FRAGMENTS OF DYNAMIC OF MÖEBIOUS MAPPINGS AND SOME APPLICATIONS. PART I

Ž. PAVIĆEVIĆ[*1,2], J. ŠUŠIĆ[1]

[1]Faculty of Sciences and Mathematics, University of Montenegro, Podgorica, Montenegro;
[2]National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), Moscow, Russia
*Corresponding author. E-mail: zarkop@ucg.ac.me

Summary. In this article we prove that a continuous mapping on a simply-connected domain of the extended complex plane, which is normal with respect to the cycle group of all conformal automorphisms of the domain with a fixed attractive point, which belongs to the domain is a constant function. Applying this result we obtain new proofs of the classical Theorem of Liouville and little Picard Theorem for holomorphic, meromorphic and harmonic functions in complex plane. We also prove some results from the dynamic of Möbius mappings.

1 INTRODUCTION

A problem of Ch. Pommerenke, formulated in [1, p. 169], is if there exists a non-constant meromorphic mapping on the unit disc of the complex plane, which is automorphic and normal with respect to any non-continuous group Γ of Möbius mappings of the unit disc. The same problem was considered by D. Mind in [2, p. 119].

In [3] P. Järvi solved this problem by constructing an open Riemann surface which does not allow non-constant normal analytic mappings.

The question of connection between normality and constancy of functions was also considered by J. Väisälä in the article [4]. In this article it is shown (Theorem 2, p. 17) that styding the normal continuous functions on the complex plane and the extended complex plane does not make sense, since the family of all continuous functions on a simply connected elliptic (parabolic) domain in the complex plane, that are normal with respect to the group of all conformal automorphisms of such kind of domains, reduce to the family of constant functions. However, in this paper a more general result is proved. We prove that if for a simply connected domain of the extended complex plane there exists conformal automorphism g which has at least one attractive fixed point in that domain which is not ∞, then any function f which is continuous in that domain and which maps that domain in the Riemann sphere $\overline{\mathbb{C}}$ or extended set of real numbers $\mathbb{R} \cup \{-\infty, +\infty\}$, for which the family $\{ f \circ g^n \mid n \in \mathbb{Z} \}$ is a normal family of functions, is a constant function in that domain (*Theorem 3.2 and Theorem 5.2*).

From this result it follows that if a domain is of the parabolic type (complex plane) or the elliptic type (extended complex plane) then a continuous mapping on that domains will be a constant, if the family of all compositions of that function with all elements of cyclic group with generating element of the hyperbolic or parabolic Möbius mapping, is a normal family on the domain. We emphasize that the position of fixed points of a conformal automorphism plays the main role in obtaining these results.

These results are later used for obtaining the simple proof of the classical Liouville Theorem and little Picard Theorem for holomorphic, meromorphic and harmonic functions. Namely, in the theory of functions of complex argument the Liouville's Theorem on constancy of entire functions and little Picard Theorem on values of holomorphic and meromorphic functions have a special place. Proofs of these Theorems often use the classical results in the theory of analytic functions such as: Cauchy integral formula, the expansion of an analytic function in Taylor series and properties of elliptic modular function (see for example [5, 6, 7, 10]). The analogies of these Theorems for harmonic functions on the complex plane are also known (see [6, 11, 12]). We would like to highlight the reference [11], where the authors give six proofs of the Liouville Theorem for harmonic functions in the complex plane. The proof of Liouville's Theorem for harmonic functions on $R^n, n \geq 2,$ from [12], is interesting, as it has not used any single mathematical symbol. Our article gives a new approach in proving these results.

As a direct consequence of considering of constancy of continuous functions, we also obtain the known results which say that fixed points of parabolic and hyperbolic Möbius transforms, which are automorphisms of the unit disc, must be on the boundary of the unit disk, and that the fixed points of elliptic automorphisms of the unit disk cannot be attraction points as well as they cannot be on the boundary of the unit disc. We find a connection between the notion of normality and discontinuity of subgroups of Möbius group of all conformal automorphisms of the Riemann sphere $\overline{\mathbb{C}}$ (see [7, 8, 10]).

2. PRELIMINARY NOTATIONS AND DEFINITIONS

With $\mathbb{R}$ we denote the set of all real numbers, $\mathbb{Z}$ will denote the set of all integers, $\mathbb{N}$ the set of all natural numbers, and $\mathbb{C} = \{z \mid z = x + iy, \ x, y \in R\mathbb{R}\}$ will be the set of all complex numbers, i.e., the complex plane, $|z| = \sqrt{z\overline{z}}$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is Riemann sphere.

For z_1 and z_2 we denote by $d(z_1, z_2) = |z_1 - z_2|$ the Euclidean metric on $\mathbb{C}$, and

$$d_s(\omega_1, \omega_2) = \frac{2|\omega_1 - \omega_2|}{\sqrt{1 + |\omega_1|} \cdot \sqrt{1 + |\omega_2|}}, \omega_1, \omega_2 \in \mathbb{C}; \quad d_s(\omega_1, \omega_2) = \frac{2}{\sqrt{1 + |\omega_1|^2}}, \ \omega_1 \in \mathbb{C}, \ \omega_2 = \infty, \quad \text{is the}$$

spherical Riemann distance on $\overline{\mathbb{C}}$.

The set $\mathbb{C}$ with the metric $d(z_1, z_2)$ is Hausdorff and complete metric space, but it is not compact.

However, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the metric $d_s(\omega_1, \omega_2)$ is a compact metric space. On the compact subset of $\mathbb{C}$ these metrics are equivalent.

The convergences are meant in these metrics.

The group $G\left(\overline{\mathbb{C}}\right) = \left\{ \dfrac{az-b}{cz-d} \,\middle|\, a,b,c,d \in \mathbb{C},\ ad - cd = 1 \right\}$ is the group of all conformal automor-phisms of the Riemann sphere $\overline{\mathbb{C}}$, and the group $G\left(\mathbb{C}\right) = \left\{ az + b \,\middle|\, a,b \in C, a \neq 0 \right\}$ is the group of all conformal automorphisms of the complex plane $\mathbb{C}$. The group $G\left(\mathbb{C}\right)$ is a subgroup of the group $G\left(\overline{\mathbb{C}}\right)$. We use notation: $G\left(\mathbb{C}\right) \Delta G\left(\overline{\mathbb{C}}\right)$. Groups, $G\left(\overline{\mathbb{C}}\right)$, i $G\left(\mathbb{C}\right)$ are Möbius groups for the Riemann sphere $\overline{\mathbb{C}}$ and the complex plane $\mathbb{C}$, respectively, and their elements are referred to as Möbius mappings.

The Möbius mappings g_1 i g_2 are equivalent if there exists a Möbius mapping $h \in G\left(\overline{\mathbb{C}}\right)$ such that $g_1\left(z\right) = \left(h \circ g_2 \circ h^{-1}\right)\left(z\right),\ z \in \overline{\mathbb{C}}.$

For every $g\left(z\right) = \dfrac{az+b}{cz+d} \in G\left(\overline{\mathbb{C}}\right)$ there exists a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the group

$$GL\left(2,\mathbb{C}\right) = \left\{ A \,\middle|\, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a,b,c,d \in \mathbb{C},\ ad - bc \neq 0 \right\}.$$

It may be shown that the group $G\left(\overline{\mathbb{C}}\right)$ is isomorphic to the group $SL\left(2,\mathbb{C}\right)/\left\{I,-I\right\}$, where l is the identity matrix, and $SL\left(2,\mathbb{C}\right)$ is the set of all matrices A such that $\det A = 1$. On the group $G\left(\overline{\mathbb{C}}\right)$ one may introduce the norm $\|g\| = \left(|a|^2 + |b|^2 + |c|^2 + |d|^2\right)$, which generates the metric on $G\left(\overline{\mathbb{C}}\right)$, which defines the topology on it. With respect to that topology, $G\left(\overline{\mathbb{C}}\right)$ is a topological group.

For $g\left(z\right) = \dfrac{az+b}{cz+d} \in G\left(\overline{\mathbb{C}}\right)$ we have: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $tr\left(g\right) = \dfrac{\left(a+d\right)^2}{\det M}$.

With the symbol O we will always denote a simply connected domain of the Riemann sphere $\overline{\mathbb{C}}$, i.e., $O \subset \overline{\mathbb{C}}$.

Let $G\left(O\right)$ be the group of all conformal automorphisms of the domain O. A point $z_0 \in O$ is *fixed point* of $g \in G\left(O\right)$ if $g\left(z_0\right) = z_0$. Then we have $g^{-1}\left(z_0\right) = z_0$, and therefore z_0 is also a fixed point for g^{-1}.

For $g \in G\left(O\right)$ we use notation $g^n\left(z\right) = \underbrace{g\ \left(g\left(...\left(g\left(z\right)...\right)\ \right)\right.}_{n\ \text{times}}$ and

$$g^{-n}(z) = \left(g^{-1}\right)^n(z) = \underbrace{g^{-1}\left(g^{-1}\left(...\left(g^{-1}(z)...\right)\right)\right)}_{n \text{ times}}, \ n \in \mathbb{N}.$$

For a fixed point z_0 of g in the group $G(O)$ the equality $g^n(z_0) = z_0$, $n \in \mathbb{Z}$ holds.

A fixed point $z_0 \in O$ of $g \in G(O)$, $g \neq i = g^0$, where i is the identity mapping, is an *attractive point of an automorphism g*, if for every $z \in O$ we have $\lim_{n\to\infty} g^n(z) = z_0$.

3. FRAGMENTS OF DYNAMIC OF MÖBIUS MAPPINGS

Further, we need to analyse the fixed points of Möbius mappings of the group $G(\overline{\mathbb{C}})$.

In [8], on p. 67, a classification of Möbius mappings in $G(\overline{\mathbb{C}})$ based on description of fixed points of their Poincare extensions on $\mathbb{R}^3 \cup \{\infty\}$ is given.

Namely, if $g \in G(\overline{\mathbb{C}})$, $g \neq i$, has one fixed point in $\mathbb{C}$, then it is called the *parabolic element of the group* $G(\overline{\mathbb{C}})$, or *parabolic Möbius mapping*.

If the Poincare extension of an element $g \in G(\overline{\mathbb{C}})$ on $\mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ has only two fixed points in $\mathbb{R}^3$, then it is called the *loxodromic Möbius mapping*.

If for a loxodromic Möbius mapping g there exists an open circle or an open half plane in the complex plane $\overline{\mathbb{C}}$ which are invariant with respect to *g*, then we call *g* the *hyperbolic element of the group* $G(\overline{\mathbb{C}})$, *or hyperbolic Möbius mapping*. In opposite case it is called the *strictly loxodromic mapping*.

If the Poincare extension of $g \in G(\overline{\mathbb{C}})$ on $\mathbb{R}^3 \cup \{\infty\}$ has infinitely many fixed points on $\mathbb{R}^3 \cup \{\infty\}$, then it is called the *elliptic element of the group* $G(\overline{\mathbb{C}})$, or *elliptical Möbius mapping*.

Theorem 3.1 ([8], see Theorem 4.3.4, p. 67). *For $g \in G(\overline{\mathbb{C}})$, $g \neq i$ we have: g is a parabolic element if and only if we have $tr^2(g) = 4$, g is an elliptic element if and only if $tr^2(g) \in [0, 4)$, g is the hyperbolic element if and only if $tr^2(g) \in (4, \infty)$ and g is strictly loxodromic element if and only if $tr^2(g) \notin (4, \infty)$.*

Any element of $G(\overline{\mathbb{C}})$, different from the identity, has one or two fixed points in $\overline{\mathbb{C}}$.

Theorem 3.2 ([8], p. 73). (i) *If $g \in G(\overline{\mathbb{C}})$, $g \neq i$, is a parabolic element with a fixed point $z_0 \in \mathbb{C}$, then for every $z \in \overline{\mathbb{C}}$ we have $\lim_{n\to\infty} g^n(z) = z_0$.*

(ii) *If g is a loxodromic element with fixed points z_0 and z_1, then for one of these points, without loss of generality z_0, that for every $z \in \overline{\mathbb{C}} \setminus \{z_1\}$ the equality $\lim_{n \to \infty} g^n(z) = z_0$ holds, but then for every $z \in \overline{\mathbb{C}} \setminus \{z_0\}$ $\lim_{n \to \infty} g^{-n}(z) = z_1$.*

(iii) *If g is an elliptic element with fixed points z_0 and z_1, then g remains invariant for every circle with respect to which the points z_0 z_1 are invertible.*

For elements of $G(\mathbb{C})$ we will use the following Lemma in the proof of Theorems in sections 5, 6 i 7.

Lemma 3.3. *For $g \in G(\mathbb{C}) = \{ g(z) = az + c \mid a \in \mathbb{C},\ a \neq 0,\ c \in \mathbb{C} \}$, $g \neq i$ the following holds:*

  (i)   *g is a parabolic Möbius mapping if and only if $a = 1$, ant it's fixed attractive point is ∞;*

  (ii)   *g is an elliptic Möbius mapping if and only if $|a| = 1$, $a \neq 1$, and it's fixed points are $\dfrac{c}{1-a}$ and ∞;*

  (iii)   *g is a hyperbolic Möbius mapping with fixed attractive points $\dfrac{c}{1-a}$ if and only if $a \in \mathbb{R}$, $0 < a < 1$, and ∞ is its repulsive fixed point;*

  (iv)   *g is a hyperbolic Möbius mapping with repulsive fixed point $\dfrac{c}{1-a}$ if and only if $a \in \mathbb{R}$, $1 < a$, and its attractive point is ∞;*

  (v)   *in other cases g is strictly loxodromic Möbius mapping.*

Proof of Lemma 3.3.   For $g(z) = az + c \in G(\mathbb{C})$, $g \neq i$, $a \in \mathbb{C}$, $a \neq 0$, $b \in \mathbb{C}$, we have $tr^2(g) = a + \dfrac{1}{a} + 2$. Since $g(z)$ is parabolic, elliptic or hyperbolic Möbius mapping we have that

$tr^2(g) \in \mathbb{R}$. If $a = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, then we have $tr^2(g) = \alpha + \dfrac{\alpha}{|a|^2} + 2 + \beta\left(1 - \dfrac{1}{|a|^2}\right)i$, so we obtain $tr^2(g) \in \mathbb{R}$ if $1 - \dfrac{1}{|\alpha|^2} = 0$ or $\beta = 0$.

I case. Assume that $1 - \dfrac{1}{|\alpha|^2} = 0$, then $|a| = 1$. It follows $a = e^{i\theta}, \theta \in [0, 2\pi),$ so

$\alpha = \cos\theta$ i $\beta = \sin\theta$. Since $tr^2(g) = 2\alpha + 2 = 2\cos\theta + 2$, from Theorem 3.1 we have that g is a parabolic Möbius mapping if and only if $2\cos\theta + 2 = 4$. Therefore, g is a parabolic Möbius mapping if and only if $\cos\theta = 1$, i.e., for $\theta = 0$. It follows that $\beta = \sin 0 = 0$ and $a = \alpha = 1$. It is easy to see that ∞ is its attraction fixed point. We have proved the part i).

From Theorem 3.1 it follows that g is an elliptic Möbius mapping if and only if $0 \le tr^2(g) < 4$. We conclude that $0 \le 2\cos\theta + 2 < 4$, which is equivalent to $-1 \le \cos\theta < 1$. Since $0 < \theta < 2\pi$, we have $-1 \le \sin\theta < 1$. This yields $a = \alpha + i\beta = \cos\theta + i\sin\theta = e^{i\theta}, 0 < \theta < 2\pi$. Therefore, g is an elliptic Moebius mapping if and only if $|a| = 1$, $a \ne 1$. It is easy to see that all fixed points of it are

$\dfrac{c}{1-a}$ and ∞. Thus, statement ii) is proved.

From Theorem 3.1 it follows that g is a hyperbolic Möbius mapping if and only if $4 < tr^2(g)$. This means that $4 < 2\cos\theta + 2$, which is equivalent to $1 < \cos\theta$, but this is impossible, this case excludes the hyperbolic Möbius mappings.
We have finished the case **I**.

II case. Let $\beta = 0$. Then $a = \alpha \in \mathbb{R} \setminus \{0\}$, and therefore $tr^2(g) = \alpha + \dfrac{1}{\alpha} + 2$.

If $tr^2(g) = \alpha + \dfrac{1}{\alpha} + 2 = 4$, then g is a parabolic Möbius mapping. The preceding equality is equivalent to $(\alpha - 1)^2 = 0$, and this is equivalent to $\alpha = 1$, thus in this case we also have i).

The mapping g is an elliptic Möbius mapping if $0 \le tr^2(g) < 4$, from which it follows that $0 \le \alpha + \dfrac{1}{\alpha} + 2 < 4$, or $-2 \le \alpha + \dfrac{1}{\alpha} < 2$, which is equivalent to $-2 \le \alpha + \dfrac{1}{\alpha}$ and $\alpha + \dfrac{1}{\alpha} < 2$.

If we would have $\alpha < 0$, then we will derive $-2\alpha \ge \alpha^2 + 1$ and $\alpha^2 + 1 > 2\alpha$, i.e., $0 \ge (\alpha + 1)^2$ and $(\alpha - 1)^2 > 0$, but this is impossible.

If we would have $\alpha > 0$, then we will derive $-2\alpha \le \alpha^2 + 1$ and $\alpha^2 + 1 < 2\alpha$, i.e., $0 \le (\alpha + 1)^2$ and $(\alpha - 1)^2 < 0$, which is also impossible.

Therefore, the case $\beta = 0.$ excludes the elliptic Möbius mappings. Therefore we have proved *ii)*.

The condition that g is an hyperbolic Möbius mapping is that $4 < tr^2(g)$. From this we obtain that $4 < \alpha + \dfrac{1}{\alpha} + 2$, or $2 < \alpha + \dfrac{1}{\alpha}$.

If we would have $\alpha < 0$, this would imply $2\alpha > \alpha^2 + 1$, i.e., $0 > (\alpha - 1)^2$, which is not true.

If $\alpha > 0$, $2\alpha < \alpha^2 + 1$, i.e., $0 < (\alpha - 1)^2$, which is true for every $\alpha \in (0,1) \cup (1, +\infty)$.

Therefore, $\alpha \in (0,1) \cup (1, +\infty)$ is necessary and sufficient for g to be a hyperbolic Moebius mapping.

If $\alpha \in (0,1)$, then form $\alpha z + c = z$ we obtain that $\dfrac{c}{1-a}$ is a fixed point for g. Since we have

$$g^n(z) = \alpha^n z + \alpha^{n-1}c + \alpha^{n-2}c + \ldots + \alpha c + c = \alpha^n z + \frac{c(1-\alpha^n)}{1-\alpha},$$ we obtain $\lim\limits_{n \to \infty} g^n(z) = \dfrac{c}{1-a}$, from this it

follows that $\dfrac{c}{1-a}$ is a fixed attractive point for the hyperbolic Möbius mapping g, and ∞ is its *repulsive fixed point.*. This is the statement of the part *iii)*.

If $\alpha \in (1, +\infty)$ then $\dfrac{c}{1-a}$ and ∞ are fixed points for the *hyperbolic Möbius mapping g*. Since

$$g^n(z) = \alpha^n z + \alpha^{n-1}c + \alpha^{n-2}c + \ldots + \alpha c + c = \alpha^n z + \frac{c(1-\alpha^n)}{1-\alpha},$$ we conclude $\lim\limits_{n \to \infty} g^n(z) = \infty$. From this

we conclude that ∞ is fixed attractive point for the hyperbolic Moebius mapping *g,* so $\dfrac{c}{1-a}$ is its *repulsive fixed point*, and we have finished the part *vi)*.

If g isn't parabolic, elliptic, or hyperbolic Möbius mapping, then it is strictly loxodromic Moebius mapping. Therefore we have *v)*. $\square$

4. THE MAIN RESULTS

We say that a family of functions $\Im = \left\{ f \mid f : O \to \overline{\mathbb{C}} \right\}$ is *normal family on the domain O, $O \subset \mathbb{C}$,* if any sequence (f_n) of $\Im$ has a subsequence (f_{n_k}) which is uniformly convergent to a function $f : O \to \overline{\mathbb{C}}$ on compacts of *O. For this type of normality of the family $\Im$ we say that it is normal in the sense of Montel.* The family of functions $\Im = \left\{ f \mid f : O \to \overline{\mathbb{C}} \right\}$ is normal in $z \in O$ if it is a normal family in a domain which contains *z*.

It is known (see for example [5, 9, 10,14]) that the family of functions $\Im = \left\{ f \middle| f : O \to \overline{\mathbb{C}} \right\}$ is a normal family on a domain O if and only if it is normal in every point of the domain O.

If $O \subset \overline{\mathbb{C}}$, i.e., if $\infty \in O$, then the family of functions $\Im = \left\{ f \middle| f : O \to \overline{\mathbb{C}} \right\}$ is normal in ∞ if the family $\Im' = \left\{ f \left(\frac{1}{z} \right) \middle| f \in \Im \right\}$ is normal in 0, and the function $\Im = \left\{ f \middle| f : O \to \overline{\mathbb{C}} \right\}$ is normal on O if it is normal in every point of the domain O.

A family $\Im$ of functions is *equicontinuous in a point* $z_0 \in O$, $O \subset \overline{\mathbb{C}}$, if for every $\varepsilon > 0$ there exists $\delta = \delta(z_0, \varepsilon) > 0$ such that for every $f \in \Im$ and every z for which $d_1(z, z_0) < \delta$ there holds $d_2(f(z), f(z_0)) < \varepsilon$, where d_1 and d_2 are previously defined metrics on $\mathbb{C}$ i $\overline{\mathbb{C}}$. We can take $d_1 = d_2 = d_s$, if we consider a domain $O \subset \overline{\mathbb{C}}$ which contains the point ∞. A family $\Im$ of functions is equicontinuous family of functions on a domain O if it is equicontinuous in every point of the domain.

Let $G(O)$ be the group of all conformal authomorphisms of the domain O. For a function $f : O \to \overline{\mathbb{C}}$ we say that it is a *normal function* on the domain O with respect to the group G if the family $\Im = \left\{ f \circ \varphi \middle| \varphi \in G \right\}$ is normal family on O, i.e., if any sequence of this family has a subsequence which is uniformly convergent on compact subsets of O.

We will need the following Theorem for the proof of our main result which is given in Theorem 3.2:

Theorem 4.1. ([5], p. 12, or [10]). *A family $\Im$ of continuous functions on a domain O is a normal family on that domain if and only if the family $\Im$ is equicontinuous in O.*

The main result in this paper is the following Theorem:

Theorem 4.2. *Let g be a conformal automorphism of simply connected domain $O \subset \overline{\mathbb{C}}$ which has a attractive fixed point $z_0 \in O$, $z_0 \neq \infty$, and let $f : O \to \overline{\mathbb{C}}$ be a continuous function on O. If the function f is normal on the domain O with respect to the cyclic group $G_g = \left\{ g^n \middle| n \in \mathbb{Z} \right\}$, which is determined by the conformal automorphism g, then f is a constant function on O.*

Proof of Theorem 4.2. Assume the contrary, i.e., that there exists a continuous function f on O which is not constant on O but is normal with respect to the group $G_g = \{g^n \mid n \in \mathbb{Z}\}$. Then there exists a point $z_1 \in O$ such that $f(z_1) \neq f(z_0)$, so $d_2(f(z_1), f(z_0)) > 0$. Denote

$$\varepsilon = \frac{d_2(f(z_1), f(z_0))}{2} > 0. \tag{1}$$

From the condition that f is normal on the domain O with respect to the cyclic group $G_g = \{g^n \mid n \in \mathbb{Z}\}$ and Theorem 4.1 it follows that the family $\{f \circ g^n \mid n \in \mathbb{Z}\}$ is equicontinuous on the domain O, so it is equicontinuous *in* $z_0 \in O$. It follows that for $\varepsilon = \dfrac{d_2(f(z_1), f(z_0))}{2}$ there exists $\delta > 0$, such that for every z for which $d_1(z, z_0) < \delta$ there holds

$$d_2(f \circ g^n(z), f \circ g^n(z_0)) < \varepsilon, \ n \in \mathbb{Z}, \tag{2}$$

where d_1 i d_2 are metrics defined before.

Let us consider the sequence (w_n), $w_n = g^n(z_1)$, $n \in \mathbb{N}$. Since z_0 is an attractive fixed point for g, we have $\lim\limits_{n \to \infty} g^n(z_1) = \lim\limits_{n \to \infty} w_n = z_0$, so for δ there exists a natural number N such that for every $n \geq N$ there holds that $d(w_n, z_0) < \delta$. From this and (2) it follows that $d_2(f \circ g^{-n}(w_n), f \circ g^{-n}(z_0)) < \varepsilon$ for every $n \geq N$, i.e., $d_2(f \circ g^{-n}(g^n(z_1)), f \circ g^{-n}(g^n(z_0))) < \varepsilon$ for every $n \geq N$, from this and from (1) it follows that $d_2(f(z_1), f(z_0)) < \dfrac{d_2(f(z_1), f(z_0))}{2}$. Which is contradiction and the Theorem follows. $\square$

For $g \in G(\mathbb{C})$, $g \neq i$, with G_g we will denote in the further exposition the cyclic group $G_g = \{g^n \mid n \in \mathbb{Z}\}$. The group G_g is a group of all conformal automorphisms of the complex plane $\mathbb{C}$, as well as the Riemanin sphere $\overline{\mathbb{C}}$. Therefore $G_g \Delta G(\mathbb{C}) \Delta G(\overline{\mathbb{C}})$.

Remark 4.3. If we take the complex plane $\mathbb{C}$ or the Riemann sphere $\overline{\mathbb{C}}$ for the domain O in Theorem 4.2, and the group G_g for the group of conformal automorphisms ,where g is a hyperbolic element from part *iii)* of Lemma 3.3, then from Theorem 4.2 we have the statement of Theorem 2 from [5], on page 17. Therefore, Theorem 4.2 is a generalization of Theorem 2 in [5].

5. APPLICATIONS ON HOLOMORPHIC AND MEROMORPHIC FUNCTIONS

In this section we prove that Liouville and little Picard Theorem may be obtained as a direct consequences of the Montel Theorem on normality of family of holomorphic and meromorphic functions.

For the following considerations we will need the local boundedness of the family of functions. A family of functions $\Im = \{f \mid f : O \to \mathbb{C}\}$ is locally bounded on a domain O if for every $z_0 \in O$ there exists a constant $M = M(z_0) > 0$ and a disc $D(z_0, r) = \{z \mid z \in \mathbb{C}, |z - z_0| < r\} \subset O, r > 0,$ such that for every $z \in D(z_0, r)$ and every $f \in \Im$ there holds $|f(z)| < M$.

Theorem 5.1 ([10], Montel's Theorem, p. 35). *If $\Im$ is a family of locally bounded holomorphic functions on a domain O, then $\Im$ is a normal family on the domain O.*

Theorem 5.2 ([7], Theorem 1.3 (Liouville's Theorem), p. 3). *A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ which is bounded on $\mathbb{C}$, must be a constant on $\mathbb{C}$.*

Proof of Theorem 5.2. Let $g \in G(\mathbb{C}), g(z) = az + c, 0 < a < 1, c \in \mathbb{C}$. From the boundedness of an holomorphic function on $\mathbb{C}$, the function f and from Theorem 5.1 it follows that the family $\Im_f = \{f \circ g^n \mid n \in \mathbb{Z}\}$ is a normal family on $\mathbb{C}$ with respect to the group G_g . The statement of Theorem 5.2 now follows straightforwardly from *iii)* of Lemma 3.3 and Theorem 4.2. □

Theorem 5.3 (see [11], p. 112, or Lemma 2.5, [7], p. 17). *A holomorphic function $f : \overline{\mathbb{C}} \to \mathbb{C}$ must be constant on $\overline{\mathbb{C}}$.*

Proof of Theorem 5.3. From the Theorem of maximum of modulus of an holomorphic function it follows that f is a bounded holomorphic function on $\overline{\mathbb{C}}$. Now from Theorem 5.1 it follows that f is normal function on $\overline{\mathbb{C}}$ with respect to the group G_g , $g(z) = az + c, 0 < a < 1, c \in \mathbb{C}$. Since G_g is a group of all conformal automorphisms of $\overline{\mathbb{C}}$, from *iii)* in Lemma 3.3 and Theorem 4.2 it follows that f a constant function on $\overline{\mathbb{C}}$. □

Theorem 5.4 ([10], Fundamental Normality Test, p. 54) *If $\Im$ is a family of holomorphic functios on a domain O that do not take two fixed values a and b in $\mathbb{C}$, then $\Im$ is normal family on O.*

Theorem 5.5([7], Theorem 2.6 (Picard's Theorem), p. 17). *An holomorphic function $f : \mathbb{C} \to \mathbb{C}$, which is not constant on $\mathbb{C}$, takes all values in $\mathbb{C}$, with at most one exception.*

Proof of Theorem 5.5. Assume that the statement of Theorem 5.5 is not correct, i.e., that there exists an holomorphic function $f : \mathbb{C} \to \mathbb{C}$, which is not constant on $\mathbb{C}$, which achieves all values in $\mathbb{C}$, except two or more fixed values in $\mathbb{C}$. Let us consider the family $\mathfrak{I}_f = \left\{ f \circ g^n \big| n \in \mathbb{Z} \right\}$, where $g(z) = az + c, 0 < a < 1, c \in \mathbb{C}$, is a hyperbolic element of the group $G(\mathbb{C})$. Then functions from the family $\mathfrak{I}_f = \left\{ f \circ g^n \big| n \in \mathbb{Z} \right\}$ does not assume two fixed values in $\mathbb{C}$, therefore from Theorem 5.4 it follows that the family $\mathfrak{I}_f = \left\{ f \circ g^n \big| n \in \mathbb{Z} \right\}$ is a normal family in $\mathbb{C}$. Therefore the function f is normal on $\mathbb{C}$ with respect to the cyclic group $G_g = \left\{ g^n \big| n \in \mathbb{Z} \right\}$. Since from *iii)* of Lemma 3.3 it follows that the element g has an attractive fixed point in $\mathbb{C}$, Theorem 4.2 yields that f is constant on $\mathbb{C}$, which is contrary to our assumption. This proves Theorem 5.5. □

Theorem 5.6 ([10], Fundamental Normality Test, p. 74)**.** *Let $\mathfrak{I}$ be a family of meromorphic functions on a domain O such that any function in the family does not take any of three fixed values a, b and c in $\mathbb{C}$. Then $\mathfrak{I}$ is a normal family on O.*

Theorem 5.7 (Picard's Theorem for meromorphic functions)**.** *A meromorphic functions $f : \mathbb{C} \to \overline{\mathbb{C}}$, which is not constant, achieves all values in $\mathbb{C}$, with possible exception of at most two values.*

Theorem 5.7 may be proved in the same fashion as Theorem 5.5 using the cyclic group $G_g = \left\{ g^n \big| n \in \mathbb{Z} \right\}$, which is generated by a hyperbolic Möbius mapping $g(z) = \alpha z + c, 0 < \alpha < 1$, $c \in \mathbb{C}$, which by *iii)* of Lemma 3.3 has an attractive fixed point in $\mathbb{C} \subset \overline{\mathbb{C}}$, but in the proof we should use Theorem 5.6 instead of Theorem 5.4.

Theorem 4.2 shows that in given proofs of Theorems of Liouville and Picard, instead of hyperbolic Möbius mapping $g(z) = az + c, 0 < a < 1$, in $G(\mathbb{C})$ we could take any hyperbolic or parabolic Möbius mapping in $G(\overline{\mathbb{C}})$, which has an attractive fixed point in $\mathbb{C}$.

Remark 5.8. The results of this section are proved in [13] using Theorem 2 from [5], p. 17. In this section for the proof of Theorem Liouville and Picard we use the weaker result which is given in Theorem 4.2, and which shows that for constancy of functions on a *simply connected domain* O, $O \subset \overline{\mathbb{C}}$, the existence of *conformal automorphism of the domain O, which has a fixed attractive point $z_0 \in O$*, $z_0 \neq \infty$ is important.

6. APPLICATIONS ON MÖBIUS MAPPINGS

Here we consider some properties of elements and subgroups of the Möbius group $G(\overline{\mathbb{C}})$.

In the sequel we denote by D the unit disc, H a half plane $\overline{\mathbb{C}}$, i.e., $D = \left\{ z \mid z \in \mathbb{C}, \ |z| < 1 \right\}$, and $H = \left\{ z \mid z = x + iy, \ x \in \mathbb{R}, \ y \geq 0 \right\}$, and with ∂D and ∂H we denote the boundary of D and H, respectively.

The next Theorem follows from the proof of Theorem 5.2.1 in [7], p. 93, which will be shown:

Theorem 6.1. *A parabolic or hyperbolic Möbius mapping in group* $G(\overline{\mathbb{C}})$ *with invariant disc D or half plane H, has fixed points that must be on the boundary* ∂D, *or on the boundary* ∂H, *but fixed points of elliptic Möbius mappings cannot be attractive fixed points.*

The statement of 6.1 may be derived directly from Theorem *4.2*. This is shown below:

Proof of Theorem 6.1. Assume the contrary i.e., that there exists a hyperbolic element $g \in G(\overline{\mathbb{C}})$, such that at least one of its fixed points is in D. Then by Theorem 4.2 g *is a constant function*, but this is not so.

Therefore, a fixed point of g cannot be in D.

In the same way one can show that a fixed point of g cannot be in $\mathbb{C} \backslash D$.

In the same way it is possible to prove a statement in the case of half plane H, and when ∞ is a fixed point, it is clear that it belongs to the boundary ∂H.

The statement for fixed points of parabolic Möbius mappings can be proved analogously.
Since fixed points of elliptic mappings belong to D or H, it follows from Theorem 4.2 that they cannot be attractive points, otherwise then it would follow that bounded analytic functions on D or H are constants, which is impossible. This statement follows from the property that an elliptic Möbius mapping is equivalent to the Möbius mapping $g(z) = |k| z, \ |k| = 1$, which are rotations with respect to 0. This follows from *(iii)*, of Theorem 3.2. □

Subgroup G of group $G(\overline{\mathbb{C}})$ is *discrete* if and only if for every $k > 0$ the set $\left\{ g \in \mathcal{G} \mid \|g\| < k \right\}$ is finite.

The subgroup G of the group $G(\overline{\mathbb{C}})$ is *discontinuous in the point* z_0 if z_0 is not in the closure of the set $\mathcal{G}(z) = \left\{ g(z) \mid g \in \mathcal{G} \right\}$, for every $z \in \overline{\mathbb{C}}$. In other words, the subgroup G of the group

$G(\overline{\mathbb{C}})$ is *discontinuous in* the point z_0 if there does not exist a sequence of mutually different elements $g_n \in \mathcal{G}$ such that for every $z \in \overline{\mathbb{C}}$ we have $g_n(z) \to z_0$, if $n \to \infty$. The group G is *discontinuous* on the set S if it is *discontinuous* in every point of the set S. The group G is *discontinuous* if it is *discontinuous* on an nonempty set.

If a subgroup G of the group $G(\overline{\mathbb{C}})$ *is discontinuous, then it is a discrete group.* The converse statement does not hold. Namely, there exists a subgroup G of the group $G(\overline{\mathbb{C}})$ which is discrete but not discontinuous. The example is the group of Picard (see [8], p. 95-103, or [10], p. 200-201).

The next Theorem shows conditions under which a discrete subgroup G of the group $G(\overline{\mathbb{C}})$ *is discontinuous* group on a domain.

Theorem 6.2. ([10], Theorem 5.5.10, p. 205). *A subgroup* G *of* $G(\overline{\mathbb{C}})$ *is discontinuous in a point* α *if and only if* G *is discrete and makes a normal family of functions in* α.

Theorem 6.2 shows that it is important to answer under which conditions a *subgroup* G of the group $G(\overline{\mathbb{C}})$ is normal, or is not normal in a point. One answer is given by the following Theorem:

Theorem 6.3. *Let a subgroup* G *of group* $G(\overline{\mathbb{C}})$ *contain a loxodromic (parabolic) element of the group* $G(\overline{\mathbb{C}})$. *Then:*

    i) the subgroup G *is a not normal family of functions in the fixed point of that loxodromic (parabolic) element,*

    ii) G *is not normal on any domain which contains a fixed point of a loxodromic (parabolic) element of subgroup* G. *In particular,* G *is not normal on* $\overline{\mathbb{C}}$.

Proof of Theorem 6.3. Assume that g a is loxodromic element of the group G and let z_0 be a fixed point of g. Then $G_g = \{g^n \,|\, n \in \mathbb{Z}\}$ is a cyclic subgroup of the group G, and z_0 is an attractive point of g or g^{-1}. Let z_0 be an attractive point for g, without loss of generality.

For every fixed $z_1 \neq z_0$ we have

$$(1) \qquad \lim_{n \to \infty} g^n(z_1) = \lim_{n \to \infty} w_n = z_0,$$

where $w_n = g^n(z_1)$, $n \in \mathbb{N}$ (see Theorem 3.2).

Assume that the family G is a normal family of functions in the point z_0.

From the normality of family G in z_0, the normality of family $G_g = \{g^n \,|\, n \in \mathbb{Z}\}$ in z_0 follows. Since g^n, $n \in \mathbb{Z}$, is continuous on $\overline{\mathbb{C}}$, the family $G_g = \{g^n \,|\, n \in \mathbb{Z}\}$ is a normal family in z_0 if and only if there exists an neighbourhood O of z_0 on which the family $G_g = \{g^n \,|\, n \in \mathbb{Z}\}$ is equicontinuous (Theorem 4.1). It follows that for every $\varepsilon > 0$ there exists $\delta = \delta(z_0, \varepsilon) > 0$, such that for every g^n, $n \in \mathbb{Z}$, and every z for which $d(z - z_0) < \delta$ the inequality

$$\varepsilon = d(z_1 - z_0) > 0. \tag{3}$$

Then for ε given in (3) there exists $\delta > 0$ such that for every z which satisfies $d(z - z_0) < \delta$ it holds

$$d(g^n(z), g^n(z_0)) < \varepsilon, \qquad n \in \mathbb{Z}. \tag{4}$$

Since $z_1 \neq z_0$, (1) yields $\lim_{n \to \infty} g^n(z_1) = z_0$. Then there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have $d(g^n(z_1) - z_0) = d(g^n(z_1) - g^n(z_0)) < \delta$. Now having in mind (4) we obtain

$$d(g^{-n}(g^n(z_1)), g^{-n}(g^n(z_0))) < \varepsilon, \quad n \geq N. \tag{5}$$

From (5) we obtain $d(z_1 - z_0) < \varepsilon$, and from (3) we conclude that $\varepsilon < \varepsilon$. This is a contradiction so the subgroup G in z_0 is not a normal family of functions in z_0, so we finish the proof of part *i*).

The part *ii*) follows directly from the part *i*).

If the above proof for g we take an parabolic element of subgroup G we derive a proof for elements of parabolic type.

Theorem *4.2* yields that G is not normal on $\overline{\mathbb{C}}$. □

From Theorems 6.2 and 6.3 we obtain the following:

Theorem 6.4. ([8], Lemma 5.3.3, p. 96)**.** *Let G be a subgroup of group* $G(\overline{\mathbb{C}})$ *and let O be an open set on Riemann sphere* $\overline{\mathbb{C}}$ *which contains a fixed point of a parabolic or loxodromic element g in G. Then G does not acts discontinuously on O.*

7. Apendix: APPLICATIONS ON HARMONIC FUNCTIONS

Further, we will need the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, i.e., the extended set of real numbers which is compactified by the two points $-\infty$ and $+\infty$. For x_1, $x_2 \in \mathbb{R}$ we will denote $d_{\mathbb{R}}(x_1, x_2) = |x_1 - x_2|$ the distance on $\mathbb{R}$, but for $x_1, x_2 \in \overline{\mathbb{R}}$ we have $d_{\overline{\mathbb{R}}}(x_1, x_2) = |arctg\ x_1 - arctg\ x_2|$, $arctg\ (+\infty) = \dfrac{\pi}{2}$, $arctg\ (-\infty) = -\dfrac{\pi}{2}$, a metric on $\overline{\mathbb{R}}$. On compact subsets of $\mathbb{R}$ the metrics $d_{\mathbb{R}}(x_1, x_2)$ and $d_{\overline{\mathbb{R}}}(x_1, x_2)$ are equivalent. The extended set of real numbers $\overline{\mathbb{R}}$ with the distance $d_{\overline{\mathbb{R}}}(x_1, x_2)$ is a Hausdorff compact and complete metric space.

In the sequel we will also need the extended Arzela-Ascoli Theorem. Assume that X and Y are two compact metric spaces, let $C_{X,Y}$ be the set of all continuous functions f which map X in Y and let for $f, g \in C_{X,Y}$ $d_{X,Y}(f, g) = \sup\limits_{x \in X} d_Y(f(x), g(x))$. With d_{XY} we have the distance functions on set $C_{X,Y}$. From the convergence of the sequence (f_n) in $C_{X,Y}$ in the metric $d_{X,Y}$ follows the uniform convergence of that sequence on compact sets in X.

In the sequel we will also need the definition of equicontinuous family of functions $\Im$, $\Im \subset C_{XY}$. Namely, a family $\Im$ is equicontinuous *on* X, if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $f \in \Im$ and all $x, y \in X$, for which $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \varepsilon$.

Theorem 7.1 ([14], *a general Arzela-Ascoli Theorem*, p. 114). *A set* $\Im$, $\Im \subset C_{X,Y}$, *is precompact, (compact, since we have compact metric spaces X and Y), i.e.,* $\Im$ *is a normal family of functions in* $C_{X,Y}$, *if and only if* $\Im$ *is equicontinuous set of functions on X.*

Using Theorem 6.1, and statement *iii) of* Lemma 3.3, one can show in a similar way as Theorem 4.2, the following Theorem:

Theorem 7.2. *Let g be a conformal automorphism of a simply connected domain* $O \subset \mathbb{C}$ *which has a fixed attractive point* $z_0 \in O$, $z_0 \neq \infty$, *and let* $f : \mathbb{C} \to \overline{\mathbb{R}}$ *be a continuous function on O. If the function f is normal on the domain O with respect to the cyclic group* $G_g = \{g^n | n \in \mathbb{Z}\}$, *generated by the conformal automorphism g, then f is a constant function on O.*

We will consider a harmonic function in a domain of the complex plane. A function $f : O \to \mathbb{R} \subset \overline{\mathbb{R}}$ is harmonic on a domain O, $O \subset \mathbb{C}$, if $f \in C^2(O)$ and f satisfies the Laplace

equation $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$ on O. If f is harmonic on a domain O and if $\varphi:O'\to O$ is a conformal

mapping of O', $O'\subset\mathbb{C}$, onto O, then a $f\circ\varphi$ is harmonic on O' (see [6]. or the [11]) .

Let $\mathcal{H}=\left\{f\,\middle|\,f:O\to\mathbb{R}\subset\overline{\mathbb{R}}\right\}$ denote a family of harmonic functions on a domain O .

Theorema 7.3 ([10], Theorem 5.4.2, p. 185). *A locally bounded family H of harmonic functions on a domain O is a normal family on that domain.*

Theorem 7.4 (Liouville's Theorem for bounded harmonic functions). *A harmonic and bounded function on the complex plane $\mathbb{C}$ is constant on $\mathbb{C}$.*

Proof of Theorem 7.4. Let $g\in G(\mathbb{C})$, $g(z)=az+c, 0<a<1, c\in\mathbb{C}$. From the boundedness of the harmonic function f on $\mathbb{C}$ and Theorem 6.3 it follows that the family $\mathfrak{I}_f=\left\{f\circ g^n\,\middle|\,n\in\mathbb{Z}\right\}$, is normal on $\mathbb{C}$ with respect to the group G_g . Since a harmonic function f on $\mathbb{C}$ is continuous on $\mathbb{C}$, the statement of Theorem 7.4 now follows directly from part *iii)* of Lemma 3.3 and Theorem 7.2. $\square$

Theorem 7.5 ([10], Theorem 5.4.3, p. 185). *The family H^+ of positive harmonic functions on a domain O is normal.*

Theorem 7.6 (Liouville's Theorem for positive harmonic functions). *A positive harmonic function on the complex plane $\mathbb{C}$ is constant on $\mathbb{C}$.*

Proof of Theorem 7.6. From the conditions of Theorem 6.6 and Theorem 6.5 it follows that $\mathfrak{I}_f=\left\{f\circ g^n\,\middle|\,n\in\mathbb{Z}\right\}$, $g(z)=az+c, 0<a<1, c\in\mathbb{C}$, is a normal family of harmonic functions on $\mathbb{C}$. Since the harmonic function f on $\mathbb{C}$ is also continuous on $\mathbb{C}$, from the part *iii)* of Lemma 3.3 and Theorem 7.2 it follows that f is a constant function on $\mathbb{C}$. $\square$

Corollary 7.7. *If a harmonic function f in the complex plane $\mathbb{C}$ is bounded above or below then f is a constant function on $\mathbb{C}$.*

Proof of Corollary 7.7. From the condition that a function f is bounded above it follows that there exists a constant M>0 such that for every $z\in\mathbb{C}$ we have $f(z)<M$. Now the function

$\varphi(z) = M - f(z)$, $z \in \mathbb{C}$, is positive and harmonic on $\mathbb{C}$. From Theorem 7.6 it follows that φ is a constant on $\mathbb{C}$, then it follows that f is also constant on $\mathbb{C}$.

Similarly one can show that if f is bounded below on $\mathbb{C}$ then it is a constant. $\square$

Theorem 7.8 ([10], Corollary 5.4.5, p. 186). *A family* H *of harmonic functions in a domain O which omit one specific real value* α *is normal.*

Theorem 7.9. (Picard's Theorem for harmonic functions). *A nonconstant harmonic function on the complex plane* $\mathbb{C}$ *takes every value in the set of real numbers* $\mathbb{R}$.

Proof of Theorem 7.9. Assume contrary, i.e., that there exists a nonconstant harmonic function f on the complex plane $\mathbb{C}$ which does not take all values in the set of real numbers $\mathbb{R}$. Then there exists $a \in \mathbb{R}$ such that for every $z \in \mathbb{C}$ the inequality $f(z) \neq a$ holds. Now, Theorem 7.8 yields that $\Im = \left\{ (f \circ g)(z) \mid g \in G(\mathbb{C}) \right\}$ is a normal family of harmonic functions on $\mathbb{C}$. From part *iii)* of Lemma 3.3 and Theorem 7.2 follows that the function f is constant on $\mathbb{C}$. We have reached a contradiction, therefore our Theorem is proved. $\square$

8. CONCLUSION

In this article we show that using theory of normal family of functions and properties of fixed points of Moebius mappings one can prove classical Theorems of Liouville and little Picard Theorem for holomorphic, meromorphic and harmonic functions in a simple way. Our proofs show why in some domains of Riemann sphere one can study properties of classes of functions (for example: class of bounded holomorphic functions). Applying our result some properties of elements and subgroups of Möbius group $G(\overline{\mathbb{C}})$ can be easily verified.

It would be of interest to further investigate if the approach given in this paper concerning the Montel normality of family of functions and properties of Möbius mappings is helpful in proving some results for holomorphic, meromorphic or harmonic functions and compare the results that are known for the Bloch principle (see [10, 13,15]).

Also it would be of interest to try to apply some of our approach in the study of functions on $\mathbb{R}^2$ and functions with a domain in $\mathbb{R}^n$, $n > 2$.

REFERENCES

[1] J. Clunie and W.K. Hayman, *Symposium on Complex Analysi*s, Canterbury, (1973), London Mathematical Society Lecture Note Series 12, Cambridge University Press, Cambridge (1974).

[2] C. D. Minda, "Marden constants for Bloch and normal functions", *J. Anal. Math*. **42**, 117-127 (1982/83).

[3] P. Järvi, "On the existence of normal analytic functions", *Ann. Acad. Sci. Fenn. Ser.A I Math*., **11**, 203-206 (1986).

[4] J. Väisälä, "On normal quasiconformal functions", *Ann. Acad. Sci. Fenn. Ser. A. I*., **266**, 33 (1959).

[5] V.L. Ahlfors, *Complex Analysis: an Introduction to the Theory of Analytic Functions of One Complex Variable*, McGraw-Hill, Inc. (1979).

[6] B.V. Shabat, *Vvedenie v kompleksnyi analiz. Chast 1*, M.: Nauka, (1985)

[7] J. Milnor, *Dynamics in One Complex Variable*, Princeton University Press, Princeton and Oxford (2006).

[8] A.F. Beardon, *The Geometry of Discrete Groups*, Springer (1995).

[9] P. Montel, *Lecons sur les Familles Normales de Functons Analytiques et leurs applica tions*, Paris, Gautnier-Villars (1927).

[10] L.J. Schiff, *Normal Families*, Springer Science + Business Media, LLC (1993).

[11] S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory*, Springer-Verlag, New York (2001).

[12] E. Nelson, "A Proof of Liouville's Theorem", *Proc. Amer. Math. Soc*, **12**(6), 995 (1961).

[13] Ž. Pavićević, "Normal Families, Theorems of Liouville and Picard and Bloch Principle", *Math. Montis.*, **43**, 5-9 (2018).

[14] A.N. Kolmogorov, S.V. Fomin, *Elementy teorii funktsiy i funktsional'nogo analiza*, Fizmatlit (2019).

[15] Walter Bergweiler, "Bloch's Principle", *Computational Methods and Function Theory*, **6** (1), 77–108 (2006).