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STUDY ON LA-RING BY THEIR INTUITIONISTIC FUZZY IDEALS

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Summary. In this paper, we extend the characterizations of Kuroki [17], by initiating the concept of intuitionistic fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals in a class of non-associative and non-commutative rings (LA-ring). We characterize regular (intra-regular, both regular and intra-regular) LA-rings in terms such ideals.

1 INTRODUCTION

In ternary operations, the commutative law is given by $abc = cba.$ Kazim and Naseerudin [7], have generalized this notion by introducing the parenthesis on the left side of this equation to get a new pseudo associative law, that is $(ab)c = (cb)a.$ This law $(ab)c = (cb)a$ is called the left invertive law. A groupoid S is called a left almost semigroup (abbreviated as LA-semi-group) if it satisfies the left invertive law. An LA-semi-group is a midway structure between a commutative semigroup and a groupoid.

A groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). An LA-semi-group is medial, but in general an LA-semi-group needs not to be paramedial. Every LA-semi-group with left identity is paramedial and also satisfies $a(bc) = b(ac),$ $(ab)(cd) = (dc)(ba).$

Kamran [16], extended the notion of LA-semi-group to the left almost group (LA-group). An LA-semi-group G is called a left almost group, if there exists a left identity $e \in G$ such that $ea = a$ for all $a \in G$ and for every $a \in G$ there exists $b \in G$ such that $ba = e.$

Shah et al. [22], by a left almost ring, mean a non-empty set R with at least two elements such that $(R,+)$ is an LA-group, $(R,\cdot)$ is an LA-semi-group, both left and right distributive laws hold. For example, from a commutative ring $(R,+,\cdot),$ we can always obtain an LA-ring $(R,\oplus,\cdot)$ by defining for all $a, b \in R,$ $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

A non-empty subset A of R is called an LA-subring of R if $a - b \in A$ and $ab \in A$ for all

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$a, b \in A.$ A is called a left (resp. right) ideal of R if $(A,+)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R.

A non-empty subset A of R is called an interior ideal of R if $(A,+)$ is an LA-group and $(RA)R \subseteq A.$ A non-empty subset A of R is called a quasi-ideal of R if $(A,+)$ is an LA-group and $AR \cap RA \subseteq A.$ An LA-subring A of R is called a bi-ideal of R if $(AR)A \subseteq A.$ A non-empty subset A of R is called a generalized bi-ideal of R if $(A,+)$ is an LA-group and $(AR)A \subseteq A.$

We will define the concept of intuitionistic fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring $R.$ We will establish a study by discussing the different properties of such ideals. We will characterize regular (resp. intra-regular, both regular and intra-regular) LA-rings by the properties of intuitionistic fuzzy (left, right, quasi-, bi-, generalized bi-) ideals such ideals.

2 INTUITIONISTIC FUZZY IDEALS IN LA-RINGS

After, the introduction of fuzzy set by Zadeh [24], several researchers explored on the generalization of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1], as a generalization of the notion of fuzzy set. Liu [18], introduced the concept of fuzzy subrings and fuzzy ideals of a ring. Many authors have explored the theory of fuzzy rings (for example [3, 9, 11-15, 18, 19-20, 23]). Gupta and Kantroo [4], gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [17], characterized regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (right, quasi, bi-) ideals.

An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : \ x \in X\},$ where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership and the degree of non-membership, respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$ [1].

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : \ x \in X\}$ in X can be identified to be an ordered pair (μ_A, γ_A) in $I^X \times I^X$, where I^X is the set of all functions from X to $[0,1].$ For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : \ x \in X\}.$

Banerjee and Basnet [2] and Hur et al. [6], initiated the notion of intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring. Subsequently many authors studied the intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring by describing the different properties (see [5]). Shah et al. [21, 22] initiated the concept of intuitionistic fuzzy normal subrings over a non-associative ring and also characterized the non-associative rings by their intuitionistic fuzzy bi-ideals in [8]. Kausar [10] explored the notion of direct product of finite intuitionistic anti fuzzy normal subrings over non-associative rings.

We initiate the notion of intuitionistic fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring R.

An intuitionistic fuzzy set (IFS) $A=(\mu_A,\gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy LA-subring of R if

(1) $\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(2) $\gamma_A(x-y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,

(3) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(4) $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$, for all $x, y \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy left ideal of R if

(1) $\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(2) $\gamma_A(x-y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,

(3) $\mu_A(xy) \geq \mu_A(y)$,

(4) $\gamma_A(xy) \leq \gamma_A(y)$, for all $x, y \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy right ideal of R if

(1) $\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(2) $\gamma_A(x-y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$,

(3) $\mu_A(xy) \geq \mu_A(x)$,

(4) $\gamma_A(xy) \leq \gamma_A(x)$, for all $x, y \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of R is called an intuitionistic fuzzy ideal of an LA-ring R if it is both an intuitionistic fuzzy left ideal and an intuitionistic fuzzy right ideal of R.
Let A be a non-empty subset of an LA-ring R. Then the intuitionistic characteristic of A is denoted by $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ and defined by

$$\mu_{\chi_A}(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases} \text{ and } \gamma_{\chi_A}(x) = \begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \notin A \end{cases}$$

We note that an LA-ring R can be considered an intuitionistic fuzzy set of itself and we write $R = I_R$, i.e., $R(x) = (\mu_R, \gamma_R) = (1, 0)$ for all $x \in R$.

Let A and B be two intuitionistic fuzzy sets of an LA-ring R. Then

(1) $A \subseteq B \Leftrightarrow \mu_A \subseteq \mu_B$ and $\gamma_A \supseteq \gamma_B$,

(2) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$,

(3) $A^c = (\gamma_A, \mu_A)$,

(4) $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B) = (\mu_{A \wedge B}, \gamma_{A \vee B})$,

(5) $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B) = (\mu_{A \vee B}, \gamma_{A \wedge B})$,

(6) $0 \approx= (0,1), 1 \approx= (1,0)$.

The product of $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ is denoted by $A \circ B = (\mu_A \circ \mu_B, \gamma_A \circ \gamma_B)$ and defined as:

$$(\mu_A \circ \mu_B)(x) = \begin{cases} \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{\mu_A(a_i) \wedge \mu_B(b_i)\}\} & \text{if } x = \sum_{i=1}^{n} a_i b_i,\ a_i, b_i \in R \\ 0 & \text{if } x \neq \sum_{i=1}^{n} a_i b_i \end{cases}$$

$$\text{and } (\gamma_A \circ \gamma_B)(x) = \begin{cases} \wedge_{x=\sum_{i=1}^{n} a_i b_i} \{\vee_{i=1}^{n} \{\gamma_A(a_i) \vee \gamma_B(b_i)\}\} & \text{if } x = \sum_{i=1}^{n} a_i b_i,\ a_i, b_i \in R \\ 1 & \text{if } x \neq \sum_{i=1}^{n} a_i b_i \end{cases}$$

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy interior ideal of R if

(1) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$,

(2) $\gamma_A(x - y) \leq \gamma_A(x) \vee \gamma_A(y)$,

(3) $\mu_A((xy)z) \geq \mu_A(y)$,

(4) $\gamma_A((xy)z) \leq \gamma_A(y)$, for all $x, y, z \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy quasi-ideal of R if

(1) $(\mu_A \circ R) \cap (R \circ \mu_A) \subseteq \mu_A$,

(2) $(\gamma_A \circ R) \cup (R \circ \gamma_A) \supseteq \gamma_A$,

(3) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$,

(4) $\gamma_A(x - y) \leq \gamma_A(x) \vee \gamma_A(y)$, for all $x, y \in R$.

An Intuitionistic fuzzy LA-subring $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy bi-ideal of R if

(1) $\mu_A((xy)z) \geq \mu_A(x) \wedge \mu_A(z)$,

(2) $\gamma_A((xy)z) \leq \gamma_A(x) \vee \gamma_A(z)$, for all $x, y, z \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy generalized bi-ideal of R if

(1) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$,

(2) $\gamma_A(x-y) \leq \gamma_A(x) \vee \gamma_A(y)$,

(3) $\mu_A((xy)z) \geq \mu_A(x) \wedge \mu_A(z)$,

(4) $\gamma_A((xy)z) \leq \gamma_A(x) \vee \gamma_A(z)$, for all $x, y, z \in R$.

An intuitionistic fuzzy ideal $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy idempotent if $\mu_A \circ \mu_A = \mu_A$ and $\gamma_A \circ \gamma_A = \gamma_A$.

Now we give some imperative properties of such ideals of an LA-ring R, which will be very helpful in later sections.

Lemma 2.1: Let R be an LA-ring. Then the following properties hold:

(1) $(A \circ B) \circ C = (C \circ B) \circ A$,

(2) $(A \circ B) \circ (C \circ D) = (A \circ C) \circ (B \circ D)$,

(3) $A \circ (B \circ C) = B \circ (A \circ C)$,

(4) $(A \circ B) \circ (C \circ D) = (D \circ B) \circ (C \circ A)$,

(5) $(A \circ B) \circ (C \circ D) = (D \circ C) \circ (B \circ A)$, for all intuitionistic fuzzy sets A, B, C and D of R.

Proof: Obvious.

Theorem 2.2: Let A and B be two non-empty subsets of an LA-ring R. then the following properties hold:

(1) If $A \subseteq B$ then $\chi_A \subseteq \chi_B$.

(2) $\chi_A \circ \chi_B = \chi_{AB}$.

(4) $\chi_A \cap \chi_B = \chi_{A \cap B}$.

Proof: (1) Suppose that $A \subseteq B$ and $a \in R$. If $a \in A$, this implies that $a \in B$. Thus $\mu_{\chi_A}(a) = 1 = \mu_{\chi_B}(a)$ and $\gamma_{\chi_A}(a) = 0 = \gamma_{\chi_B}(a)$, i.e., $\chi_A \subseteq \chi_B$.

If $a \notin A$, and $a \notin B$. Thus $\mu_{\chi_A}(a) = 0 = \mu_{\chi_B}(a)$ and $\gamma_{\chi_A}(a) = 1 = \gamma_{\chi_B}(a)$, i.e., $\chi_A \subseteq \chi_B$.

If $\alpha \notin A$ and $\alpha \in B$. Thus $\mu_{\chi A}(\alpha) = 0$ and $\mu_{\chi B}(\alpha) = 1$ and $\gamma_{\chi A}(\alpha) = 1$ and $\gamma_{\chi B}(\alpha) = 0$, i.e.,

(2) Let $x \in R$ and $x \in AB$. This means that $x=ab$ for some $\alpha \in A$ and $b \in B$.

Now

$$(\mu_{\chi_A} \circ \mu_{\chi_B})(x) = \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{\mu_{\chi_A}(a_i) \wedge \mu_{\chi_B}(b_i)\}\}$$
$$\geq \mu_{\chi_A}(a) \wedge \mu_{\chi_B}(b) = 1 \wedge 1 = 1 = \mu_{\chi_{AB}}(x)$$

and

$$(\gamma_{\chi_A} \circ \gamma_{\chi_B})(x) = \wedge_{x=\sum_{i=1}^{n} a_i b_i} \{\vee_{i=1}^{n} \{\gamma_{\chi_A}(a_i) \vee \gamma_{\chi_B}(b_i)\}\}$$
$$\leq \gamma_{\chi_A}(a) \vee \gamma_{\chi_B}(b) = 0 \vee 0 = 0 = \gamma_{\chi_{AB}}(x).$$

If $x \notin AB$, i.e., $x \neq ab$ for all $a \in A$ and $b \in B$. Then there are two cases.

(i) If $x = uv$ for some $u, v \in R,$ then

$$(\mu_{\chi_A} \circ \mu_{\chi_B})(x) = \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{\mu_{\chi_A}(a_i) \wedge \mu_{\chi_B}(b_i)\}\}$$
$$= 0 \wedge 0 = 0 = \mu_{\chi_{AB}}(x)$$
$$\text{and } (\gamma_{\chi_A} \circ \gamma_{\chi_B})(x) = \wedge_{x=\sum_{i=1}^{n} a_i b_i} \{\vee_{i=1}^{n} \{\gamma_{\chi_A}(a_i) \vee \gamma_{\chi_B}(b_i)\}\}$$
$$= 1 \vee 1 = 1 = \gamma_{\chi_{AB}}(x).$$

(ii) If $x \neq uv$ for all $u, v \in R,$ then obviously $(\chi_A \circ \chi_B)(x) = 0 = \chi_{AB}(x).$ Hence $\chi_A \circ \chi_B = \chi_{AB}.$

Similarly, we can prove (3) and (4).

Theorem 2.3: Let A be a non-empty subset of an LA-ring $R.$ then the following properties hold.

(1) A is an LA-subring of R if and only if χ_A is an intuitionistic fuzzy LA-subring of $R.$

(2) A is a left (resp. right, two-sided) ideal of R if and only if χ_A is an intuitionistic fuzzy left (resp. right, two-sided) ideal of $R.$

Proof: (1) Let A be an LA-subring of R and $a, b \in R.$ If $a, b \in A,$ then by definition $\mu_A(a) = 1 = \mu_A(b)$ and $\gamma_A(a) = 0 = \gamma_A(b).$ Since $a - b$ and $ab \in A,$ A being an LA-subring of $R,$ this implies that $\mu_A(a-b) = 1 = \mu_A(ab)$ and $\gamma_A(a-b) = 0 = \gamma_A(ab).$ Thus

$$\mu_A(a-b) \geq \mu_A(a) \wedge \mu_A(b), \ \mu_A(ab) \geq \mu_A(a) \wedge \mu_A(b)$$
$$\text{and } \gamma_A(a-b) \leq \gamma_A(a) \vee \gamma_A(b), \gamma_A(ab) \leq \gamma_A(a) \vee \gamma_A(b).$$

Similarly, we have

$$\mu_A(a-b) \geq \mu_A(a) \wedge \mu_A(b), \ \mu_A(ab) \geq \mu_A(a) \wedge \mu_A(b).$$
$$\text{and } \gamma_A(a-b) \leq \gamma_A(a) \vee \gamma_A(b), \gamma_A(ab) \leq \gamma_A(a) \vee \gamma_A(b).$$

when $a, b \notin A.$ Hence χ_A is an intuitionistic fuzzy LA-subring of $R.$

Conversely, suppose that χ_A is an intuitionistic fuzzy LA-subring of R and let $a, b \in A.$ This means that $\mu_A(a) = 1 = \mu_A(b)$ and $\gamma_A(a) = 0 = \gamma_A(b).$ Since

$$\mu_A(a-b) \geq \mu_A(a) \wedge \mu_A(b) = 1 \wedge 1 = 1,$$
$$\mu_A(ab) \geq \mu_A(a) \wedge \mu_A(b) = 1 \wedge 1 = 1,$$
$$\gamma_A(a-b) \leq \gamma_A(a) \vee \gamma_A(b) = 0 \vee 0 = 0,$$
$$\gamma_A(ab) \leq \gamma_A(a) \vee \gamma_A(b) = 0 \vee 0 = 0,$$

χ_A being an intuitionistic fuzzy LA-subring of $R.$ Thus $\mu_A(a-b) = 1 = \mu_A(ab)$ and $\gamma_A(a-b) = 0 = \gamma_A(ab),$ i.e., $a - b$ and $ab \in A.$ Hence A is an LA-subring of $R.$

(2) Let A be a left ideal of R and $a,b \in R.$ If $a,b \in A,$ then by definition $\mu_A(a) = 1 = \mu_A(b)$ and $\gamma_A(a) = 0 = \gamma_A(b).$ Since $a - b$ and $ab \in A,$ A being a left ideal of $R,$ this implies that $\mu_A(a-b) = 1 = \mu_A(ab)$ and $\gamma_A(a-b) = 0 = \gamma_A(ab).$ Thus

$$\mu_A(a-b) \geq \mu_A(a) \wedge \mu_A(b), \ \mu_A(ab) \geq \mu_A(b)$$
$$\text{and } \gamma_A(a-b) \leq \gamma_A(a) \vee \gamma_A(b), \gamma_A(ab) \leq \gamma_A(b).$$

Similarly, we have

$$\mu_A(a-b) \geq \mu_A(a) \wedge \mu_A(b), \ \mu_A(ab) \geq \mu_A(b)$$
$$\text{and } \gamma_A(a-b) \leq \gamma_A(a) \vee \gamma_A(b), \gamma_A(ab) \leq \gamma_A(b).$$

when $a,b \notin A.$ Therefore χ_A is an intuitionistic fuzzy left ideal of $R.$

Conversely, assume that χ_A is an intuitionistic fuzzy left ideal of R and let $a,b \in A$ and $z \in R.$ This means that $\mu_A(a) = 1 = \mu_A(b)$ and $\gamma_A(a) = 0 = \gamma_A(b).$ Since

$$\mu_A(a-b) \geq \mu_A(a) \wedge \mu_A(b) = 1 \wedge 1 = 1,$$
$$\mu_A(zb) \geq \mu_A(b) = 1,$$
$$\gamma_A(a-b) \leq \gamma_A(a) \vee \gamma_A(b) = 0 \vee 0 = 0,$$
$$\gamma_A(zb) \leq \gamma_A(b) = 0,$$

χ_A being an intuitionistic fuzzy left ideal of $R.$ Thus $\mu_A(a-b) = 1 = \mu_A(zb)$ and $\gamma_A(a-b) = 0 = \gamma_A(zb),$ i.e., $a - b$ and $zb \in A.$ Therefore A is a left ideal of $R.$

Remark 2.4: (i) A is an additive LA-subgroup of R if and only if χ_A is an intuitionistic fuzzy additive LA-subgroup of $R.$

(ii) A is an LA-subsemigroup of R if and only if χ_A is an intuitionistic fuzzy LA-subsemigroup of $R.$

Lemma 2.5: If A and B are two intuitionistic fuzzy LA-subrings (resp. (left, right, two-sided) ideals) of an LA-ring $R,$ then $A \cap B$ is also an intuitionistic fuzzy LA-subring (resp. (left, right, two-sided) ideal) of $R.$

Proof: Obvious.

Lemma 2.6: If A and B are two intuitionistic fuzzy LA-subrings of an LA-ring $R,$ then $A \circ B$ is also an intuitionistic fuzzy LA-subring of $R.$

Proof: Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy LA-subrings of $R.$ We have to show that $A \circ B$ is also an intuitionistic fuzzy LA-subring of $R.$ Now

$$(\mu_A \circ \mu_B)^2 = (\mu_A \circ \mu_B) \circ (\mu_A \circ \mu_B) = (\mu_A \circ \mu_A) \circ (\mu_B \circ \mu_B) \subseteq \mu_A \circ \mu_B$$
$$\text{and } (\gamma_A \circ \gamma_B)^2 = (\gamma_A \circ \gamma_B) \circ (\gamma_A \circ \gamma_B) = (\gamma_A \circ \gamma_A) \circ (\gamma_B \circ \gamma_B) \supseteq \gamma_A \circ \gamma_B.$$

Since $\mu_B - \mu_B \subseteq \mu_B$ and $\gamma_B - \gamma_B \supseteq \gamma_B,$ $B = (\mu_B, \gamma_B)$ being an intuitionistic fuzzy LA-subring of $R.$ This implies that $\mu_A \circ (\mu_B - \mu_B) \subseteq \mu_A \circ \mu_B$ and

$\gamma_A \circ (\gamma_B - \gamma_B) \supseteq \gamma_A \circ \gamma_B$, i.e., $\mu_A \circ \mu_B - \mu_A \circ \mu_B \subseteq \mu_A \circ \mu_B$ and $\gamma_A \circ \gamma_B - \gamma_A \circ \gamma_B \supseteq \gamma_A \circ \gamma_B$. Therefore $A \circ B$ is an intuitionistic fuzzy LA-subring of R.

Remark 2.7: If A is an intuitionistic fuzzy LA-subring of an LA-ring R, then $A \circ A$ is also an intuitionistic fuzzy LA-subring of R.

Lemma 2.8: Let R be an LA-ring with left identity e. Then $RR = R$ and $eR = R = \mathrm{Re}$.

Proof: Since $RR \subseteq R$ and $x = ex \in RR$, where $x \in R$, i.e., $RR = R$. Since e is the left identity of R, i.e., $eR = R$. Now $\mathrm{Re} = (RR)e = (eR)R = RR = R$.

Lemma 2.9: Let R be an LA-ring with left identity e. Then every intuitionistic fuzzy right ideal of R is an intuitionistic fuzzy ideal of R.

Proof: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of R and $x, y \in R$. Now

$$\mu_A(xy) = \mu_A((ex)y) = \mu_A((yx)e) \geq \mu_A(yx) \geq \mu_A(y)$$
$$\text{and } \gamma_A(xy) = \gamma_A((ex)y) = \gamma_A((yx)e) \leq \gamma_A(yx) \leq \gamma_A(y).$$

Thus A is an intuitionistic fuzzy ideal of R.

Lemma 2.10: If A and B are two intuitionistic fuzzy left (resp. right) ideals of an LA-ring R with left identity e, then $A \circ B$ is also an intuitionistic fuzzy left (resp. right) ideal of R.

Proof: Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy left ideals of R. We have to show that $A \circ B$ is also an intuitionistic fuzzy left ideal of R. Since $\mu_A \circ \mu_B - \mu_A \circ \mu_B \subseteq \mu_A \circ \mu_B$ and $\gamma_A \circ \gamma_B - \gamma_A \circ \gamma_B \supseteq \gamma_A \circ \gamma_B$. Now

$$R \circ (\mu_A \circ \mu_B) = (R \circ R) \circ (\mu_A \circ \mu_B) = (R \circ \mu_A) \circ (R \circ \mu_B) \subseteq (\mu_A \circ \mu_B)$$
$$\text{and } R \circ (\gamma_A \circ \gamma_B) = (R \circ R) \circ (\gamma_A \circ \gamma_B) = (R \circ \gamma_A) \circ (R \circ \gamma_A) \supseteq (\gamma_A \circ \gamma_B).$$

Hence $A \circ B$ is an intuitionistic fuzzy left ideal of R.

Remark 2.11: If A is an intuitionistic fuzzy left (resp. right) ideal of an LA-ring R with left identity e, then $A \circ A$ is an intuitionistic fuzzy ideal of R.

Lemma 2.12: If A and B are two intuitionistic fuzzy ideals of an LA-ring R, then $A \circ B \subseteq A \cap B$.

Proof: Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy ideals of R and $x \in R$. If x cannot expressible as $x = \sum_{i=1}^{n} a_i b_i$, where $a_i, b_i \in R$ and n is any positive integer, then obviously $A \circ B \subseteq A \cap B$, otherwise we have

$$(\mu_A \circ \mu_B)(x) = \vee_{x = \sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{\mu_A(a_i) \wedge \mu_B(b_i)\}\}$$
$$\leq \vee_{x = \sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{\mu_A(a_i b_i) \wedge \mu_B(a_i b_i)\}\}$$
$$= \vee_{x = \sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} (\mu_A \cap \mu_B)(a_i b_i)\} = (\mu_A \cap \mu_B)(x).$$
$$\Rightarrow \mu_A \circ \mu_B \subseteq \mu_A \cap \mu_B.$$

Similarly, we can prove $\gamma_A \circ \gamma_B \supseteq \gamma_A \cup \gamma_B$.

Therefore $A \circ B \subseteq A \cap B$ for all intuitionistic fuzzy ideals A and B of $R.$

Remark 2.13: If A is an intuitionistic fuzzy ideal of an LA-ring $R,$ then $A \circ A \subseteq A.$

Lemma 2.14: Let R be an LA-ring. Then $A \circ B \subseteq A \cap B$ for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy left ideal B of $R.$

Proof: Same as Lemma 2.12

Theorem 2.15: Let A be a non-empty subset of an LA-ring $R.$ Then A is an interior (resp. quasi-, bi-, generalized bi-) ideal of R if and only if χ_A is an intuitionistic fuzzy interior (resp. quasi-, bi-, generalized bi-) ideal of $R.$

Proof: Let A be an interior ideal of $R,$ this implies that A is an additive LA-subgroup of $R.$ Then χ_A is an intuitionistic fuzzy additive LA-subgroup of R by the Remark 2.4. Let $x, y, a \in R.$ If $a \in A,$ then by definition $\mu_{\chi_A}(a) = 1$ and $\gamma_{\chi_A}(a) = 0.$ Since $(xa)y \in A,$ A being an interior ideal of $R,$ this means that $\mu_{\chi_A}((xa)y) = 1$ and $\gamma_{\chi_A}((xa)y) = 0.$ Thus $\mu_{\chi_A}((xa)y) \geq \mu_{\chi_A}(a)$ and $\gamma_{\chi_A}((xa)y) \leq \gamma_{\chi_A}(a).$ Similarly, we have $\mu_{\chi_A}((xa)y) \geq \mu_{\chi_A}(a)$ and $\gamma_{\chi_A}((xa)y) \leq \gamma_{\chi_A}(a),$ when $a \notin A.$ Hence χ_A is an intuitionistic fuzzy interior ideal of $R.$

Conversely, suppose that χ_A is an intuitionistic fuzzy interior ideal of $R,$ this means that χ_A is an intuitionistic fuzzy additive LA-subgroup of $R.$ Then A is an additive LA-subgroup of R by the Remark 2.4. Let $x, y \in R$ and $a \in A,$ so $\mu_{\chi_A}(a) = 1$ and $\gamma_{\chi_A}(a) = 0.$ Since $\mu_{\chi_A}((xa)y) \geq \mu_{\chi_A}(a) = 1$ and $\gamma_{\chi_A}((xa)y) \leq \gamma_{\chi_A}(a) = 0,$ χ_A being an intuitionistic fuzzy interior ideal of $R.$ Thus $\mu_{\chi_A}((xa)y) = 1$ and $\gamma_{\chi_A}((xa)y) = 0,$ i.e., $(xa)y \in A.$ Hence A is an interior ideal of $R.$ Similarly, we can prove for (quasi-, bi-, generalized bi-) ideal.

Lemma 2.16: If A and B are two intuitionistic fuzzy bi-(resp. generalized bi-, quasi-, interior) ideals of an LA-ring $R,$ then $A \cap B$ is also an intuitionistic fuzzy bi- (resp. generalized bi-, quasi-, interior) ideal of $R.$

Proof: Obvious.

Lemma 2.17: If A and B are two intuitionistic fuzzy bi- (resp. generalized bi-, interior) ideals of an LA-ring R with left identity $e,$ then $A \circ B$ is also an intuitionistic fuzzy bi- (resp. generalized bi-, interior) ideal of $R.$

Proof: Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy bi-ideals of $R.$ We have to show that $A \circ B$ is also an intuitionistic fuzzy bi-ideal of $R.$ Since A and B are two intuitionistic fuzzy LA-subrings of $R,$ then $A \circ B$ is also an intuitionistic fuzzy LA-subring of R by the Lemma 2.6. Now

$$\begin{aligned} ((\mu_A \circ \mu_B) \circ R) \circ (\mu_A \circ \mu_B) &= ((\mu_A \circ \mu_B) \circ (R \circ R)) \circ (\mu_A \circ \mu_B) \\ &= ((\mu_A \circ R) \circ (\mu_B \circ R)) \circ (\mu_A \circ \mu_B) \\ &= ((\mu_A \circ R) \circ \mu_A) \circ ((\mu_B \circ R) \circ \mu_B) \\ &\subseteq \mu_A \circ \mu_B. \end{aligned}$$

Similarly, we have $((\gamma_A \circ \gamma_B) \circ R) \circ (\gamma_A \circ \gamma_B) \supseteq \gamma_A \circ \gamma_B$. Therefore $A \circ B$ is an intuitionistic fuzzy bi-ideal of R.

Lemma 2.18: Every intuitionistic fuzzy ideal of an LA-ring R is an intuitionistic fuzzy interior ideal of R. The converse is not true in general.

Proof: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of R and $x, y, z \in R$. Thus $\mu_A((xy)z) = \mu_A(xy) \geq \mu_A(y)$ and $\gamma_A((xy)z) = \gamma_A(xy) \leq \gamma_A(y)$. Hence A is an intuitionistic fuzzy interior ideal of R.

The converse is not true in general, giving an example:

Example 2.19: Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is an LA-ring.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	0	3	1	6	4	7	5
2	1	3	0	2	5	7	4	6
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	6	4	7	5	2	0	3	1
6	5	7	4	6	1	3	0	2
7	7	6	5	4	3	2	1	0

and

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	4	4	0	0	4	4	0
2	0	4	4	0	0	4	4	0
3	0	0	0	0	0	0	0	0
4	0	3	3	0	0	3	3	0
5	0	7	7	0	0	7	7	0
6	0	7	7	0	0	7	7	0
7	0	3	3	0	0	3	3	0

Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R. We define

$$\mu_A(0) = \mu_A(4) = 0.7, \quad \mu_A(1) = \mu_A(2) = \mu_A(3) = \mu_A(5) = \mu_A(6) = \mu_A(7) = 0$$

and $\gamma_A(0) = \gamma_A(4) = 0, \quad \gamma_A(1) = \gamma_A(2) = \gamma_A(3) = \gamma_A(5) = \gamma_A(6) = \gamma_A(7) = 0.7.$

$A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal of R, but not an intuitionistic fuzzy ideal of R, because A is not an intuitionistic fuzzy right ideal of R, as

$$\mu_A(41) = \mu_A(3) = 0.$$
$$\mu_A(4) = 0.7.$$
$$\Rightarrow \mu_A(41) \ngeq \mu_A(4).$$
$$\text{and } \gamma_A(41) = \gamma_A(3) = 0.7.$$
$$\gamma_A(4) = 0.$$
$$\Rightarrow \gamma_A(41) \nleq \gamma_A(4).$$

Proposition 2.20: Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R with left identity e. Then A is an intuitionistic fuzzy ideal of R if and only if A is an intuitionistic fuzzy interior ideal of R.

Proof: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy interior ideal of R and $x, y \in R$. Thus $\mu_A(xy) = \mu_A((ex)y) \geq \mu_A(x)$ and $\gamma_A(xy) = \gamma_A((ex)y) \leq \gamma_A(x)$, i.e., A is an intuitionistic fuzzy right ideal of R. Hence A is an intuitionistic fuzzy ideal of R by the Lemma 2.9. Converse is true by the Lemma 2.18.

Lemma 2.21: Every intuitionistic fuzzy left $($resp. right, two-sided$)$ ideal of an LA-ring R is an intuitionistic fuzzy bi-ideal of R.

Proof: Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R and $x, y, z \in R$. Thus

$$\mu_A((xy)z) = \mu_A(xy) \geq \mu_A(x) \text{ and } \mu_A((xy)z) = \mu_A((zy)x) \geq \mu_A(zy) \geq \mu_A(z),$$

this implies that $\mu_A((xy)z) \geq \mu_A(x) \wedge \mu_A(z)$. Similarly, we have $\gamma_A((xy)z) \leq \gamma_A(x) \vee \gamma_A(z)$. Therefore A is an intuitionistic fuzzy bi-ideal of R.

The converse is not true in general, giving an example:

Using Example 2.19, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of R, but not an intuitionistic fuzzy right ideal of R, as

$$\begin{aligned} \mu_A(41) &= \mu_A(3) = 0. \\ \mu_A(4) &= 0.7. \\ &\Rightarrow \mu_A(41) \ngeq \mu_A(4). \\ \text{and } \gamma_A(41) &= \gamma_A(3) = 0.7. \\ \gamma_A(4) &= 0. \\ &\Rightarrow \gamma_A(41) \nleq \gamma_A(4). \end{aligned}$$

Lemma 2.22: Every intuitionistic fuzzy bi-ideal of an LA-ring R is an intuitionistic fuzzy generalized bi-ideal of R.

Proof: Obvious.

Lemma 2.23: Every intuitionistic fuzzy left $($ resp. right, two-sided$)$ ideal of an LA-ring R is an intuitionistic fuzzy quasi-ideal of R.

Proof: Assume that $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy left ideal of R. Now $\mu_A \circ R \cap R \circ \mu_A \subseteq R \circ \mu_A \subseteq \mu_A$ and $\gamma_A \circ R \cup R \circ \gamma_A \supseteq R \circ \gamma_A \supseteq \gamma_A$. So A is an intuitionistic fuzzy quasi-ideal of R.

Lemma 2.24: Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then every intuitionistic fuzzy quasi-ideal of R is an intuitionistic fuzzy bi-ideal of R.

Proof: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy quasi-ideal of R and $A \circ A \subseteq A$ by the Proposition 2.20. Now

$$\begin{aligned} (\mu_A \circ R) \circ \mu_A &\subseteq (R \circ R) \circ \mu_A \subseteq R \circ \mu_A \\ \text{and } (\mu_A \circ R) \circ \mu_A &\subseteq (\mu_A \circ R) \circ R = (\mu_A \circ R) \circ (e \circ R) \\ &= (\mu_A \circ e) \circ (R \circ R) \subseteq (\mu_A \circ e) \circ R = \mu_A \circ R. \\ &\Rightarrow (\mu_A \circ R) \circ \mu_A \subseteq \mu_A \circ R \cap R \circ \mu_A \subseteq \mu_A. \end{aligned}$$

Similarly, $(\gamma_A \circ R) \circ \gamma_A \supseteq \gamma_A \circ R \cup R \circ \gamma_A \supseteq \gamma_A$. Hence A is an intuitionistic fuzzy bi-ideal of R.

3 REGULAR LA-RINGS

An LA-ring R is called a regular if for every $x \in R$, there exists an element $a \in R$ such that $x = (xa)x.$ In this section, we characterize regular LA-rings by the properties of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals.

Lemma 3.1: Every intuitionistic fuzzy right ideal of a regular LA-ring R is an intuitionistic fuzzy ideal of $R.$

Proof: Suppose that $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy right ideal of $R.$ Let $x, y \in R,$ this implies that there exists an element $a \in R,$ such that $x=(xa)x.$ Thus

$$\mu_A(xy)=\mu_A(((xa)x)y)=\mu_A((yx)(xa))\geq \mu_A(yx)\geq \mu_A(y)$$

and

$$\gamma_A(xy)=\gamma_A(((xa)x)y)=\gamma_A((yx)(xa))\leq \gamma_A(yx)\leq \gamma_A(y).$$

Hence A is an intuitionistic fuzzy ideal of $R.$

Lemma 3.2: Let $A=(\mu_A,\gamma_A)$ be an IFS of a regular LA-ring $R.$ Then A is an intuitionistic fuzzy ideal of R if and only if A is an intuitionistic fuzzy interior ideal of $R.$

Proof: Consider that $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy interior ideal of $R.$ Let $x, y \in R,$ then there exists an element $a \in R,$ such that $x=(xa)x.$ Thus

$$\mu_A(xy)=\mu_A(((xa)x)y)=\mu_A((yx)(xa))\geq \mu_A(x)$$

and

$$\gamma_A(xy)=\gamma_A(((xa)x)y)=\gamma_A((yx)(xa))\leq \gamma_A(x),$$

i.e., A is an intuitionistic fuzzy right ideal of $R.$ So A is an intuitionistic fuzzy ideal of R by the Lemma 3.1. Converse is true by the Lemma 2.18.

Remark 3.3: The concept of intuitionistic fuzzy (interior, two-sided) ideals coincides with the same concept in regular LA-rings.

Proposition 3.4: Let R be a regular LA-ring. Then $(A\circ R)\cap(R\circ A)=A$ for every intuitionistic fuzzy right ideal A of $R.$

Proof: Suppose that $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy right ideal of $R.$ This implies that $(A\circ R)\cap(R\circ A)\subseteq A,$ because every intuitionistic fuzzy right ideal of R is an intuitionistic fuzzy quasi-ideal of R by the Lemma 2.23. Let $x \in R,$ this implies that there exists an element $a \in R,$ such that $x=(xa)x.$ Thus

$$(\mu_A\circ R)(x)=\vee_{x=\sum_{i=1}^{n}a_ib_i}\{\wedge_{i=1}^{n}\{\mu_A(a_i)\wedge R(b_i)\}\}$$
$$\geq \mu_A(xa)\wedge R(x)\geq \mu_A(x)\wedge 1=\mu_A(x)$$
$$\text{and } (\gamma_A\circ R)(x)=\wedge_{x=\sum_{i=1}^{n}a_ib_i}\{\vee_{i=1}^{n}\{\gamma_A(a_i)\vee R(b_i)\}\}$$
$$\leq \gamma_A(xa)\vee R(x)\leq \gamma_A(x)\vee 0=\gamma_A(x).$$
$$\Rightarrow A\subseteq A\circ R.$$

Similarly, we have $A \subseteq R \circ A,$ i.e., $A \subseteq (A \circ R) \cap (R \circ A).$ Hence $(A \circ R) \cap (R \circ A) = A.$

Lemma 3.5: Let R be a regular LA-ring. Then $D \circ L = D \cap L$ for every intuitionistic fuzzy right ideal D and every intuitionistic fuzzy left ideal L of $R.$

Proof: Since $D \circ L \subseteq D \cap L,$ for every intuitionistic fuzzy right ideal $D = (\mu_D, \gamma_D)$ and every intuitionistic fuzzy left ideal $L = (\mu_L, \gamma_L)$ of R by the Lemma 2.14. Let $x \in R,$ this means that there exists an element $a \in R$ such that $x = (xa)x.$ Thus

$$
\begin{aligned}
(\mu_D \circ \mu_L)(x) &= \vee_{x = \sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{\mu_D(a_i) \wedge \mu_L(b_i)\}\} \\
&\geq \mu_D(xa) \wedge \mu_L(x) \geq \mu_D(x) \wedge \mu_L(x) = (\mu_D \cap \mu_L)(x) \\
\text{and } (\gamma_D \circ \gamma_L)(x) &= \wedge_{x = \sum_{i=1}^{n} a_i b_i} \{\vee_{i=1}^{n} \{\gamma_D(a_i) \vee \gamma_L(b_i)\}\} \\
&\leq \gamma_D(xa) \vee \gamma_L(x) \leq \gamma_D(x) \vee \gamma_L(x) = (\gamma_D \cup \gamma_L)(x).
\end{aligned}
$$

Therefore $D \circ L = D \cap L.$

Lemma 3.6: Let R be an LA-ring with left identity $e.$ Then Ra is the smallest left ideal of R containing $a.$

Proof: Let $x, y \in Ra$ and $r \in R.$ This implies that $x = r_1 a$ and $y = r_2 a,$ where $r_1, r_2 \in R.$ Now

$$
\begin{aligned}
x - y &= r_1 a - r_2 a = (r_1 - r_2)a \in Ra \\
\text{and } rx &= r(r_1 a) = (er)(r_1 a) = ((r_1 a)r)e = ((r_1 a)(er))e \\
&= ((r_1 e)(ar))e = (e(ar))(r_1 e) = (ar)(r_1 e) \\
&= ((r_1 e)r)a \in Ra.
\end{aligned}
$$

Since $a = ea \in Ra$. Thus Ra is a left ideal of R containing $a.$ Let I be another left ideal of R containing $a.$ Since $ra \in I,$ where $ra \in Ra,$ i.e., $Ra \subseteq I.$ Hence Ra is the smallest left ideal of R containing $a.$

Lemma 3.7: Let R be an LA-ring with left identity $e.$ Then aR is a left ideal of $R.$

Proof: Straight forward.

Proposition 3.8: Let R be an LA-ring with left identity $e.$ Then $aR \cup Ra$ is the smallest right ideal of R containing $a.$

Proof: Let $x, y \in aR \cup Ra,$ this means that $x, y \in aR$ or $Ra.$ Since aR and Ra both are left ideals of $R,$ so $x - y \in aR$ and $Ra,$ i.e., $x - y \in aR \cup Ra.$ We have to show that $(aR \cup Ra)R \subseteq (aR \cup Ra).$ Now

$$
\begin{aligned}
(aR \cup Ra)R &= (aR)R \cup (Ra)R = (RR)a \cup (Ra)(eR) \\
&\subseteq Ra \cup (\mathrm{Re})(aR) = Ra \cup R(aR) \\
&= Ra \cup a(RR) \subseteq Ra \cup aR = aR \cup Ra. \\
&\Rightarrow (aR \cup Ra)R \subseteq aR \cup Ra.
\end{aligned}
$$

Since $a \in Ra$, i.e., $a \in aR \cup Ra$. Let I be another right ideal of R containing a. Since $aR \in IR \subseteq I$ and $Ra = (RR)a = (aR)R \in (IR)R \subseteq IR \subseteq I$, i.e., $aR \cup Ra \subseteq I$. Therefore $aR \cup Ra$ is the smallest right ideal of R containing a.

Theorem 3.9: Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

(1) R is a regular.

(2) $D \cap L = D \circ L$ for every intuitionistic fuzzy right ideal D and every intuitionistic fuzzy left ideal L of R.

(3) $C = (C \circ R) \circ C$ for every intuitionistic fuzzy quasi-ideal C of R.

Proof: Suppose that (1) holds and $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy quasi-ideal of R. Then $(C \circ R) \circ C \subseteq C$, because every intuitionistic fuzzy quasi-ideal of R is an intuitionistic fuzzy bi-ideal of R by the Lemma 2.24. Let $x \in R$, this implies that there exists an element a of R such that $x = (xa)x$. Thus

$$\begin{aligned}((\mu_C \circ R) \circ \mu_C)(x) &= \vee_{a = \sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{(\mu_C \circ R)(a_i) \wedge \mu_C(b_i)\}\} \\ &\geq (\mu_C \circ R)(xa) \wedge \mu_C(x) \\ &= \vee_{xa = \sum_{i=1}^{n} p_i q_i} \{\wedge_{i=1}^{n} \{\mu_C(p_i) \wedge R(q_i)\}\} \wedge \mu_C(x) \\ &\geq \mu_C(x) \wedge R(a) \wedge \mu_C(x) = \mu_C(x). \\ &\Rightarrow \mu_C \subseteq (\mu_C \circ R) \circ \mu_C.\end{aligned}$$

Similarly, we have $\gamma_C \supseteq (\gamma_C \circ R) \circ \gamma_C$, i.e., $C = (C \circ R) \circ C$. Hence (1) implies (3). Assume that (3) holds. Let D be an intuitionistic fuzzy right ideal and L be an intuitionistic fuzzy left ideal of R. This means that D and L be intuitionistic fuzzy quasi-ideals of R by the Lemma 2.23, so $D \cap L$ be also an intuitionistic fuzzy quasi-ideal of R. Then by our assumption, $D \cap L = ((D \cap L) \circ R) \circ (D \cap L) \subseteq (D \circ R) \circ L \subseteq D \circ L$, i.e., $D \cap L \subseteq D \circ L$. Since $D \circ L \subseteq D \cap L$. Therefore $D \circ L = D \cap L$, i.e., $(3) \Rightarrow (2)$. Suppose that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 3.7 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.8. This implies that χ_{Ra} is an intuitionistic fuzzy left ideal and $\chi_{aR \cup Ra}$ is an intuitionistic fuzzy right ideal of R, by the Theorem 2.3. Then by our supposition

$$\chi_{aR \cup Ra} \cap \chi_{Ra} = \chi_{aR \cup Ra} \circ \chi_{Ra}, \text{ i.e., } \chi_{(aR \cup Ra) \cap Ra} = \chi_{(aR \cup Ra) Ra}$$

by the Theorem 2.2. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in (aR)(Ra) \cup (Ra)(Ra)$. This implies that

$$a \in (aR)(Ra) \text{ or } a \in (Ra)(Ra).$$

If $a \in (aR)(Ra)$, then

$a = (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a = ((xe)(ay))a = (a((xe)y))a$ for any $x, y \in R.$

If $a \in (Ra)(Ra),$ then $(Ra)(Ra) = ((\mathrm{Re})a)(Ra) = ((ae)R)(Ra) = (aR)(Ra),$ i.e., $a \in (aR)(Ra).$ So a is a regular, i.e., R is a regular. Hence $(2) \Rightarrow (1).$

Theorem 3.10: Let R be an LA-ring with left identity $e,$ such that $(xe)R = xR$ for all $x \in R.$ Then the following conditions are equivalent.

(1) R is a regular.

(2) $A = (A \circ R) \circ A$ for every intuitionistic fuzzy quasi-ideal A of $R.$

(3) $B = (B \circ R) \circ B$ for every intuitionistic fuzzy bi-ideal B of $R.$

(4) $C = (C \circ R) \circ C$ for every intuitionistic fuzzy generalized bi-ideal C of $R.$

Proof: $(1) \Rightarrow (4),$ is obvious. Since $(4) \Rightarrow (3),$ every intuitionistic fuzzy bi-ideal of R is an intuitionistic fuzzy generalized bi-ideal of R by the Lemma 2.22. Since $(3) \Rightarrow (2),$ every intuitionistic fuzzy quasi-ideal of R is an intuitionistic fuzzy bi-ideal of R by the Lemma 15. $(2) \Rightarrow (1),$ by the Theorem 3.9.

Theorem 3.11: Let R be an LA-ring with left identity $e,$ such that $(xe)R = xR$ for all $x \in R.$ Then the following conditions are equivalent.

(1) R is a regular.

(2) $A \cap I = (A \circ I) \circ A$ for every intuitionistic fuzzy quasi-ideal A and every intuitionistic fuzzy ideal I of $R.$

(3) $B \cap I = (B \circ I) \circ B$ for every intuitionistic fuzzy bi-ideal B and every intuitionistic fuzzy ideal I of $R.$

(4) $C \cap I = (C \circ I) \circ C$ for every intuitionistic fuzzy generalized bi-idea C and every intuitionistic fuzzy ideal I of $R.$

Proof: Assume that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal and $I = (\mu_I, \gamma_I)$ be an intuitionistic fuzzy ideal of $R.$ Now $(C \circ I) \circ C \subseteq (R \circ I) \circ R \subseteq I \circ R \subseteq I$ and $(C \circ I) \circ C \subseteq (C \circ R) \circ C \subseteq C,$ i.e., $(C \circ I) \circ C \subseteq C \cap I.$ Let $x \in R,$ this means that there exists an element $a \in R$ such that $x = (xa)x.$ Now $xa = ((xa)x)a = (ax)(xa) = x((ax)a).$ Thus

$$
\begin{aligned}
((\mu_C \circ \mu_I) \circ \mu_C)(x) &= \vee_{x = \sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{(\mu_C \circ \mu_I)(a_i) \wedge \mu_C(b_i)\}\} \\
&\geq (\mu_C \circ \mu_I)(xa) \wedge \mu_C(x) \\
&= \vee_{xa = \sum_{i=1}^{n} p_i q_i} \{\wedge_{i=1}^{n} \{\mu_C(p_i) \wedge \mu_I(q_i)\}\} \wedge \mu_C(x) \\
&\geq \mu_C(x) \wedge \mu_I((ax)a) \wedge \mu_C(x) \\
&\geq \mu_C(x) \wedge \mu_I(x) = (\mu_C \cap \mu_I)(x). \\
&\Rightarrow \mu_C \cap \mu_I \subseteq (\mu_C \circ \mu_I) \circ \mu_C .
\end{aligned}
$$

Similarly, we have $\gamma_C \cup \gamma_I \supseteq (\gamma_C \circ \gamma_I) \circ \gamma_C .$ Hence $C \cap I = (C \circ I) \circ C,$ i.e., $(1) \Rightarrow (4).$ It is clear that $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2).$ Suppose that (2) is true. Then

$A \cap R = (A \circ R) \circ A,$ where R itself is an intuitionistic fuzzy two-sided ideal of $R.$ So $A = (A \circ R) \circ A,$ because every intuitionistic fuzzy two-sided ideal of R is an intuitionisitc fuzzy quasi-ideal of $R.$ Hence R is a regular by the Theorem 3.9, i.e., $(2) \Rightarrow (1).$

Theorem 3.12: Let R be an LA-ring with left identity $e,$ such that $(xe)R = xR$ for all $x \in R.$ Then the following conditions are equivalent.

(1) R is a regular.

(2) $A \cap D \subseteq D \circ A$ for every intuitionistic fuzzy quasi-ideal A and every intuitionistic fuzzy right ideal D of $R.$

(3) $B \cap D \subseteq D \circ B$ for every intuitionistic fuzzy bi-ideal B and every intuitionistic fuzzy right ideal D of $R.$

(4) $C \cap D \subseteq D \circ C$ for every intuitionistic fuzzy generalized bi-ideal C and every intuitionistic fuzzy right ideal D of $R.$

Proof: $(1) \Rightarrow (4),$ is obvious. It is clear that $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2).$ Suppose that (2) holds, this implies that $D \cap A = A \cap D \subseteq D \circ A,$ where A is an intuitionistic fuzzy left ideal of $R.$ Since $D \circ A \subseteq D \cap A,$ i.e., $D \cap A = D \circ A.$ Hence R is a regular by the Theorem 3.9, i.e., $(2) \Rightarrow (1).$

4 IINTRA-REGULAR LA-RINGS

An LA-ring An LA-ring R is called an intra-regular if for every $x \in R,$ there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n} (a_i x^2) b_i.$ In this section, we characterize intra-regular LA-rings by the properties of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals.

Lemma 4.1: Every intuitionistic fuzzy left (right) ideal of an intra-regular LA-ring R is an intuitionistic fuzzy ideal of $R.$

Proof: Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of $R.$ Let $x, y \in R,$ this implies that there exist elements $a_i, b_i \in R,$ such that $x = \sum_{i=1}^{n} (a_i x^2) b_i.$ Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A(((a_i x^2) b_i) y) = \mu_A((y b_i)(a_i x^2)) \\ &\geq \mu_A(y b_i) \geq \mu_A(y) \\ \text{and } \gamma_A(xy) &= \gamma_A(((a_i x^2) b_i) y) = \gamma_A((y b_i)(a_i x^2)) \\ &\leq \gamma_A(y b_i) \leq \gamma_A(y). \end{aligned}$$

Hence A is an intuitionistic fuzzy ideal of $R.$

Proposition 4.2: Let A be an IFS of an intra-regular LA-ring R with left identity $e.$ Then A is an intuitionistic fuzzy ideal of R if and only if A is an intuitionistic fuzzy interior ideal of $R.$

Proof: Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal of $R.$ Let $x, y \in R,$ this implies that there exist elements $a_i, b_i \in R,$ such that $x = \sum_{i=1}^{n} (a_i x^2) b_i.$ Thus

$$\begin{aligned}\mu_A(xy) &= \mu_A(((a_i x^2)b_i)y) = \mu_A((yb_i)(a_i x^2))\\ &= \mu_A((yb_i)(a_i(xx))) = \mu_A((yb_i)(x(a_i x)))\\ &= \mu_A((yx)(b_i(a_i x))) \geq \mu_A(x).\end{aligned}$$

Similarly, we have $\gamma_A(xy) \leq \gamma_A(x)$, i.e., A is an intuitionistic fuzzy right ideal of R. Hence A is an intuitionistic fuzzy ideal of R by the Lemma 4.1. Converse is true by the Lemma 2.18.

Remark 4.3: The concept of intuitionistic fuzzy (interior, two-sided) ideals coincides in intra-regular LA-rings with left identity.

Lemma 4.4: Let R be an intra-regular LA-ring with left identity e. Then $D \cap L \subseteq L \circ D$ for every intuitionistic fuzzy left ideal L and every intuitionistic fuzzy right ideal D of R.

Proof: Let $L = (\mu_L, \gamma_L)$ be an intuitionistic fuzzy left ideal and $D = (\mu_D, \gamma_D)$ be an intuitionistic fuzzy right ideal of R. Let $x \in R$, this means that there exist elements such that $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Now

$$\begin{aligned}x &= (a_i x^2)b_i = (a_i(xx))b_i = (x(a_i x))b_i\\ &= (x(a_i x))(eb_i) = (xe)((a_i x)b_i) = (a_i x)((xe)b_i).\end{aligned}$$

Thus

$$\begin{aligned}(\mu_L \circ \mu_D)(x) &= \vee_{x=\sum_{i=1}^{n} p_i q_i}\{\wedge_{i=1}^{n}\{\mu_L(p_i) \wedge \mu_D(q_i)\}\}\\ &\geq \mu_L(a_i x) \wedge \mu_D((xe)b_i) \geq \mu_L(x) \wedge \mu_D(x)\\ &= \mu_D(x) \wedge \mu_L(x) = (\mu_D \cap \mu_L)(x)\\ \text{and } (\gamma_L \circ \gamma_D)(x) &= \wedge_{x=\sum_{i=1}^{n} p_i q_i}\{\vee_{i=1}^{n}\{\gamma_L(p_i) \vee \gamma_D(q_i)\}\}\\ &\leq \gamma_L(a_i x) \vee \gamma_D((xe)b_i) \leq \gamma_L(x) \vee \gamma_D(x)\\ &= \gamma_D(x) \vee \gamma_L(x) = (\gamma_D \cup \gamma_L)(x).\\ &\Rightarrow D \cap L \subseteq L \circ D.\end{aligned}$$

Theorem 4.5: Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

(1) R is an intra-regular.

(2) $D \cap L \subseteq L \circ D$ for every intuitionistic fuzzy left ideal L and every intuitionistic fuzzy right ideal D of R.

Proof: $(1) \Rightarrow (2)$ is true by the Lemma 4.4. Suppose that (2) holds. Let $a \in R$, then Ra is a left ideal of R containing a by the Lemma 3.6 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.8. So χ_{Ra} is an intuitionistic fuzzy left ideal and $\chi_{aR \cup Ra}$ is an intuitionistic fuzzy right ideal of R, by the Theorem 1.3. By our supposition

$$\chi_{aR\cup Ra} \cap \chi_{Ra} \subseteq \chi_{Ra} \circ \chi_{aR\cup Ra}, \text{ i.e., } \chi_{(aR\cup Ra)\cap Ra} \subseteq \chi_{(Ra)(aR\cup Ra)}$$

by the Theorem 1.2. Thus $(aR \cup Ra) \cap Ra \subseteq Ra(aR \cup Ra)$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in Ra(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra)$. This implies that $a \in (Ra)(aR)$ or $a \in (Ra)(Ra)$. If $a \in (Ra)(aR)$, then

$$\begin{aligned}(Ra)(aR) &= (Ra)((ea)(RR)) = (Ra)((RR)(ae)) \\ &= (Ra)(((ae)R)R) = (Ra)((aR)R) \\ &= (Ra)((RR)a) = (Ra)(Ra) = ((Ra)a)R \\ &= ((Ra)(ea))R = ((\mathrm{Re})(aa))R = (Ra^2)R.\end{aligned}$$

So $a \in (Ra^2)R$. If $a \in (Ra)(Ra)$, then obvious $a \in (Ra^2)R$. This implies that a is an intra-regular. Hence R is an intra-regular, i.e., $(2) \Rightarrow (1)$.

Theorem 4.6: Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

(1) R is an intra-regular.

(2) $A \cap I = (A \circ I) \circ A$ for every intuitionistic fuzzy quasi-ideal A and every intuitionistic fuzzy ideal I of R.

(3) $B \cap I = (B \circ I) \circ B$ for every intuitionistic fuzzy bi-ideal B and every intuitionistic fuzzy ideal I of R.

(4) $C \cap I = (C \circ I) \circ C$ for every intuitionistic fuzzy generalized bi-ideal C and every intuitionistic fuzzy ideal I of R.

Proof: Suppose that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal and $I = (\mu_I, \gamma_I)$ be an intuitionistic fuzzy ideal of R. Now $(C \circ I) \circ C \subseteq (R \circ I) \circ R \subseteq I \circ R \subseteq I$ and $(C \circ I) \circ C \subseteq (C \circ R) \circ C \subseteq C$, thus $(C \circ I) \circ C \subseteq C \cap I$. Let $x \in R$, this implies that there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n} (a_i x^2) b_i$. Now

$$x = (a_i x^2) b_i = (a_i (xx)) b_i = (x(a_i x)) b = (b_i (a_i x)) x.$$

$$\begin{aligned}b_i (a_i x) &= b_i (a_i ((a_i x^2) b_i)) = b_i ((a_i x^2)(a_i b_i)) = b_i ((a_i x^2) c_i) \\ &= (a_i x^2)(b_i c_i) = (a_i x^2) d_i = (a_i x^2)(e d_i) = (d_i e)(x^2 a_i) \\ &= m_i (x^2 a_i) = x^2 (m_i a_i) = (xx) l_i = (l_i x) x = (l_i x)(ex) \\ &= (xe)(x l_i) = x((xe) l_i).\end{aligned}$$

Thus

$$\begin{aligned}((\mu_C \circ \mu_I) \circ \mu_C)(x) &= \vee_{x = \sum_{i=1}^{n} p_i q_i} \{\wedge_{i=1}^{n} \{(\mu_C \circ \mu_I)(p_i) \wedge \mu_C(q_i)\}\} \\ &\geq (\mu_C \circ \mu_I)(b_i (a_i x)) \wedge \mu_C(x) \\ &= \vee_{b_i (a_i x) = \sum_{i=1}^{n} m_i n_i} \{\wedge_{i=1}^{n} \{\mu_C(m_i) \wedge \mu_I(n_i)\}\} \wedge \mu_C(x) \\ &\geq \mu_C(x) \wedge \mu_I((xe) l_i) \wedge \mu_C(x) \\ &\geq \mu_C(x) \wedge \mu_I(x) = (\mu_C \cap \mu_I)(x). \\ &\Rightarrow \mu_C \cap \mu_I \subseteq (\mu_C \circ \mu_I) \circ \mu_C.\end{aligned}$$

Similarly, we have $\gamma_C \cup \gamma_I \supseteq (\gamma_C \circ \gamma_I) \circ \gamma_C$. Hence $C \cap I = (C \circ I) \circ C$, i.e. $(1) \Rightarrow (4)$. It is clear that $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$. Assume that (2) is true. Let A be an intuitionistic fuzzy right ideal and I be an intuitionistic fuzzy two-sided ideal of R. Since every intuitionistic fuzzy right ideal of R is an intuitionistic fuzzy quasi-ideal of R by the Lemma 2.23, so A is an intuitionistic fuzzy quasi-ideal of R. By our assumption $A \cap I = (A \circ I) \circ A \subseteq (R \circ I) \circ A \subseteq I \circ A$, i.e., $A \cap I \subseteq I \circ A$. Hence R is an intra-regular by the Theorem 4.5, i.e., $(2) \Rightarrow (1)$.

Theorem 4.7: Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

(1) R is an intra-regular.

(2) $A \cap L \subseteq L \circ A$ for every intuitionistic fuzzy quasi-ideal A and every intuitionistic fuzzy left ideal L of R.

(3) $B \cap L \subseteq L \circ B$ for every intuitionistic fuzzy bi-ideal B and every intuitionistic fuzzy left ideal L of R.

(4) $C \cap L \subseteq L \circ C$ for every intuitionistic fuzzy generalized bi-ideal C and every intuitionistic fuzzy left ideal L of R.

Proof: Assume that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal and $L = (\mu_L, \gamma_L)$ be an intuitionistic fuzzy left ideal of R. Let $x \in R$, this means that there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n} (a_i x^2) b_i$. Now $x = (a_i (xx)) b_i = (x(a_i x)) b_i = (b_i (a_i x)) x$. Thus

$$\begin{aligned}(\mu_L \circ \mu_C)(x) &= \vee_{x = \sum_{i=1}^{n} p_i q_i} \{\wedge_{i=1}^{n} \{\mu_L(p_i) \wedge \mu_C(q_i)\}\} \\ &\geq \mu_L(b_i(a_i x)) \wedge \mu_C(x) \geq \mu_L(x) \wedge \mu_C(x) \\ &= \mu_C(x) \wedge \mu_L(x) = (\mu_C \cap \mu_L)(x). \\ &\Rightarrow \mu_C \cap \mu_L \subseteq \mu_L \circ \mu_C.\end{aligned}$$

Similarly, we have $\gamma_C \cup \gamma_L \supseteq \gamma_L \circ \gamma_C$. Hence $C \cap L \subseteq L \circ C$, i.e., $(1) \Rightarrow (4)$. It is clear that $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$. Suppose that (2) holds. Let A be an intuitionistic fuzzy right ideal and L be an intuitionistic fuzzy left ideal of R. Since every intuitionistic fuzzy right ideal of R is an intuitionistic fuzzy quasi-ideal of R, this implies that A is an intuitionistic fuzzy quasi-ideal of R. By our supposition $, A \cap L \subseteq L \circ A$. Thus R is an intra-regular by the Theorem 4.5, i.e., $(2) \Rightarrow (1)$.

5 CONCLUSION

Our ambition is to inspire the study and maturity of non associative algebraic structure (LA-ring). The objective is to explain original methodological developments on ordered LA-rings, which will be very helpful for upcoming theory of algebraic structure. The ideal of fuzzy set to the characterizations of LA-rings are captivating a great attention of algebraist.

The aim of this paper is to investigate, the study of (regular, intra-regular) LA-rings by using of fuzzy left (right, interior, quasi-, bi-, generalized bi-) ideals.

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