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ON (APPROXIMATE) HOMOLOGICAL NOTIONS OF CERTAIN BANACH ALGEBRAS

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Summary. In this paper, we study the notion of ϕ-biflatness, ϕ-biprojectivity, approximate biprojectivity and Johnson pseudo-contractibility for a new class of Banach algebras. Using this class of Banach algebras, we give some examples which are approximately biprojective. Also some Banach algebras are given among matrix algebras which are never Johnson pseudo-contractible.

1 INTRODUCTION

Given a Banach algebra A, Kamyabi-Gol *et al.* in [4] defined a new product on A which is denoted by $*$. In fact $a * b = aeb$, for each $a,b \in A$ where e is an element of the closed unit ball $\overline{B_1^0}$ of A. A Banach algebra A equipped with $*$ as its product is denoted by A_e. They studied some properties like amenability and Arens regularity of A_e. In [6] some homological properties of A_e like biflatness, biprojectivity and ϕ −amenability discussed.

New notions of ϕ −amenability and approximate notions of homological Banach theory introduced and studied for Banach algebras see[14], [15] and [5]. In fact a Banach algebra a Banach algebra A is called approximate ϕ −contractible if there exists a net (m_α) in A such that $am_\alpha - \phi(a)m_\alpha \to 0$. and $\phi(m_\alpha) = 1$, for every $a \in A$. where ϕ is a multiplicative linear functional on A. For more information see [2]. Also a Banach algebra A is called approximate biprojective if there exists a net of bounded linear maps from A into $A \otimes_p A$, say $(\rho_\alpha)_{\alpha \in I}$, such that

1. $a \cdot \rho_\alpha(b) - \rho_\alpha(ab) \xrightarrow{||\cdot||} 0$,
2. $\rho_\alpha(ba) - \rho_\alpha(b) \cdot a \xrightarrow{||\cdot||} 0$,
3. $\pi_A \circ \rho_\alpha(a) - a \to 0$,

for every $a,b \in A$. In [1] the structure of approximate biprojective Banach algebras and its nilpotent ideals and also the relation with other notions of amenability are discussed.

We present some standard notations and definitions that we shall need in this paper. Let A be a Banach algebra. Throughout this work, the character space of A is denoted by $\Delta(A)$, that

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is, all non-zero multiplicative linear functionals on A. For each $\phi \in \Delta(A)$ there exists a unique extension $\tilde{\phi}$ to A^{**} which is defined $\tilde{\phi}(F) = F(\phi)$. It is easy to see that $\tilde{\phi} \in \Delta(A^{**})$. The projective tensor product $A \otimes_p A$ is a Banach A-bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c. (b \otimes c) \cdot a = b \otimes ca \quad (a.b.c \in A).$$

The product morphism $\pi_A: A \otimes_p A \to A$ is given by $\pi_A(a \otimes b) = ab,$ for every $a.b \in A$. Let A and B be Banach algebras. We denote by $\phi \otimes \psi$ a map defined by $\phi \otimes \psi(a \otimes b) = \phi(a)\psi(b)$ for all $a \in A$ and $b \in B$. It is easy to see that $\phi \otimes \psi \in \Delta(A \otimes_p B)$.
Let X and Y be Banach A −bimodules. The map $T: X \to Y$ is called A −bimodule morphism, if

$$T(a \cdot x) = a \cdot T(x). \quad T(x \cdot a) = T(x) \cdot a. \qquad (a \in A, x \in X).$$

Also a net of (T_α) of maps from X into Y is called approximate A −bimodule morphism, if

$$T_\alpha(a \cdot x) - a \cdot T_\alpha(x) \to 0. \quad T_\alpha(x \cdot a) - T_\alpha(x) \cdot a \to 0. \qquad (a \in A. x \in X).$$

The content of the paper is as follows. In section 2 we study ϕ −homological properties of A_e like ϕ −biflatness and ϕ −biprojectivity. Approximate biprojectivity and Johnson pseudo-contractibility are two important notions of Banach homology theory, which we discuss for A_e in section 3. We give some examples of matrix algebras to illustrate the paper.

2. ϕ - HOMOLOGICAL PROPERTIES OF CERTAIN BANACH ALGEBRAS

This section is devoted to the concepts of Banach homology related to a character ϕ.

Proposition 2.1 *[4, Proposition 2.3] Let A be a Banach algebra and* $e \in \overline{B_1^0}$ *. Then* A_e *is unital if and only if A is unital and e is invertible.*

Proposition 2.2 *[4, Proposition 2.4] Let A be a Banach algebra and* $e \in \overline{B_1^0}$ *.Then the followings hold:*

1. If ϕ is a multiplicative linear functional on A, then $\phi(e)\phi$ is a multiplicative linear functional on A_e.
2. If A_e is unital and ψ is a multiplicative linear functional on A_e, then $\phi(a) = \psi(e^{-1}a)$ is a multiplicative linear functional on A.

Proposition 2.3 *[6, Proposition 2.3] Let A be a Banach algebra and* $e \in \overline{B_1^0}$*. If* A_e *is unital then* $(A_e)_{e^{-2}} = A$*, (isometrically isomorphism).*

Proposition 2.4 *Suppose that A is a Banach algebra and also suppose that* $e \in \overline{B_1^0}$ *and* $\phi \in \Delta(A)$*. Then the followings hold:*

1. If A is approximate ϕ −contractible and $\phi(e) \neq 0$, then A_e is approximately ψ-contractible, where $\psi = \phi(e)\phi$.
2. If A_e is unital and approximate ψ −contractible, then A is approximate ϕ-contractible, where $\phi(a) = \psi(e^{-1}a)$ for each $a \in A$.

Proof. Suppose that A is approximately ϕ −contractible. So there is a net (m_α) in A such that

$$am_\alpha - \phi(a)m_\alpha \to 0. \quad \phi(m_\alpha) = 1. \qquad (a \in A).$$

Define $n_\alpha = \frac{m_\alpha}{\phi(e)}$. Since $\psi(a) = \phi(ae) = \phi(ea)$, we have

$$\begin{aligned} a * n_\alpha - \psi(a)n_\alpha &= aen_\alpha - \psi(a)n_\alpha \\ &= ae\frac{m_\alpha}{\phi(e)} - \psi(a)\frac{m_\alpha}{\phi(e)} \\ &= ae\frac{m_\alpha}{\phi(e)} - \phi(ae)\frac{m_\alpha}{\phi(e)} + \phi(ae)\frac{m_\alpha}{\phi(e)} - \psi(a)\frac{m_\alpha}{\phi(e)} \to 0. \qquad (a \in A_e). \end{aligned}$$

Also

$$\psi(n_\alpha) = \psi(\tfrac{m_\alpha}{\phi(e)}) = \phi(e)\phi(\tfrac{m_\alpha}{\phi(e)}) = \phi(m_\alpha) = 1.$$

It follows that A_e is approximate ϕ −contractible.

Suppose that $\phi(a) = \psi(e^{-1}a)$ and also suppose that A_e is unital and approximately left ψ −contractible. It is easy to see that $\psi(a) = \phi(ea)$. Let (m_α) be a net in A_e such that

$$a * m_\alpha - \psi(a)m_\alpha \to 0. \quad \psi(m_\alpha) = 1. \qquad (a \in A_e).$$

Since

$$\begin{aligned} a * m_\alpha - \psi(a)m_\alpha &= aem_\alpha - \psi(a)m_\alpha \\ &= aem_\alpha - \phi(ea)m_\alpha \\ &= aem_\alpha - \phi(e)\phi(a)m_\alpha \\ &= aem_\alpha - \phi(a)\phi(e)m_\alpha \\ &= aem_\alpha - \phi(ae)m_\alpha, \end{aligned}$$

we have

$$a * m_\alpha - \psi(a)m_\alpha = aem_\alpha - \phi(ae)m_\alpha \to 0$$

for each $a \in A$. Replacing a with ae^{-1} we have $am_\alpha - \phi(a)m_\alpha \to 0$. Regarding

$$1 = \psi(m_\alpha) = \phi(em_\alpha) = \phi(e)\phi(m_\alpha).$$

we may suppose that $\phi(m_\alpha) \neq 0$, for each α. Now define $n_\alpha = \frac{m_\alpha}{\phi(m_\alpha)}$. Clearly $\phi(n_\alpha) = 1$. Also

$$an_\alpha - \phi(a)n_\alpha = a\frac{m_\alpha}{\phi(m_\alpha)} - \phi(a)\frac{m_\alpha}{\phi(m_\alpha)} \to 0.$$

It finishes the proof.

Example 2.5 *In this example we show that there exists a Banach algebra A_e which is not approximate ψ-contractible. Let* $A = \{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} | a_{ij} \in \mathbb{C}\}$ *and suppose that* $e =$

$\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$. *Clearly A with matrix operations and ℓ^1-norm is a Banach algebra. We know that e is invertible and by Proposition 2.1, A_e is unital. Define $\phi: A \to \mathbb{C}$ by*

$$\phi(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}) = a_{33}.$$

Clearly ϕ is a character(multiplicative linear functional) and $\phi(e) \neq 0$. Suppose conversely that A_e is approximate ψ −contractible. By previous Proposition(2), A becomes approximate ϕ −contractible. On the other hand by the same arguments as in the proof of [7, Theorem 5.1] A is not approximate ϕ −contractible, which is a contradiction.

Let A be a Banach algebra and $\phi \in \Delta(A)$. A is called ϕ-biprojective, if there exists a bounded A-bimodule morphism $\rho: A \to A \otimes_p A$ such that $\phi \circ \pi_A \circ \rho = \phi$. Also A is called ϕ-biflat if there exists a bounded A-bimodule morphism $\rho: A \to (A \otimes_p A)^{**}$ such that $\tilde{\phi} \circ \pi_A^{**} \circ \rho = \phi$. For more information about ϕ −biflatness and ϕ −biprojectivity, the reader refers to [8] and [9].

Theorem 2.6 *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that $e \in \overline{B_1^0}$ and $\phi(e) \neq 0$. If A is ϕ-biprojective, then A_e is $\psi = \phi(e)\phi$-biprojective.*

Proof. Since A is ϕ-biprojective, there exists a bounded A −bimodule morphism $\rho: A \to A \otimes_p A$ such that $\phi \circ \pi_A \circ \rho = \phi$. Define $\tilde{\rho} = \frac{1}{\phi(e)}\rho$. We show that $\tilde{\rho}$ is a bounded A_e-bimodule morphism. To see this, consider

$$\begin{aligned} \tilde{\rho}(a * b) = \frac{1}{\phi(e)}\rho(a * b) = \frac{1}{\phi(e)}\rho(aeb) \quad &= ae\frac{1}{\phi(e)}\rho(b) \\ &= a * \frac{1}{\phi(e)}\rho(b) \\ &= a * \tilde{\rho}(b). \qquad (a.b \in A_e). \end{aligned}$$

Also

$$\begin{aligned} \tilde{\rho}(a * b) = \frac{1}{\phi(e)}\rho(a * b) = \frac{1}{\phi(e)}\rho(aeb) \quad &= \frac{1}{\phi(e)}\rho(a)be \\ &= \frac{1}{\phi(e)}\rho(a) * b \\ &= \tilde{\rho}(a) * b. \qquad (a.b \in A_e). \end{aligned}$$

On the other hand, since

$$\psi \circ \pi_{A_e} \circ \tilde{\rho} = \phi(e)\phi \circ \pi_A \circ \rho.$$

we have

$$\psi \circ \pi_{A_e} \circ \tilde{\rho}(a) = \phi(e)\phi \circ \pi_A \circ \rho(a) = \phi(e)\phi(a) = \psi(a). \quad (a \in A_e).$$

So A_e is ψ −biprojective.

Using the similar arguments as in the proof of the previous theorem, we have the following corollary:

Corollary 2.7 *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that $e \in \overline{B_1^0}$ and $\phi(e) \neq 0$. If A is ϕ-biflat, then A_e is $\psi = \phi(e)\phi$-biflat.*

Let A be a Banach algebra and $\phi \in \Delta(A)$. A is called ϕ −amenable if there exists a bounded net (m_α) in A such that $am_\alpha - \phi(a)m_\alpha \to 0$ and $\phi(m_\alpha) = 1$, for every $a \in A$. see [5].

Corollary 2.8 *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that $e \in \overline{B_1^0}$ and $\phi(e) \neq 0$. If A is ϕ-biflat and A has a left approximate identity, then A_e is approximate $\psi = \phi(e)\phi$-contractible.*

Proof. Since A is ϕ-biflat and A has a left approximate identity, by similar arguments as in the proof of [7, Theorem 2.2] A is ϕ-amenable. It is easy to see that ϕ-amenability of A implies that A is approximate ϕ −contractible. Applying Proposition 2.4, A_e becomes approximate ψ −contractible.

Let A b a Banach algebra and $\phi \in \Delta(A)$. Then A is called approximate left ϕ-biprojective if there exists a net of bounded linear maps from A into $A \otimes_p A$, say $(\rho_\alpha)_{\alpha \in I}$, such that

1. $\rho_\alpha(ab) - \phi(a)\rho_\alpha(b) \xrightarrow{||\cdot||} 0,$
2. $\rho_\alpha(ba) - \rho_\alpha(b) \cdot a \xrightarrow{||\cdot||} 0,$
3. $\pi_A \circ \rho_\alpha(a) - a \to 0,$

for every $a, b \in A$, see [12].

Theorem 2.9 *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that $e \in \overline{B_1^0}$ and $\phi(e) \neq 0$. If A is approximate left ϕ-biprojective, then A_e is approximate left $\psi = \phi(e)\phi$-biprojective.*

Proof. Since A is approximate left ϕ-biprojective, there exists a net of bounded linear maps (ρ_α) from A into $A \otimes_p A$ such that

$$\rho_\alpha(ab) - \phi(a)\rho_\alpha(b) \to 0, \quad \rho_\alpha(ab) - \rho_\alpha(a) \cdot b \to 0, \quad \phi \circ \pi_A \circ \rho(a) - \phi(a) \to 0.$$

Define $\tilde{\rho}_\alpha = \frac{1}{\phi(e)}\rho_\alpha$. We show that there exists a net of bounded linear maps $(\tilde{\rho}_\alpha)$ from A_e in to $A_e \otimes_p A_e$ such that

$$\tilde{\rho}_\alpha(a * b) - \psi(a)\tilde{\rho}_\alpha(b) \to 0. \quad \tilde{\rho}_\alpha(a * b) - \tilde{\rho}_\alpha(a) * b \to 0. \quad \psi \circ \pi_A \circ \tilde{\rho}(a) - \psi(a) \to 0.$$

To see this, consider

$$\begin{aligned}
\tilde{\rho}_\alpha(a * b) - \psi(a)\tilde{\rho}_\alpha(b) &= \tilde{\rho}_\alpha(aeb) - \phi(a)\phi(e)\tilde{\rho}_\alpha(b) \\
&= \frac{1}{\phi(e)}(\rho_\alpha(aeb) - \phi(a)\phi(e)\rho_\alpha(b)) \\
&= \frac{1}{\phi(e)}(\rho_\alpha(aeb) - \phi(ae)\rho_\alpha(b) + \phi(ae)\rho_\alpha(b) - \phi(a)\phi(e)\rho_\alpha(b)) \\
&\to 0
\end{aligned}$$

Also

$$\tilde{\rho}_\alpha(a * b) - \tilde{\rho}_\alpha(a) * b = \frac{1}{\phi(e)}\rho_\alpha(aeb) - \frac{1}{\phi(e)}\rho_\alpha(a)eb \to 0$$

On the other hand, since

$$\psi \circ \pi_{A_e} \circ \tilde{\rho}_\alpha = \phi(e)\phi \circ \pi_A \circ \rho_\alpha.$$

we have for $(a \in A_e$

$$\psi \circ \pi_{A_e} \circ \tilde{\rho}_\alpha(a) - \psi(a) = \phi(e)\phi \circ \pi_A \circ \rho_\alpha(a) - \phi(e)\phi(a) \to \phi(e)\phi(a) - \phi(e)\phi(a) = 0.$$

So A_e is approximate left ψ −biprojective.

Remark 2.10 *Let A and B be Banach algebras and $e_A \in \overline{B_1^0}^A$ and $e_B \in \overline{B_1^0}^B$. Then there exist two sequences (x_n) and (y_n) in the unit ball A and the unit ball B such that $x_n \to e_A$ and $y_n \to e_B$. respectively. Since*

$$||x_n \otimes y_n - e_A \otimes e_B|| \leq ||x_n \otimes y_n - e_A \otimes y_n|| + ||e_A \otimes y_n - e_A \otimes e_B|| \to 0.$$

we have $e_A \otimes e_B \in \overline{B_1^0}^{A\otimes_p B}$. Define $T: A_{e_A} \otimes_p B_{e_B} \to A \otimes_p B_{e_A \otimes e_B}$ by $T(a \otimes b) = a \otimes b$ for every $a \in A$ and $b \in B$. It is easy to see that T is an isometric algebra isomorphism. Also T is a bounded $A \otimes_p B_{e_A\otimes e_B}$ −bimodule morphism.

Proposition 2.11 *Let A and B be Banach algebras and $e_A \in \overline{B_1^0}^A$ and $e_B \in \overline{B_1^0}^B$. Suppose that $\phi_A \in \Delta(A)$ and $\phi_B \in \Delta(B)$ which $\phi_A(e_A) \neq 0$ and $\phi_B(e_B) \neq 0$. If A and B are ϕ_A −biprojective and ϕ_B −biprojective, respectively, then $A \otimes_p B_{e_A\otimes e_B}$ is $\phi_A(e_A)\phi_A \otimes \phi_B(e_B)\phi_B$ −biprojective.*

Proof. Since A and B are ϕ_A −biprojective and ϕ_B −biprojective, respectively, then by Theorem 2.9, A_e and B_e are $\phi_A(e_A)\phi_A$ −biprojective and $\phi_B(e_B)\phi_B$ −biprojective, respectively. So there exist a A_{e_A} −bimodule morphism $\rho_0: A_{e_A} \to A_{e_A} \otimes_p A_{e_A}$ and a B_{e_B}-bimodule morphism $\rho_1: B_{e_B} \to B_{e_B} \otimes_p B_{e_B}$ such that $\phi_A(e_A)\phi_A \circ \pi_A \circ \rho_0 = \phi_A(e_A)\phi_A$ and $\phi_B(e_B)\phi_B \circ \pi_B \circ \rho_1 = \phi_B(e_B)\phi_B$.

Define $\theta: (A_{e_A} \otimes_p A_{e_A}) \otimes_p (B_{e_B} \otimes_p B_{e_B}) \to (A_{e_A} \otimes_p B_{e_B}) \otimes_p (A_{e_A} \otimes_p B_{e_B})$ by

$$(a_1 \otimes a_2) \otimes (b_1 \otimes b_2) \mapsto (a_1 \otimes b_1) \otimes (a_2 \otimes b_2).$$

where $a_1.a_2 \in A$ and $b_1.b_2 \in B$. Clearly θ is an isometric algebra isomorphism. Set $\rho = (T \otimes T) \circ \theta \circ (\rho_0 \otimes \rho_1) \circ T^{-1}$, where T is the map defined as in Remark 2.10. We know that ρ is a bounded linear map from $A \otimes_p B_{e_A\otimes e_B}$ into $(A \otimes_p B_{e_A\otimes e_B}) \otimes_p (A \otimes_p B_{e_A\otimes e_B})$. Consider

$$\pi_{A\otimes_p B_{e_A\otimes e_B}} \circ \theta(a_1 \otimes a_2 \otimes b_1 \otimes b_2) = \pi_{A\otimes_p B_{e_A\otimes e_B}}(a_1 \otimes b_1 \otimes a_2 \otimes b_2)$$
$$= \pi_{A_{e_A}}(a_1 \otimes a_2) \otimes \pi_{B_{e_B}}(b_1 \otimes b_2).$$

then clearly one can show that $\pi_{A\otimes_p B_{e_A\otimes e_B}} \circ \theta = \pi_{A_{e_A}} \otimes \pi_{B_{e_B}}$. Hence,

$$\pi_{A\otimes_p B_{e_A\otimes e_B}} \circ \theta(\rho_0(a) \otimes \rho_1(b)) = \pi_{A_{e_A}} \circ \rho_0(a) \otimes \pi_{B_{e_B}} \circ \rho_1(b)$$

and it is easy to see that

$$\phi_A(e_A)\phi_A \otimes \phi_B(e_B)\phi_B \circ \pi_{A\otimes_p B} \circ \theta(\rho_0 \otimes \rho_1)(a \otimes b) = \phi_A(e_A)\phi_A \otimes \phi_B(e_B)\phi_B(a \otimes b).$$

the proof is complete.

3 APPROXIMATE HOMOLOGICAL PROPERTIES OF CERTAIN BANACH ALGEBRAS

In this section we investigate approximate biprojectivity and Johnson pseudo-contractibility of A_e.

Theorem 3.1 *Suppose that A is a Banach algebra and also suppose that $e \in \overline{B_1^0}$. Then the followings hold:*

1. If A is approximately biprojective and A_e is unital then A_e is approximately biprojective.
2. If A_e is unital and approximately biprojective, then A is approximately biprojective.

Proof. To show (1), suppose that A is approximately biprojective and A_e is unital. It follows that there is an approximately A −bimodule morphism (ρ_α) from A into $A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(a) - a \to 0$ for each $a \in A$. Note that

$$\begin{aligned}\rho_\alpha(a * b) - a * \rho_\alpha(b) &= \rho_\alpha(aeb) - a * \rho_\alpha(b)\\ &= \rho_\alpha(aeb) - ae\rho_\alpha(b) + ae\rho_\alpha(b) - a * \rho_\alpha(b) \to 0.\end{aligned}$$

and

$$\begin{aligned}\rho_\alpha(a * b) - \rho_\alpha(a) * b &= \rho_\alpha(aeb) - \rho_\alpha(a) * b\\ &= \rho_\alpha(aeb) - \rho_\alpha(a)eb + \rho_\alpha(a)eb - \rho_\alpha(a) * b \to 0,\end{aligned}$$

for each $a \in A_e$. It implies that (ρ_α) from A_e into $A_e \otimes_p A_e$ is an approximately A_e-bimodule morphism. Define $T: A_e \otimes_p A_e \to A_e \otimes_p A_e$ by $T(a \otimes b) = ae^{-1} \otimes b$. Note that using **Proposition 2.1**, the definition of T makes sense. It is easy to see that

$$T(a * (c \otimes d)) = a * T(c \otimes d). \quad T((c \otimes d) * a) = T(c \otimes d) * a. \qquad (a.c.d \in A).$$

Set $\tilde{\rho}_\alpha = T \circ \rho_\alpha$. Using direct calculations we can see that

$$\pi_{A_e} \circ \tilde{\rho}_\alpha = \pi_A \circ \rho_\alpha$$

It follows that $\pi_{A_e} \circ \tilde{\rho}_\alpha - a = \pi_A \circ \rho_\alpha - a \to 0. \qquad (a \in A_e)$.

Thus A_e is approximately biprojective.

To show (2), suppose that A_e is unital and approximately biprojective. By Proposition 2.3, we know that $A = (A_e)_{e^{-2}}$. Now applying (1) it is easy to see that A is approximately biprojective.

A Banach algebra A is called biprojective if there exists a bounded A −bimodule morphism $\rho: A \to A \otimes_p A$ such that $\pi_A \circ \rho(a) = a$ for each $a \in A$. see [13].

Example 3.2 *In this example we give a Banach algbra A_e which is approximately biprojective. Let* $A = \{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} | a_{ij} \in \mathbb{C}\}$. *With the matrix operations and ℓ^1-norm, A becomes a Banach algebra. Suppose that* $e = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$. *Clearly e is invertible and A is unital. So by Proposition 2.1, A_e is unital. It is well-known that A is biprojective, see [13]. So A is approximately biprojective. Applying previous theorem A_e becomes approximately biprojective.*

Definition 3.3 *We say that a Banach algebra A has approximate (F)-property(or A is AFP) if there is an approximate A −bimodule morphsim (ρ_α) from A into $(A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho_\alpha(a) - a \to 0$. for each $a \in A$.*

For the motivation of this definition see [3].

Proposition 3.4 *If A is AFP and A_e is unital, then A_e is approximately biprojective.*

Proof. Since A is AFP, there exists an approximate A −bimodule morphsim (ρ_α) from A into $(A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho_\alpha(a) - a \to 0$, for each $a \in A$. It is easy to see that (ρ_α) is an approximate A_e −bimodule morphsim from A_e into $(A_e \otimes_p A_e)^{**}$ such that $\pi_{A_e}^{**} \circ \rho_\alpha(a) - a \to 0$. for each $a \in A_e$. Let $T: A_e \otimes_p A_e \to A_e \otimes_p A_e$ be the same map as in the proof of Theorem 3.1. Clearly T is A_e-module morphism, so is T^{**}. Similar to the proof of Theorem 3.1, for the net $(T^{**} \circ \rho_\alpha)$ is an approximate A_e −bimodule morphism from A_e into $(A_e \otimes_p A_e)^{**}$ such that

$$\pi_{A_e}^{**} \circ T^{**} \circ \rho_\alpha(a) - a = \pi_A^{**} \circ \rho_\alpha(a) - a \to 0. \quad (a \in A).$$

We denote the identity of A_e with a_0 and define $m_\alpha = \rho_\alpha(a_0)$. Clearly (m_α) is a net in $(A_e \otimes_p A_e)^{**}$ which satisfies

$$a * m_\alpha - m_\alpha * a \to 0. \qquad \pi_{A_e}^{**}(m_\alpha) * a - a \to 0. \quad (a \in A_e).$$

Take $\epsilon > 0$ and arbitrary finite subsets $F \subseteq A_e$, $\Lambda \subseteq (A_e \otimes_p A_e)^*$ and $\Gamma \subseteq A_e^*$. Then we have

$$||a * m_\alpha - m_\alpha * a|| < \epsilon. \quad ||\pi_{A_e}^{**}(m_\alpha) * a - a|| < \epsilon. \quad (a \in F).$$

It is well-known that for each α, there exists a net $(n_\beta^\alpha)_\beta$ in $A_e \otimes_p A_e$ such that $n_\beta^\alpha \xrightarrow{w^*} m_\alpha$. Since $\pi_{A_e}^{**}$ is a w^*-continuous map, we have $\pi_{A_e}(n_\beta^\alpha) = \pi_{A_e}^{**}(n_\beta^\alpha) \xrightarrow{w^*} \pi_A^{**}(m_\alpha)$.

Thus we have $|a * n_\beta^\alpha(f) - a * m_\alpha(f)| < \frac{\epsilon}{K_0}$. $|n_\beta^\alpha * a(f) - m_\alpha * (f)| < \frac{\epsilon}{K_0}$

and $$|\pi_{A_e}(n_\beta^\alpha)(g) - \pi_{A_e}^{**}(m_\alpha)(g)| < \frac{\epsilon}{K_1}.$$

for each $a \in F$, $f \in \Lambda$ and $g \in A^*$, where $K_0 = \sup\{||f||: f \in \Lambda\}$ and $K_1 = \sup\{||g||: g \in \Gamma\}$. Since $a * m_\alpha - m_\alpha * a \to 0$ and $\pi_{A_e}^{**}(m_\alpha) * a - a \to 0$. we can find $\beta = \beta(F.\Lambda.\Gamma.\epsilon)$ such that

$$|a * n_\beta^\alpha(f) - n_\beta^\alpha * a(f)| < c\frac{\epsilon}{K_0} \quad |\pi_{A_e}(n_\beta^\alpha) * a(g) - a(g)| < \frac{\epsilon}{K_1}. \quad (a \in F. f \in \Lambda. g \in \Gamma)$$

for some $c \in \mathbb{R}^+$. Using Mazur's lemma, we have a net $(n_{(F.\Lambda.\Gamma.\epsilon)})$ in $A_e \otimes_p A_e$ such that

$$||a * n_{(F.\Lambda.\Gamma.\epsilon)} - n_{(F.\Lambda.\Gamma.\epsilon)} * a|| \to 0. \quad ||\pi_A(n_{(F.\Lambda.\Gamma.\epsilon)}) * a - a|| \to 0. \quad (a \in F).$$

Define $\rho_{(F.\Lambda.\Gamma.\epsilon)}: A_e \to A_e \otimes_p A_e$ by $\rho_{(F.\Lambda.\Gamma.\epsilon)}(a) = a * n_{(F.\Lambda.\Gamma.\epsilon)}$ for each $a \in A_e$. It is clear that $\rho_{(F.\Lambda.\Gamma.\epsilon)}(a * b) = a * \rho_{(F.\Lambda.\Gamma.\epsilon)}(b)$ for each $a, b \in A$. Also

$$\begin{aligned} ||\rho_{(F.\Lambda.\epsilon)}(a * b) - \rho_{(F.\Lambda.\Gamma.\epsilon)}(a) * b|| &= ||ab * n_{(F.\Lambda.\Gamma.\epsilon)} - a * (n_{(F.\Lambda.\Gamma.\epsilon)} * b)|| \\ &\leq ||a||||b * n_{(F.\Lambda.\Gamma.\epsilon)} - n_{(F.\Lambda.\Gamma.\epsilon)} * b|| \to 0. \end{aligned} \tag{3.1}$$

for each $a.b \in A_e$. Also

$$\begin{aligned} ||\pi_{A_e} \circ \rho_{(F.\Lambda.\Gamma.\epsilon)}(a) - a|| &= ||\pi_{A_e}(a * n_{(F.\Lambda.\Gamma.\epsilon)}) - a|| \\ &= ||\pi_{A_e}(a * n_{(F.\Lambda.\Gamma.\epsilon)}) - \pi_{A_e}(n_{(F.\Lambda.\Gamma.\epsilon)} * a) + \pi_{A_e}(n_{(F.\Lambda.\Gamma.\epsilon)} * a) - a|| \\ &\leq ||\pi_{A_e}(a * n_{(F.\Lambda.\Gamma.\epsilon)}) - \pi_{A_e}(n_{(F.\Lambda.\Gamma.\epsilon)} * a)|| + ||\pi_{A_e}(n_{(F.\Lambda.\Gamma.\epsilon)}) * a - a|| \\ &\to 0. \end{aligned}$$

for each $a \in F$. Thus with respect to the net $(\rho_{(F.\Lambda.\Gamma.\epsilon)})_{(F.\Lambda.\Gamma.\epsilon)}$. A_e becomes approximately biprojective.

A Banach algebra A is called Johnson pseudo-contractible, if there exists a not necessarily bounded net (m_α) in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a - a \to 0$. for every $a \in A$, see [11] and [10].

A Banach algebra A is called biflat, if there is a bounded A −bimodule morphsim ρ from A into $(A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho_\alpha(a) = a$, for each $a \in A$, see [13].

Proposition 3.5 *Let A be a Banach algebra and $e \in \overline{B_1^0}$. Suppose that A_e is unital. Then A is Johnson pseudo-contractible if and only if A_e is Johnson pseudo-contractible.*

Proof. Since A_e is unital, by Proposition 2.1 A is unital. So using [3, Theorem 2.1], Johnson pseudo-contractibility of A implies that A is amenable. Thus by [13, Exercise 4.3.15], A is biflat. Then by [6, Theorem 2.4] A_e is biflat. Since A_e is unital, biflatness of A_e gives the amenability of A_e.

For converse, suppose that A_e is Johnson pseudo-contractible. Since A_e is unital by [3, Theorem 2.1] A_e is amenable, so is biflat. Applying [6, Theorem 2.4] follows that A is biflat.

Using Proposition 2.1, A is unital, thus by [13, Exercise 4.3.15] A is amenable. So [11, Lemma 2.1] implies that A is Johnson pseudo-contractible.

Example 3.6 *We give a Banach algebra A_e which is not Johnson pseudo-contractible. Let*

$$A = \{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} | a_{ij} \in \mathbb{C}\} \text{ and suppose that } e = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}. \text{ Clearly } e \text{ is invertible}$$

and A is unital. So by Proposition 2.1 A_e is unital. Using [11, Theorem 2.5] we know that A is not Johnson pseudo-contractible. So by previous proposition A_e is not Johnson pseudo-contractible.

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