MATHEMATICA MONTISNIGRI
Vol XLVI (2019)

A BREAF SURVEY ON ARMENDARIZ AND CENTRAL ARMENDARIZ RINGS

DUŠAN JOKANOVIĆ*

University of East Sarajevo, Production and Management Faculty Trebinje
*Corresponding author. E-mail: dusan.jokanovic@fpm.ues.rs.ba

DOI: 10.20948/mathmontis-2019-46-1

Summary. In this paper R is a ring with unity, and σ is endomorphism of the ring. We deal with central Armendariz rings as a natural generalization of Armendariz rings. We investigate a posibility of extending central Armendariz property from a ring to coresponding polynomial or matrix extension. At the end of this paper we consider an interesting note on reduced rings.

1 INTRODUCTION

Throughout this article R denotes a ring with unity, $R[x]$ is coresponding polynomial ring, σ denotes an endomorphism of R, $R[x;\sigma]$ denotes skew polynomial ring with the ordinary addition and the multiplication subject to the relation $xr=\sigma(r)x$, and $R[[x;\sigma]]$ denotes power series ring. The notion of Armendariz ring is introduced by Rege and Chhawchharia (see [2]). They defined a ring R to be Armendariz if $f(x)g(x)=0$ implies $a_ib_j=0$, for all polynomials $f(x)=\sum_{i=0}^{n}a_ix^i$ and $g(x)=\sum_{j=0}^{m}b_jx^j$ from $R[x]$. The motivation for those rings comes from the fact that Armendariz had shown that the above reesult can be extended to a class of reduced rings, i.e., rings without non-zero nilpotent elements. In [1] authors introduced a class of central Armendariz rings. A ring R is called central Armendariz ring if $f(x)g(x)=0$ implies $a_ib_j\in C(R)$, for all polynomials $f(x)=\sum_{i=0}^{n}a_ix^i$ and $g(x)=\sum_{j=0}^{m}b_jx^j$ from $R[x]$, where $C(R)$ is center of a ring R. Clearly Armendariz rings are central Armendariz rings. It is known from [1] that a class of central Armendariz rings is closed for polynomial extensions and localizations, and that the central hhArmendariz rings are strictly between Armendariz rings and abelian rings. As a generalization of σ-skew Armendariz rings, Onyang (see [4]) introduced a notion of weak σ-skew Armendariz ring (see [3],[4].[5]). A weak σ-skew Armendariz ring R is a ring in which $f(x)g(x)=0$ implies $a_i\sigma^i(b_j)$ is the nilpotent element of R for all $f(x)=\sum_{i=0}^{n}a_ix^i$ and $g(x)=\sum_{j=0}^{m}b_jx^j$ from $R[x;\sigma]$. Chain and Tong (see [5]) have proved that if R and S are rings and σ is and isomorphism of rings R and S and R is α-skew Armendariz ring, then S is $\sigma\alpha\sigma^{-1}$-skew Armendariz ring. In this paper we give (see [3]) a variant of this theorem for weak skew-Armendariz rings. In our main result we give an example of central Armendariz matrix ring $T(n,R)$, for reduced ring R.

2 EXTENDING OF ARMENDARIZ PROPERTY

In this section we deal with posibility of extending the Armendariz property under ring isomorphism (see [3]). From universal algebra we know that every homomorphism σ of rings R and S can be extended to the homomorphism of the corresponding rings of polynomials $R[x]$ and $S[x]$ by $\sum_{i=0}^{m}a_ix^i\mapsto\sum_{i=0}^{m}\sigma(a_i)x^i$, which we also denote by σ. Chain and Tong in

2010 Mathematics Subject Classification: 16U80.
Key words and Phrases: Armendariz rings, reduced rings.

[5] prove that if σ is ring isomorphism of rings R and S and R is α-skew Armendariz, then S is $\sigma\alpha\sigma^{-1}$ skew Armendariz ring. We prove the weak skew Armendariz variant of this theorem.

Theorem 2.1 ([3]) *Let R and S be rings with a ring isomorphism $\sigma: R \to S$. If R is weak α -skew Armendariz then S is weak $\sigma\alpha\sigma^{-1}$ -skew Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be polynomials from the ring $S[x; \sigma\alpha\sigma^{-1}]$. We have to prove that $f(x)g(x) = 0$ implies $a_i(\sigma\alpha\sigma^{-1})^i b_j \in nil(S)$, for all i and j.

As we noted, σ extends to the isomorphism of corresponding polynomial rings, so that there exist polynomials $f_1(x) = \sum_{i=0}^{m} a_i' x^i$ and $g_1(x) = \sum_{j=0}^{m} b_j' x^j$ from $R[x]$ such that $f(x) = \sigma(\ f_1(x)) = \sum_{i=0}^{m} \sigma(a_i')x^{i,}$ and $g(x) = \sigma(g_1(x)) = \sum_{j=0}^{m} \sigma(b_j')x^j$.

First, we shall show that $f(x)g(x) = 0$ implies $f_1(x)g_1(x) = 0$. If $f(x)g(x) = 0$, we have

$$a_0 b_k + a_1(\sigma\alpha\sigma^{-1})(b_{k-1}) + \ldots + a_k(\sigma\alpha\sigma^{-1})^k(b_0) = 0,$$

for any $0 \le k \le m$. From the definition of $f_1(x)$ and $g_1(x)$, we have

$$\sigma(a_0')\sigma(b_k') + \sigma(a_1')(\sigma\alpha\sigma^{-1})\sigma(b_{k-1}') + \ldots + \sigma(a_k')(\sigma\alpha\sigma^{-1})^k\sigma(b_0') = 0,$$

and using $(\sigma\alpha\sigma^{-1})^t = \sigma\alpha^t\sigma^{-1}$ we obtain

$$a_0' b_k' + a_1'\alpha(b_{k-1}') + \ldots + a_k'\alpha^k(b_0') = 0,$$

which means that $f_1(x)g_1(x) = 0$ in the ring $R[x; \alpha]$.

It remains to prove that $f_1(x)g_1(x) = 0$ implies $a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in nil(S)$. From the fact that R is weak α-skew Armendariz we have $a_i'\alpha^i(b_j') \in nil(R)$, and since

$a_i' = \sigma^{-1}(a_i), b_j' = \sigma^{-1}(b_j)$, we have $\sigma^{-1}(a_i)\alpha^i\sigma^{-1}(b_j) \in nil(R)$.

This implies

$$\sigma^{-1}(a_i)\sigma^{-1}\sigma\alpha^i\sigma^{-1}(b_j) = \sigma^{-1}(a_i(\sigma\alpha\sigma^{-1})^i(b_j)) \in nil(R)$$

and finally we obtain

$$a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in nil(S),\ 0 \le i \le m, 0 \le j \le n.$$

Hence S is weak $\sigma\alpha\sigma^{-1}$-skew Armendariz.

3 MATRIX CENTRAL ARMENDARIZ RING $T(R, n)$

In this section we give an example of matrix central Armendariz ring. For a ring R consider a following set of triangular matrices

$$T_n(R) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \mid a_{ij} \in R \right\}.$$

We also consider the following set of triangular matrices over ring R

$$T(R,n) = \left\{ \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix} \mid a_i \in R \right\},$$

which is subring of $T_n(R)$. It is well known that $T_n(R)$ and $T(R,n)$ are subrings of the triangular matrix rings with matrix addition and multiplication. Let α be endomorphism of ring R. It is well known that endomorphism α can be naturally extended to an endomorphism

$$\overline{\alpha}: T_n(R) \to T_n(R),$$

and

$$\overline{\alpha}: T(R,n) \to T(R,n),$$

with:

$$\overline{\alpha}\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \right) = \begin{bmatrix} \alpha(a_{11}) & \alpha(a_{12}) & \alpha(a_{13}) & \cdots & \alpha(a_{1n}) \\ 0 & \alpha(a_{22}) & \alpha(a_{23}) & \cdots & \alpha(a_{2n}) \\ 0 & 0 & \alpha(a_{33}) & \cdots & \alpha(a_{3n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha(a_{nn}) \end{bmatrix},$$

and

$$\overline{\alpha}\left(\begin{bmatrix} a_0 & a_1 & a_{13} & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix} \right) = \begin{bmatrix} \alpha(a_0) & \alpha(a_1) & \alpha(a_2) & \cdots & \alpha(a_{n-1}) \\ 0 & \alpha(a_0) & \alpha(a_1) & \cdots & \alpha(a_{n-2}) \\ 0 & 0 & \alpha(a_{33}) & \cdots & \alpha(a_{3n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha(a_0) \end{bmatrix}.$$

Theorem 3.1 *If R is reduced ring then $T(R,n)$ is central Armendariz ring.*

Proof. From [1] we obtain that for reduced ring R, the factor ring $R[x]/(x^n)$ is central Armendariz, for all $n \geq 2$. We use the ring isomorphism $f: R[x]/(x^n) \to T(R,n)$ given by

$$f(a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + (x^n)) = (a_0, a_1, \ldots, a_{n-1}),$$

where (x^n) is ideal in $R[x]$ generated with x^n, and $(a_0, a_1, \ldots, a_{n-1})$ is a breaf representation for a matrix from $T(R,n)$. Therefore $T(R,n)$ is central Armendariz ring.

We end this section with our result from [3], in which we give sufficient condition for the power series ring $R[[x;\sigma]]$ to be reduced.

Theorem 3.2 *If an endomorphism σ of a reduced ring R satisfies so called compatibi-lity condition: $a\sigma(b) = 0 \Leftrightarrow ab = 0$, then the power series ring $R[[x;\sigma]]$ is reduced.*

Proof. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $(f(x))^2 = 0$. It is clear that $a_0^2 = 0$, so that $a_0 = 0$. Now from the compatibility condition $a_1\sigma(a_1) = 0$ implies $a_1^2 = 0$, but since R is reduced we have $a_1 = 0$. By induction argument we have $a_i = 0$ for all i. This means that $f(x) = 0$ and so $R[[x;\sigma]]$ is reduced.

Without compatibility condition the previous theorem is not true. Since for the ring $R = Z_2 \oplus Z_2$ and σ defined by $\sigma(a,b) = (b,a)$, it is easy to check that $R[[x;\sigma]]$ is not reduced. Observe that $(1,0)(0,1) = (0,0)$ but $(1,0)\sigma(0,1) \neq (0,0)$.

REFERENCES

[1] N. Agayev, G. Gungoroglu, A. Harmanci and S. Halicioglu, "Central Armendariz Rings", *Bull. Malays. Math. Sci. Soc.*, **2**(34(1)), 137-145 (2011).

[2] M. Rege, and S. Chhawchharia, "Armendariz rings", *Proc. Japan Acad. Ser. A. Math. Sci.*, **73**, 14-17 (1997).

[3] D. Jokanović, "Properties of Armendariz rings and weak Armendariz rings", *Pubications de l'Institute Mathematique, Nouvelle serie*, **85**(99), 131-137 (2009).

[4] L. Ouyang, "Extensions of generalized α-rigid rings", *International Journal of Algebra*, **3**, 105-116 (2008).

[5] W. Chen and W. Tong, "On skew Armendariz and rigid rings", *Houston Journal of Mathematics*, **22**, (2007).

Received November 15, 2019