$ represents the total number of "events" that have occurred in the interval $(0, t)$. The most important types of counting processes are Wiener process and the Poisson process.

5.1 Wiener Process

A random process $(X(t), t \geq 0)$ is called a Wiener process if

1. X(t) has stationary independent increments.
2. The increment $X(t) - X(s)$ $(t\ >\ s)$ is normally distributed.
3. $E[X(t)]\ = 0$.
4. $X(0)\ =\ 0$.

The Wiener process is also known as the Brownian motion process, since it originates as a model for *Brownian motion*, the motion of particles suspended in a fluid. These process is an example of an independent stationary increments process. For $0 \leq s \leq t$ the increment $X(t) - X(s)$ is a Gaussian random variable with zero mean and variance $t - s$.

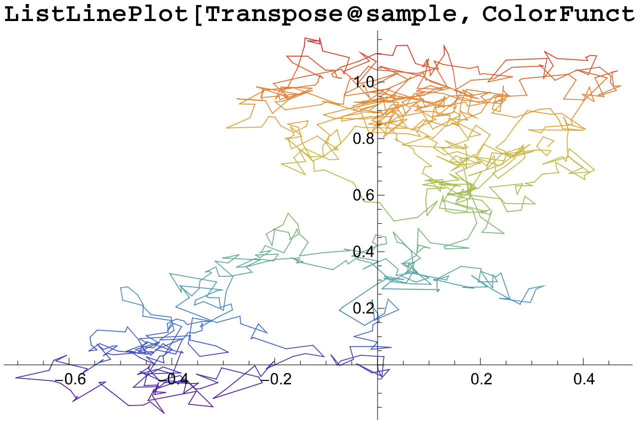
A random process $(X(t), t \geq 0)$ is called a Wiener process with drift coefficient μ if

1. $X(t)$ has stationary independent increments.
2. $X(t)$ is normally distributed with mean μt.
3. $X(0)\ =\ 0$.

EXAMPLE 2. Simulation of a Wiener process is enabled by functions WienerProcess[μ, σ], where μ represents drift and σ is volatility. The parameter μ can be any real number and the parameter σ can be any positive real number. The use of these functions is exemplified by the following example. The example shows simulatation of Brownian motion in two dimensions.

```
In[11]:= SeedRandom[151];
sample = RandomFunction[WienerProcess[], {0, 1, .001}, 2]["States"];
ListLinePlot[Transpose@sample, ColorFunction → "Rainbow"]
```

Out[11]=



A process $X(t)$ on a probability space (Ω, F, P) is a *Brownian motion process* if and only if:

1. Sample paths $X_\omega(t)$ are continuous functions of t for almost all ω.
2. $X_\omega(0) = 0$ for almost all ω.

3. For $0 \le s \le t$, the increment $X(t) - X(s)$ is a Gaussian random variable with zero mean and variance $t - s$.
4. Random variables $X(t_0), X(t_1) - X(t_0), \cdots, X(t_k) - X(t_{k-1})$ are independent for every $k \ge 1$ and $0 = t_0 \le t_1 \le \ldots \le t_k$.

An $\mathbb{R}^d$-valued process $X(t) = (X^1(t), \cdots, X^d(t))$ is said to be a (standard) d-dimensional Brownian motion if its components $X^1(t), \cdots, X^d(t)$ are independent one-dimensional Brownian motions.

A *Brownian bridge* is a continuous stochastic process with a probability distribution that is the conditional distribution of a Wiener process given prescribed values at the beginning and end of the process. A function which is used for simulating Brownian bridge is BrownianBridgeProcess[σ, {t_1, a}, {t_2, b}]. This function represents the Brownian bridge process from value a at time t_1 to value b at time t_2 with volatility σ. The following example displays displays 10 paths of a Brownian bridge process connecting two values, at the beginning (which is 0) and at the end (also 0).

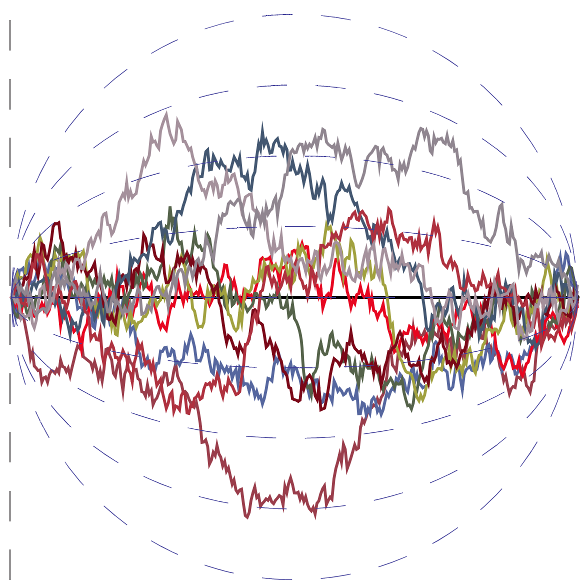


Figure 1: Brownian bridge

5.1 Poisson Process

The Poisson process is one of the two most fundamental stochastic processes. The other one is Brownian motion, with which Poisson process shares a number of common properties (both are examples of a Levy process).

A counting process $X(t)$ is said to be a Poisson process with rate (or intensity) $\lambda > 0$ if

1. $X(0) = 0$,
2. $X(t)$ has independent increments,
3. The number of events in any interval of length t is Poisson distributed with mean λt, that is, for all $s, t > 0$

$$P[X(t+s)-X(s)=n]=e^{-\lambda x}\frac{(\lambda t)^n}{n!}, n=0,1,2,\cdots. \tag{13}$$

From the condition 3 it follows that a Poisson process has stationary increments and that

$$E[X(t)]=\lambda t. \tag{14}$$

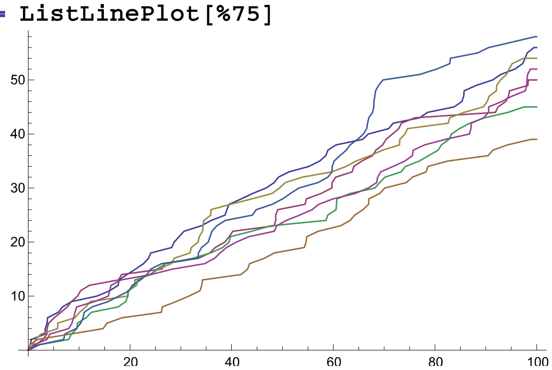
EXAMPLE 3. This example shows sample trajectories of seven Poisson processes with intensity $\boldsymbol{\lambda}=\frac{\mathbf{1}}{\mathbf{2}}$. The function which is used here is PoissonProcess[$\boldsymbol{\mu}$]. PoissonProcess[$\boldsymbol{\mu}$] simulate a continuous-time and discrete-state random process and allows $\boldsymbol{\mu}$ to be any positive real number.

```
In[75]:= p = RandomFunction[PoissonProcess[.5], {0, 100}, 7]

Out[75]= TemporalData[ 7 ]

In[76]:= ListLinePlot[%75]
```

Out[76]=



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Received September 20, 2013