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CONGRUENCES INVOLVING SUMS OF HARMONIC NUMBERS AND BINOMIAL COEFFICIENTS

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Summary. Congruences involving sums of Harmonic numbers and binomial coefficients are considered in this paper. Recently, many great mathematicians have been interested to find congruences and relationships between these numbers such Sun & Tauraso, Koparal & Ömür, Mao & Sun and Meštrović & Andjić. In the present paper, some new combinatorial congruences are proved. These congruences are mainly determined modulo p^2 or p^3 (p in any prime) and they are motivated by a recent paper by Meštrović and Andjić. The first main result (Theorem 1) presents the congruence modulo p^2 ($p > 3$ is any prime) involving sum of products of two binomial coefficients and Harmonic numbers. Two interesting congruences modulo a prime $p > 3$ (Corollary 2) involving Harmonic numbers H_k, Catalan numbers C_k and Fermat quotient $q_2 \coloneqq (2^{p-1} - 1)/p$ are obtained as consequences of Theorem 1. The second main result (Theorem 2) presents the congruence modulo p^3 ($p > 3$ is any prime) involving sum of products of two binomial coefficients and Harmonic numbers.

1. INTRODUCTION AND MAIN RESULTS

The harmonic number and the congruence in the ring of p integer $\mathbb{Z}_p$ ply important role in mathematics. Recall that harmonic numbers are to be

$$H_0 = 0, \qquad H_n = \sum_{k=1}^{n} \frac{1}{k}, \qquad n \geq 1,$$

$\mathbb{Z}_p$ is the set of rational numbers having denominators not divisible by p and the unit group $U(\mathbb{Z}_p)$ is the set of rational numbers having denominators and numerators not divisible by p.

We define, for all prime number p and for all numbers $x, y \in \mathbb{Z}_p$

$$x \equiv y \ (mod\ p) \Leftrightarrow numerator(x - y) \equiv 0 \ (mod\ p).$$

This shows when $x, y \in U(\mathbb{Z}_p)$ that

$$x \equiv y \ (mod\ p) \Leftrightarrow \tfrac{1}{x} \equiv \tfrac{1}{y} \ (mod\ p).$$

Congruences involving sums of Harmonic numbers and binomial coefficients in the ring of p integer have been studied recently by many mathematicians and a considerable amount of research results has been produced, such in 2011 Sun and Tauraso [9] proved, that for any prime $p \geq 5$, the following congruences hold

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$$\sum_{k=1}^{p-1} \binom{2k}{k} \equiv -\left(\frac{p}{3}\right) \ (mod\ p^2), \qquad (1)$$

$$\sum_{k=1}^{p-1} \binom{2k}{k} H_k \equiv -\left(\frac{p}{3}\right) q_3 \ (mod\ p), \qquad (2)$$

where $\left(\frac{p}{3}\right)$ denotes the Legendre symbol and $q_a := (a^{p-1}-1)/p$ is the Fermat quotient with a prime p and an integer a. Also, in 2016 Mao and Sun [5] established, that for a prime $p > 3$, the following congruences

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_k}{k} \equiv \frac{1}{3}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \ (mod\ p), \qquad (3)$$

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k}}{k} \equiv \frac{7}{12}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \ (mod\ p), \qquad (4)$$

where $B_n(.)$ is the n-th Bernoulli polynomial. In 2016 Koparal and Ömür [2] proved that

$$\sum_{k=1}^{(p-1)/2} (-1)^k \binom{2k}{k} H_{k-1} \equiv \frac{2^p}{p}\left(2F_{p+1} - 5^p\right) (mod\ p), \qquad (5)$$

$$\sum_{k=1}^{(p-1)/2} \frac{C_k H_k}{(-4)^k} \equiv 2\frac{Q_{p+1}}{p} - \frac{2^{p+1}}{p}\left(1 + 2^{(p+1)/2}\right) (mod\ p) \qquad (6)$$

and if $\left(\frac{5}{p}\right) = 1$ they also proved that

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k} \frac{H_{k-1}}{(-4)^k} \equiv \frac{1}{p}\left(F_{2p+1} - F_{p+2}\right) - \frac{2^p}{p} F_{p-1} \ (mod\ p), \qquad (7)$$

where F_n is the Fibonacci numbers, $\left(\frac{a}{p}\right)$ denotes the Legendre symbol, $\{Q_n\}$ is the Pell-Lucas sequence and $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

We have the following two theorems and corollaries.

Theorem 1. Let $p > 3$ be a prime number and $m \in \{1,2,\cdots,(p-1)/2\}$. We have

$$\sum_{k=1}^{(p-1)/2} (-1)^{k-1} \binom{k+m}{k}^{-1} \binom{(p-1)/2}{k} H_k \equiv \frac{m}{2m-1}\left(T(m) + S(m)p\right) (mod\ p^2), \qquad (8)$$

where

$$T(m) = -4q_2 + H_{2m-2} - 4H_{m-1} \; and$$

$$S(m) = 2q_2^2 + H_{m-1,2} - 4H_{2m-2,2} + 4\frac{q_2 + H_{m-1} - H_{2m-2}}{2m-1}.$$

Reducing the modulus in this congruence to get

$$\sum_{k=1}^{(p-1)/2} \binom{k+m}{k}^{-1} \binom{2k}{k} \frac{H_k}{4^k} \equiv \frac{4m}{2m-1}(q_2 - H_{2m-2} + H_{m-1}) \; (mod\; p). \tag{9}$$

By the congruence (16), the congruence (9) for $m \in \{1,2\}$ and by the fact that

$$\binom{k+2}{k}^{-1} = \frac{2}{k+1} - \frac{2}{k+2},$$

we may state:

Corollary 2. For each prime number $p > 3$ we have

$$\sum_{k=1}^{(p-1)/2} \frac{C_k H_k}{4^k} \equiv 4\mathrm{q}_2 \; (\mathrm{mod\ p}), \tag{10}$$

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k} \frac{H_k}{4^k(\mathrm{k}+2)} \equiv \frac{2}{3} + \frac{8}{3}\mathrm{q}_2 \; (\mathrm{mod\ p}). \tag{11}$$

Theorem 3. Let $p > 3$ be a prime number and $m \in \{1,2,\cdots,(p-3)/2\}$. We have

$$\sum_{k=1}^{(p-1)/2} (-1)^{k-1} \binom{(p-1)/2}{k} \binom{\mathrm{k}}{\mathrm{m}} \mathrm{H}_{2\mathrm{k}} \equiv \frac{(-1)^{\mathrm{m}}}{\mathrm{p}-1-2\mathrm{m}} \left(1 + \frac{(-1)^{(p-1)/2}}{2^{2\mathrm{m}+\mathrm{p}-1}} \binom{2\mathrm{m}}{\mathrm{m}}\right) (\mathrm{mod\ p^3}). \tag{12}$$

Reducing this modulus to obtain

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k} \binom{k}{m} \frac{H_{2k}}{4^k} \equiv \frac{(-1)^m}{2m+1} \left(1 + \frac{(-1)^{(p-1)/2}}{2^{2\mathrm{m}}} \binom{2\mathrm{m}}{\mathrm{m}}\right) (\mathrm{mod\ p}). \tag{13}$$

Corollary 4. For each prime $p > 3$ we have

$$\sum_{k=1}^{(p-1)/2} (-1)^{k-1} \binom{(p-1)/2}{k} \mathrm{H}_{2\mathrm{k}} \equiv \frac{1}{\mathrm{p}-1} \left(1 + \frac{(-1)^{(p-1)/2}}{2^{\mathrm{p}-1}}\right) (\mathrm{mod\ p^3}),$$

$$\sum_{k=1}^{(p-1)/2} (-1)^{k-1} \binom{(p-1)/2}{k} \mathrm{kH}_{2\mathrm{k}} \equiv -\frac{1}{\mathrm{p}-3} \left(1 + \frac{(-1)^{(p-1)/2}}{2^{\mathrm{p}}}\right) (\mathrm{mod\ p^3})$$

and

$$\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{H_{2k}}{4^k}\equiv 1+(-1)^{(p-1)/2}\ (\text{mod p}),$$
$$\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{kH_{2k}}{4^k}\equiv -\frac{1}{3}\left(1+\frac{1}{2}(-1)^{(p-1)/2}\right)(\text{mod p}).$$

To prove Theorem 1, we give the following two lemmas.

Lemma 5. [8, Eq. 19] Let $n\geq 0$ and $m\geq 1$ be integers. The following identity holds

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{k+m}{k}^{-1}\binom{n}{k}H_k=\frac{m}{m+n}(H_{m+n-1}-H_{m-1}).\qquad (14)$$

Lemma 6. Let $p>3$ be a prime number. Then for $m\in\{1,2,\cdots,(p-1)/2\}$ we have

$$H_{\frac{p-1}{2}+m-1}\equiv -2q_2+2H_{2m-2}-H_{m-1}-p\left(2H_{2m-2,2}-\frac{1}{2}H_{m-1,2}-q_2^2\right)(mod\ p^2).\quad (15)$$

Reducing the modulus in this congruence to obtain

$$H_{\frac{p-1}{2}+m-1}\equiv -2q_2+2H_{2m-2}-H_{m-1}(mod\ p).$$

Proof. We have

$$\begin{aligned}
H_{\frac{p-1}{2}+m-1} &= \sum_{k=1}^{(p-1)/2}\frac{1}{k}+\sum_{k=1}^{m-1}\frac{1}{\frac{p-1}{2}+k}\\
&= \sum_{k=1}^{(p-1)/2}\frac{1}{k}+2\sum_{k=1}^{m-1}\frac{1}{p-(1-2k)}\\
&= H_{\frac{p-1}{2}}+2\sum_{k=1}^{m-1}\frac{p+(1-2k)}{p^2-(1-2k)^2}\\
&\equiv H_{\frac{p-1}{2}}-2\sum_{k=1}^{m-1}\frac{p+(1-2k)}{(1-2k)^2}\\
&= H_{\frac{p-1}{2}}-2p\sum_{k=1}^{m-1}\frac{1}{(2k-1)^2}+2\sum_{k=1}^{m-1}\frac{1}{2k-1}\\
&= H_{\frac{p-1}{2}}-2p\left(H_{2m-2,2}-\frac{1}{4}H_{m-1,2}\right)+2\left(H_{2m-2}-\frac{1}{2}H_{m-1}\right),
\end{aligned}$$

which, by the result congruence of Lehmer [4] $H_{\frac{p-1}{2}}\equiv -2q_2+pq_2^2\ (mod\ p^2)$, the proof is complete.

Proof of Theorem 1. From the relation (14), we can write

$$\sum_{k=1}^{(p-1)/2} (-1)^{k-1} \binom{k+m}{k}^{-1} \binom{(p-1)/2}{k} H_k = \frac{m}{m+\frac{p-1}{2}} \left(H_{\frac{p-1}{2}+m-1} - H_{m-1} \right)$$

$$= \frac{2m}{2m+p-1} \left(H_{\frac{p-1}{2}+m-1} - H_{m-1} \right).$$

So, by the congruences (15) and

$$\frac{1}{2m+p-1} \equiv \frac{1}{2m-1} - \frac{p}{(2m-1)^2} (mod\ p^2)$$

we obtain

$$\sum_{k=1}^{(p-1)/2} (-1)^{k-1} \binom{k+m}{k}^{-1} \binom{(p-1)/2}{k} H_k$$

$$= \left(\frac{2m}{2m-1} - \frac{2mp}{(2m-1)^2} \right) \left(-2q_2 + 2H_{2m-2} - H_{m-1} - p \left(2H_{2m-2,2} - \frac{1}{2} H_{m-1,2} - q_2^2 \right) - H_{m-1} \right)$$

$$= \left(\frac{2m}{2m-1} - \frac{2mp}{(2m-1)^2} \right) \left(-2q_2 + 2H_{2m-2} - 2H_{m-1} - p \left(2H_{2m-2,2} - \frac{1}{2} H_{m-1,2} - q_2^2 \right) \right)$$

$$= \frac{m}{2m-1} (-4q_2 + 4H_{2m-2} - 4H_{m-1})$$

$$+ \frac{m}{2m-1} \left(2q_2^2 + H_{m-1,2} - 4H_{2m-2,2} + 4 \frac{q_2 + H_{m-1} - H_{2m-2}}{2m-1} \right) p\ (mod\ p^2).$$

To prove the relationship (9) we use the known congruence [2]

$$\binom{(p-1)/2}{k} \equiv \frac{1}{(-4)^k} \binom{2k}{k}\ (mod\ p). \tag{16}$$

Proof of Theorem 2. Let $n = ((p-1)/2)$ in the identity of Corollary 2.2 [1]

$$\sum_{k=m}^{n} (-1)^{k-1} \binom{n}{k} \binom{k}{m} H_{2k} = \frac{(-1)^m}{n-m} \left(\frac{1}{2} + \frac{2^{2n-2m-2} \binom{2m}{m}}{\binom{2n-1}{n-1}} \right).$$

Then, we have

$$\sum_{k=1}^{(p-1)/2} (-1)^{k-1} \binom{(p-1)/2}{k} \binom{k}{m} \mathrm{H}_{2\mathrm{k}} = \frac{(-1)^m}{\frac{p-1}{2} - m} \left(\frac{1}{2} + 2^{p-2m-3} \binom{2m}{m} \binom{p-1-1}{(p-1)/2-1}^{-1} \right)$$

$$= \frac{2(-1)^m}{p-1-2m} \left(\frac{1}{2} + 2^{p-2m-2} \binom{2m}{m} \binom{p-1}{(p-1)/2}^{-1} \right)$$

$$= \frac{(-1)^m}{p-1-2m} \left(1 + 2^{p-2m-1} \binom{2m}{m} \binom{p-1}{(p-1)/2}^{-1} \right).$$

From the known congruence [7]

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{\frac{p-1}{2}} 4^{p-1} \ (mod\ p^3),$$

we have

$$\sum_{k=1}^{(p-1)/2} (-1)^{k-1} \binom{(p-1)/2}{k} \binom{\mathrm{k}}{\mathrm{m}} \mathrm{H}_{2\mathrm{k}} = \frac{(-1)^{\mathrm{m}}}{\mathrm{p}-1-2\mathrm{m}} \left(1 + (-1)^{\frac{p-1}{2}} \frac{2^{p-2m-1}\binom{2m}{m}}{4^{p-1}} \right)$$

$$= \frac{(-1)^{\mathrm{m}}}{\mathrm{p}-1-2\mathrm{m}} \left(1 + (-1)^{\frac{p-1}{2}} 2^{-2m} \binom{2m}{m} \left(\frac{1}{2}\right)^{p-1} \right) (\mathrm{mod}\ \mathrm{p}^3).$$

To prove the relationship (13), use the congruence (16) and Fermat little theorem.

2. CONCLUSION

The principal results of this paper given by Theorems 1 and 2 represent an interesting contribution in congruences. They are obtained upon using technical operations on the binomial coefficients, harmonic numbers and Catalan numbers. To extend our results using the useful technics or methods to study congruences in the ring of p-integers may be, in general, difficult. A first question on the extension of these congruences is: how can us generalize the obtained congruences modulo some successive powers of a prime number p?. A second question on such extensions of Theorems 1 and 2 can be viewed as generalizations on using the q-Binomial coefficients instead of the binomial coefficients or the hyper-harmonic numbers instead of the harmonic numbers. These seem to be interesting and require technical calculus and some mathematical tools based on number theory and on complex integration.

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