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ESTIMATION OF NOISE IN CALCULATION OF SCATTERING MEDIUM LUMINANCE BY MCRT

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Summary. We investigate new method for calculation of radiance of scattering medium by bi-directional Monte-Carlo ray tracing with photon maps. Usually photons are collected by an integration spheres at the ends of camera ray segments, or a cylinder along that segments. Meanwhile in our method several integration spheres are distributed at random along the first camera ray segment. The rest segments do not collecting photons. The method optimal for a particular scene is the one which produces the least noise, so one need to be able to estimate it. In this paper an analytic calculation of noise in the general bi-directional Monte-Carlo ray tracing is derived and then applied to the proposed method. Then the analytic estimates of noise can be used to find optimal parameters and/or to choose between single integration sphere, multiple integration spheres and integration cylinders.

1 INTRODUCTION

A bi-directional Monte Carlo ray tracing is well-known as a powerful method of calculation of a virtual camera image. The forward part of the method traces the light rays from light sources and creates photon maps which allow calculating luminance and illuminance of scene surfaces. Then the backward part traces the light rays from camera, estimates the luminance in the hit point and then accumulates it along the camera ray path into the pixel.

The method can be implemented via several techniques [1, 2, 3, 4] of which the photon map visualization approach [5, 6, 7, 8] is the mostly usable now. Most of these methods calculate global illumination in the form of photon maps and then visualize them as the luminance of secondary and caustic illumination [5, 7, 9]. There is an approach that implements the reverse calculation scheme, i.e. generates a visibility map as spheres of the illuminance integration in the direction of observation, which are "filled" with the light photons related to the caustic and the secondary illumination [6, 8]. In these methods the camera ray is traced stochastically until it terminates due to some criterion, for example, after the given number of diffuse events is reached. An integration sphere to

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collect Forward Monte Carlo ray tracing (FMCRT) rays is set in each point of scattering. After a forward ray hits that sphere we calculate the surface luminance for the view direction equal to the camera ray direction before the sphere center. This luminance is then scaled by the camera ray attenuation accumulated to this point and added to the pixel luminance, [1, 6, 10].

In scattering medium the integration spheres are distributed over the volume, not only on boundaries, see e.g. [11], [12]. Frequently it is advantageous to use integration volumes other that spheres, e.g. cylinders [13]. One can find comparison and analysis in [5, 12, 14, 13]; [15]. Alternatively, the integration volume can be a union of spheres [16].

Usually the bi-directional ray tracing with photon maps is applied progressively, i.e. ray tracing goes iteration by iteration, and the error vanishes as time goes on. In each iteration we trace camera rays and place integration spheres in the ends of the segments. Then during FMCRT phase we estimate illuminance in the centers of integration spheres from the photons that hit them.

It is important to have a reliable analytic estimate of the noise in the stochastic rendering. Firstly this allows optimizing parameters of the method so that the noise remaining after the fixed simulation time was minimal (or do that with the constraint on the RAM used). Secondly there are several other inaccuracies besides the noise in the method, e.g. bias. This latter is unavoidable because of the final size of the integration area used for the illuminance estimation. And it grows rather quickly with area size. Therefore it is advantageous to decrease this size. But in such a case number of used photons is also reduced and thus the noise is increased. Therefore there exists the optimal size which minimizes the total error. And to calculate it one needs to predict the noise.

In [17] the authors operated the simplest case when the integration spheres are in the first camera rays hit so that position of the integration sphere for the given pixel is not random. In [18] the camera ray is allowed to be "glossy" (nearly specular diffuse) scattered before, so position of the integration sphere and view direction for it is random. This introduces additional source of noise which is analyzed. The authors consider progressive rendering when the radius of integration spheres decrease from iteration to iteration, so after large simulation time the results are for a very small radius. They derive the estimates of the bias and variance (noise) for some class of the radius decrease rule and show that there is a range of parameters for which both errors vanish in course of time.

They however assume that there is only one camera ray per pixel in each iteration. This is good when the integration sphere is set prior to the first random scattering of the camera ray as in [17] or when this scattering is "weakly diffuse" ("glossy") as in [18]. But this strategy is not optimal when the camera ray undergoes a wide diffuse scattering before setting the integration sphere. Indeed, now the centers of integration spheres for adjacent pixels are not close but can spread over the whole scene and even beyond the visible area. The density of integration spheres can then be low and they capture only a small fraction of the forward photons. In other words a substantial part of calculations in the FMCRT phase are lost because these photons miss the sparsely distributed integration spheres. The natural remedy is to increase the number of camera rays through pixel so that the integration spheres to cover a decent fraction of illuminated area. This introduces an additional control parameter: the number of backward rays per pixel. In principle it can vary across the image and change with time. Setting it too low is not advantageous

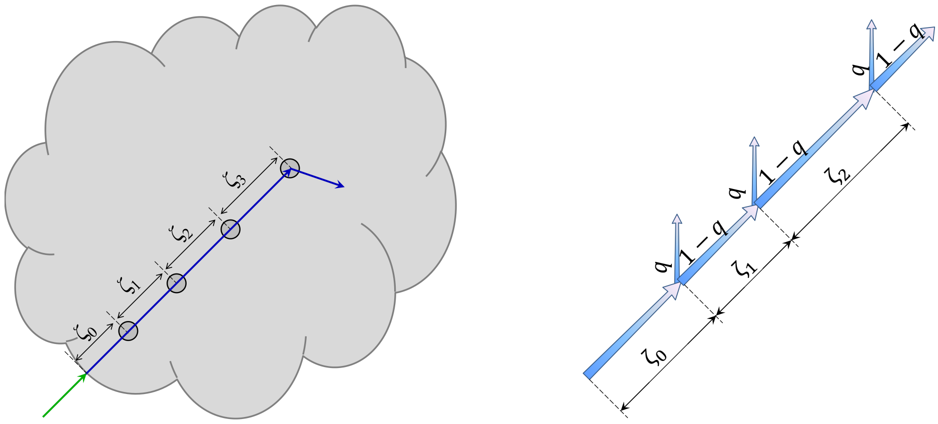

Figure 1: Several integration spheres along the camera ray segment. Left: positioning of the spheres. Right: Camera ray propagation with sub-steps: it strides by ζ_m then with probability q it undergoes extinction otherwise goes the same direction

because a many FMCRT rays whose tracing took time are not used. But using too much camera rays per pixel is also bad because this increases simulation time while the effective number of FMCRT rays used in the given pixel saturates. Therefore there exists an optimal relation between the number of forward and backward rays.

In this paper we derive an estimation of the noise as a function of the number of camera rays per pixel which applies to a general case of bi-directional MCRT with progressive photon maps. Then we apply it to calculation of luminance of turbid medium with multiple integration spheres [16] and derive an analytic dependence of noise as a function of the control parameters of the method: the number of forward and backward rays, the average number of integration spheres per camera ray segment and so on. This analytic estimate can be used to find optimal parameters.

2 MULTIPLE INTEGRATION SPHERES

In [16] we suggested to use integration volume composed of *several* integration spheres distributed randomly over a camera ray segment. The basic idea is that when tracing an camera ray we perform many "sub-steps" of length ζ so that the ray goes straight during several of them and only then an extinction which can be scattering or absorption occurs, see Figure 1.

These sub-steps are independent from each other, the density of ζ being always the same $p_\zeta(\zeta)$.

After the ray propagated the next step length ζ its further destiny is decided at random. With probability q there is an extinction event (scattering or absorption is then decided at random); otherwise the next step is made *retaining the ray direction*. Absorption is processed as a "Russian roulette" killing the ray at random.

In [16] it was obtained that step length must be distributed as

$$p_\zeta(\zeta) = \alpha e^{-\alpha\zeta} \tag{1}$$

$$\alpha \equiv \sigma_{ext}/q \tag{2}$$

The total ray length after n such steps then has the density

$$p_n(s) \quad = \quad \alpha e^{-\alpha s}\frac{(\alpha s)^n}{n!} \tag{3}$$

The contribution of an FMCRT ray segment that crosses the BMCRT ray at distance s to the pixel luminance $C(s)$ is

$$C(s) = q\mathscr{C}(s) \times \sigma_{ext} e^{-\sigma_{ext} s} ds \tag{4}$$

see [16], notice this applies to both *single segment* and *whole forward ray trajectory.* Here

$$\mathscr{C}(s) = \frac{\sigma_{sc} f(-\boldsymbol{u}, \boldsymbol{v})}{\sigma_{ext}\pi R^2}\mathcal{F} \tag{5}$$

and $\mathcal{F}$ is the total flux (sum over all light sources), f is the phase function and σ_{sc} is scattering of the medium and σ_{ext} is its extinction.

Here and below we use the term “luminance” as an equivalent of “radiance”

3 GENERAL ESTIMATE OF NOISE IN BI-DIRECTIONAL MCRT

In bi-directional MCRT each forward ray interacts with each camera ray, and the luminance of a pixel is

$$L = \frac{1}{N_F}\sum_{i=1}^{N_F}\frac{1}{N_B}\sum_{j=1}^{N_B}\hat{C}(i,j) \tag{6}$$

where i enumerates forward rays, j enumerates backward rays, $\hat{C}(i,j)$ is the increase of pixel luminance from interaction of the i-th forward with the j-th camera ray, N_F is the number of forward rays and N_B is the number of camera rays ***traced through this pixel***.

The above sum L is a random variable, and its *noise* is $\langle L^2\rangle - \langle L\rangle^2$ where the average is taken over the ensembles of rays. It can be thus understood as a *repeated* average over the forward and backward ensembles. Let us square (6)

$$\begin{aligned}(N_F N_B)^2 L^2 \quad &= \quad \sum_{i=1}^{N_F}\sum_{j=1}^{N_B}\hat{C}^2(i,j) + \sum_{i=1}^{N_F}\sum_{j\neq j'}^{N_B}\hat{C}(i,j)\hat{C}(i,j') \\ &+ \sum_{i\neq i'}^{N_F}\sum_{j=1}^{N_B}\hat{C}(i,j)\hat{C}(i',j) + \sum_{i\neq i'}^{N_F}\sum_{j\neq j'}^{N_B}\hat{C}(i,j)\hat{C}(i',j')\end{aligned}$$

and average over the forward ray ensemble and over the backward ray ensemble. These averaging are denoted as $\langle\cdot\rangle_F$ and $\langle\cdot\rangle_B$ and their order is arbitrary. This gives

$$\begin{aligned}\langle\langle L^2\rangle\rangle \quad &= \quad \frac{1}{N_F N_B}\langle\langle\hat{C}^2\rangle_F\rangle_B + \frac{N_F(N_B^2 - N_B)}{(N_F N_B)^2}\langle\langle\hat{C}\rangle_B^2\rangle_F \\ &+ \frac{N_B(N_F^2 - N_F)}{(N_F N_B)^2}\langle\langle\hat{C}\rangle_F^2\rangle_B + \frac{(N_F^2 - N_F)(N_B^2 - N_B)}{(N_F N_B)^2}\Big(\langle\langle\hat{C}\rangle_F\rangle_B\Big)^2\end{aligned}$$

where we used the obvious fact that

$$
\begin{array}{rcll}
\langle \hat{C}(i,j)\hat{C}(i',j')\rangle_F & = & \langle \hat{C}\rangle_F(j)\langle \hat{C}\rangle_F(j') & \text{if } i' \neq i \\
\langle \hat{C}(i,j)\hat{C}(i',j')\rangle_B & = & \langle \hat{C}\rangle_B(i)\langle \hat{C}\rangle_B(i') & \text{if } j' \neq j
\end{array}
$$

The noise is therefore

$$
\begin{aligned}
\langle\langle L^2\rangle\rangle - \langle\langle L\rangle\rangle^2 &= \frac{1}{N_F N_B}\left(\langle\langle \hat{C}^2\rangle_B\rangle_F - \langle\langle \hat{C}\rangle_B\rangle_F^2\right) && (7)\\
&+\frac{1}{N_B}\left(1-\frac{1}{N_F}\right)\left(\langle\langle \hat{C}\rangle_F^2\rangle_B - \langle\langle \hat{C}\rangle_B\rangle_F^2\right) && (8)\\
&+\frac{1}{N_F}\left(1-\frac{1}{N_B}\right)\left(\langle\langle \hat{C}\rangle_B^2\rangle_F - \langle\langle \hat{C}\rangle_B\rangle_F^2\right) && (9)\\
\langle\langle L\rangle\rangle &= \langle\langle \hat{C}\rangle_B\rangle_F^2 && (10)
\end{aligned}
$$

4 ESTIMATION OF NOISE FOR MULTIPLE SPHERES

The noise is as sum of three terms, see (7), (8), (9), (10) where now $\hat{C}$ is the contribution of one forward ray to *all* n integration spheres set by the given camera ray:

$$
\hat{C} = \sum_{m=0}^{n} C(\hat{\zeta}_m), \tag{11}
$$

$C(s)$ is contribution from *one* FMCRT ray to the single integration sphere at distance s from the camera ray start,

$$
\hat{\zeta}_m \equiv \sum_{i=0}^{m} \zeta_i \tag{12}
$$

is position of the m-th sphere set after m sub-steps. Here and below ζ_0, ζ_1, ζ_2, ... are successive ray sub-steps between the 1st, 2nd, ..., integration spheres.

Notice the above estimate assume that each of N_F forward rays is checked for interaction with each of N_B camera rays. If the whole process runs progressively, iteration by iteration and for the next iteration *both* forward and backward rays are all new, then this estimates applies to one iteration only. The relative variance obtained in M (identical) iterations will be M times lower (instead of replacing $N_F \mapsto MN_F$, $N_B \mapsto MN_B$).

4.1 Calculation of the averages $\langle \hat{C}\rangle_B$ and $\langle \hat{C}^2\rangle_B$

The contribution from one forward ray (11), averaged over the BMCRT ensemble i.e. sphere positions i.e. the $\langle C\rangle_B$ is obviously

$$\begin{aligned}\langle\hat{C}\rangle_B &= \sum_{n=0}^{\infty}P(n)\int_0^{\infty}\cdots\int_0^{\infty}\hat{C}p_\zeta(\zeta_0)\cdots p_\zeta(\zeta_n)d\zeta_0\cdots d\zeta_n\\ &= q\sum_{n=0}^{\infty}(1-q)^n\int_0^{\infty}\cdots\int_0^{\infty}\left(\sum_{m=0}^{n}C(\hat{\zeta}_m)\right)p_\zeta(\zeta_0)\cdots p_\zeta(\zeta_m)d\zeta_0\cdots d\zeta_m\end{aligned}$$

where $\hat{\zeta}_m$ is given by (12).

The outer sum is over the number of integration spheres n and changing the order of summation we arrive at

$$\langle\hat{C}\rangle_B = \sum_{m=0}^{\infty}(1-q)^m\int_0^{\infty}\cdots\int_0^{\infty}C(\hat{\zeta}_m)p_\zeta(\zeta_0)\cdots p_\zeta(\zeta_m)d\zeta_0\cdots d\zeta_m$$

The sum of steps $\hat{\zeta}_m$ has density $\frac{(\alpha\hat{\zeta}_m)^m}{m!}\alpha e^{-\alpha\hat{\zeta}_m}$ (3), and using (2) and (4) we obtain

$$\langle\hat{C}\rangle_B = \int_0^{\infty}\mathscr{C}(t)\sigma_{ext}e^{-\sigma_{ext}t}dt \tag{13}$$

Similarly, the *squared* contribution of one FMCRT ray, averaged over the BMCRT ensemble i.e. the $\langle\hat{C}^2\rangle_B$ is

$$\begin{aligned}\langle\hat{C}^2\rangle_B &= \sum_{n=0}^{\infty}P(n)\int_0^{\infty}\cdots\int_0^{\infty}\left(\sum_{m=0}^{n}C(\hat{\zeta}_m)\right)^2 p_\zeta(\zeta_0)\cdots p_\zeta(\zeta_n)d\zeta_0\cdots d\zeta_n\\ &= q\sum_{n=0}^{\infty}(1-q)^n\sum_{m=0}^{n}\int_0^{\infty}\cdots\int_0^{\infty}C^2(\hat{\zeta}_m)p_\zeta(\zeta_0)\cdots p_\zeta(\zeta_m)d\zeta_0\cdots d\zeta_m\\ &\quad+2q\sum_{n=0}^{\infty}(1-q)^n\int_0^{\infty}\cdots\int_0^{\infty}\sum_{m=1}^{n}\sum_{m'=0}^{m-1}C(\hat{\zeta}_m)C(\hat{\zeta}_{m'})p_\zeta(\zeta_0)\cdots p_\zeta(\zeta_n)d\zeta_0\cdots d\zeta_n\end{aligned}$$

or, changing the order of summation,

$$\begin{aligned}\langle\hat{C}^2\rangle_B &= \sum_{m=0}^{\infty}(1-q)^m\int_0^{\infty}\cdots\int_0^{\infty}C^2(\hat{\zeta}_m)p_\zeta(\zeta_0)\cdots p_\zeta(\zeta_m)d\zeta_0\cdots d\zeta_m\\ &\quad+2\sum_{m=1}^{\infty}\sum_{m'=0}^{m-1}(1-q)^m\int_0^{\infty}\cdots\int_0^{\infty}C(\hat{\zeta}_m)C(\hat{\zeta}_{m'})p_\zeta(\zeta_0)\cdots p_\zeta(\zeta_n)d\zeta_0\cdots d\zeta_n\end{aligned}$$

The sum of steps $\hat{\zeta}_{m'}$ has density $\frac{(\alpha\hat{\zeta}_m m'}{(m')!}\alpha e^{-\alpha\hat{\zeta}_{m'}}$ (3). Then, $\zeta_m = \zeta_{m'} + \zeta_{m'+1} + \cdots + \zeta_m$ is a sum of two independent random variables, $\zeta_{m'}$ and $\zeta_{m'+1} + \cdots + \zeta_m$. Since the all sub-steps are equally distributed, the density of the sum of $m - m' - 1$ of them is $p_{m-m'-1}(s) = \alpha e^{-\alpha s}\frac{(\alpha s)^{m-m'-1}}{(m-m'-1)!}$, so

$$\begin{aligned}\langle\hat{C}^2\rangle_B &= \int_0^\infty C^2(t)\alpha e^{-q\alpha t}dt + 2(1-q) \\ &\int_0^\infty\int_0^\infty C(t)C(t+s)\sum_{m=0}^\infty\sum_{m'=0}^m \frac{(\alpha(1-q)t)^{m'}}{(m')!}\frac{(\alpha(1-q)s)^{m-m'}}{(m-m')!}\alpha^2 e^{-\alpha(s+t)}dsdt\end{aligned}$$

and using the obvious identity

$$\sum_{m=0}^\infty\sum_{m'=0}^m \frac{x^{m'}}{(m')!}\frac{y^{m-m'}}{(m-m')!} = \sum_{m'=0}^\infty \frac{x^{m'}}{(m')!}\sum_{m=m'}^\infty \frac{y^{m-m'}}{(m-m')!} = \sum_{m'=0}^\infty \frac{x^{m'}}{(m')!}\sum_{m=0}^\infty \frac{y^m}{m!} = e^x e^y$$

together with (2) and (4) we arrive at

$$\langle\hat{C}^2\rangle_B = q\int_0^\infty \mathscr{C}^2(t)\sigma_{ext}e^{-\sigma_{ext}t}dt + 2(1-q)\int_0^\infty\int_0^\infty \mathscr{C}(s)\mathscr{C}(s+t)\sigma_{ext}^2 e^{-\sigma_{ext}(s+t)}dsdt \quad (14)$$

4.2 Combined averages

Averaging (13) and (14) over the FMCRT ensemble, we arrive at

$$\langle\langle\hat{C}\rangle_B\rangle_F = \int_0^\infty \langle\mathscr{C}\rangle(t)\sigma_{ext}e^{-\sigma_{ext}t}dt \quad (15)$$

$$\begin{aligned}\langle\langle\hat{C}^2\rangle_B\rangle_F &= q\int_0^\infty \langle\mathscr{C}^2\rangle(t)\sigma_{ext}e^{-\sigma_{ext}t}dt \\ &+2(1-q)\int_0^\infty\int_0^\infty \langle\mathscr{C}(s)\mathscr{C}(s+t)\rangle\sigma_{ext}^2 e^{-\sigma_{ext}(s+t)}dsdt \quad (16)\end{aligned}$$

where $\langle\mathscr{C}\rangle(t)$ (respectively $\langle\mathscr{C}^2\rangle(t)$) is contribution (respectively squared contribution) of a single forward ray to the integration sphere located at distance t from the camera ray start averaged over the forward ray ensemble.

Notice that because of linearity of averaging, $\langle\langle\hat{C}\rangle_B\rangle_F = \langle\langle\hat{C}\rangle_F\rangle_B$ and $\langle\langle\hat{C}^2\rangle_B\rangle_F = \langle\langle\hat{C}^2\rangle_F\rangle_B$.

For (8) and (9) we also need the averages $\langle\langle\hat{C}\rangle_F^2\rangle_B$ and $\langle\langle\hat{C}\rangle_B^2\rangle_F$. The former can be calculated as in the above Section just replacing $C(x)$ with $\langle C\rangle_F(x)$, so (14) transforms into

$$\begin{aligned}\langle\langle C\rangle_F^2\rangle_B &= q\int_0^\infty \langle\mathscr{C}\rangle^2(t)\sigma_{ext}e^{-\sigma_{ext}t}dt \\ &+2(1-q)\int_0^\infty\int_0^\infty \langle\mathscr{C}\rangle(s)\langle\mathscr{C}\rangle(s+t)\sigma_{ext}^2 e^{-\sigma_{ext}(s+t)}dsdt \quad (17)\end{aligned}$$

The remaining $\langle\langle\hat{C}\rangle_B^2\rangle_F$ is calculated even simpler. Squaring the $\langle C\rangle_B$ is given by (13) and averaging over the FMCRT ensemble yields

$$\langle\langle\hat{C}\rangle_B^2\rangle_F = \left\langle\left(\int_0^\infty \mathscr{C}(t)\sigma_{ext}e^{-\sigma_{ext}t}dt\right)^2\right\rangle = \int_0^\infty\int_0^\infty \langle\mathscr{C}(t)\mathscr{C}(s)\rangle\sigma_{ext}^2 e^{-\sigma_{ext}(t+s)}dtds \quad (18)$$

4.3 Averages over the forward ray ensemble and correlations

For the above, we need contribution (or squared contribution) of a forward ray to the *given* integration sphere at point t averaged over the forward ray ensemble,

If we assume a forward ray hits the given integration sphere no more than once (which is so if the latter is small), then one ray contribution *if it hits the sphere* is (5) and 0 if it misses the sphere, so

$$\langle \mathscr{C} \rangle(t) = \mathcal{F} H_1(t) \tag{19}$$

$$\langle \mathscr{C}^2 \rangle(t) = \frac{\mathcal{F}^2}{\pi R^2} H_2(t) \tag{20}$$

where

$$H_m(t) \equiv \int_0^{\pi} \int_0^{2\pi} f^m \frac{F}{\mathcal{F}} \sin\vartheta d\vartheta d\varphi$$

and F is the angular distribution of radiance in the sphere center t.

Besides, we need correlations, i.e. average of the product of contributions of *the same* forward ray to two integration spheres with centers at s and at $s+t$, $\langle \mathscr{C}(s)\mathscr{C}(s+t)\rangle$.

The product $\mathscr{C}(s)\mathscr{C}(s+t)$ is not 0 only if the forward ray hits *both* spheres. In principle it is possible that it hits one sphere, then bounces somewhere, returns and hits the second one, but in case of small spheres it has too small probability (about $O(R^4)$), so we neglect this and consider only the case when the same *one segment* hits both spheres. The probability of that event equals the area of overlap of the projection of the two spheres onto the plane perpendicular to the segment times attenuation of the ray between the spheres. The above area is the of overlap of two circles of radius R whose centers are separated by $l \equiv t\sin\vartheta$ where ϑ is the angle between the view direction and the forward ray. This area is $\left(\arccos\frac{l}{2R} - \frac{l}{2R}\sqrt{1-\left(\frac{l}{2R}\right)^2}\right)/\pi$ times lower than the whole sphere projection πR^2, and attenuation of the ray is $e^{-\sigma_{ext}t|\cos\vartheta|}$, so the probability that $\mathscr{C}(s)\mathscr{C}(s+t) \neq 0$ is

$$R^2 \times \left(\arccos\frac{l}{2R} - \frac{l}{2R}\sqrt{1-\left(\frac{l}{2R}\right)^2}\right) e^{-2\sigma_{ext}t|\cos\vartheta|}$$

and if this happens, $\mathscr{C}(s)\mathscr{C}(s+t)$ is the square of (5).

Recalling that $\langle \mathscr{C}^2 \rangle$ is that same squared (5) times probability of hitting the sphere i.e. πR^2, we obtain

$$\langle \mathscr{C}(s)\mathscr{C}(s+t)\rangle = \frac{\mathcal{F}^2}{\pi^2 R^2} \int_0^{2\pi} \int_0^{\pi} A\left(\frac{t\sin\vartheta}{R}\right) e^{-\sigma_{ext}t|\cos\vartheta|} h(\vartheta,\varphi,p) \sin\vartheta d\vartheta d\varphi \tag{21}$$

where φ is the azimuthal angle of rotation about the view direction,

$$A(x) \equiv \begin{cases} \arccos x - x\sqrt{1-x^2}, & |x| \leq 1 \\ 0, & x > 1 \end{cases}$$

and

$$h \equiv f^2 \frac{F}{\mathcal{F}}$$

is taken at the center of sphere which the forward ray with direction (ϑ, φ) hits *first* i.e. at the camera ray point p

$$p = \begin{cases} s, & \cos\vartheta \geq 0 \\ t, & \cos\vartheta \leq 0 \end{cases} \tag{22}$$

4.4 The total noise

Substituting (15), (16), (17), (18), (19) and (20) into (7)–(9) we obtain

$$\begin{aligned} \langle\langle L^2\rangle\rangle - \langle\langle L\rangle\rangle^2 \;=\; & q\frac{\int_0^\infty \langle\mathscr{C}^2\rangle(t)\sigma_{ext}e^{-\sigma_{ext}t}dt}{N_F N_B} \\ & +2(1-q)\frac{\int_0^\infty\int_0^\infty \langle\mathscr{C}(s)\mathscr{C}(s+t)\rangle\sigma_{ext}^2 e^{-\sigma_{ext}(s+t)}dsdt}{N_F N_B} \\ & +\frac{q}{N_B}\left(1-\frac{1}{N_F}\right)\int_0^\infty \langle\mathscr{C}\rangle^2(t)\sigma_{ext}e^{-\sigma_{ext}t}dt \\ & +\frac{2(1-q)}{N_B}\left(1-\frac{1}{N_F}\right)\int_0^\infty\int_0^\infty \langle\mathscr{C}\rangle(s)\langle\mathscr{C}\rangle(s+t)\sigma_{ext}^2 e^{-\sigma_{ext}(s+t)}dsdt \\ & +\frac{1}{N_F}\left(1-\frac{1}{N_B}\right)\int_0^\infty\int_0^\infty \langle\mathscr{C}(s)\mathscr{C}(t)\rangle\sigma_{ext}^2 e^{-\sigma_{ext}(t+s)}dtds \\ & -\frac{1}{N_F N_B}\left(\int_0^\infty \langle\mathscr{C}\rangle(t)\sigma_{ext}e^{-\sigma_{ext}t}dt\right)^2 \end{aligned}$$

It contains two integrals of correlations which for $R \to 0$ are estimated as (25) and (26).

Using (19) and (20), we have

$$\begin{aligned} \frac{\langle\langle L^2\rangle\rangle - \langle\langle L\rangle\rangle^2}{\langle\langle L\rangle\rangle^2} \;\approx\; & \frac{1}{R^2 N_F}\left(\frac{qD_1 + \sigma_{ext}R(1-q)D_2}{N_B} + \left(1-\frac{1}{N_B}\right)\sigma_{ext}RD_3\right) \\ & +\frac{1}{N_B}\left(1-\frac{1}{N_F}\right)(qD_4 + (1-q)D_5) - \frac{1}{N_F N_B} \end{aligned}$$

where D_i are some constants composed from space integrals of

$$
\begin{aligned}
H_m(t) &\equiv \int_0^{\pi}\int_0^{2\pi} f^m \frac{F}{\mathcal{F}} \sin\vartheta d\vartheta d\varphi \\
\tilde{H}_m(s) &\equiv \int_0^{\pi}\int_0^{2\pi} f^m \frac{F}{\mathcal{F}} d\vartheta d\varphi
\end{aligned}
$$

and are thus independent from any ray tracing parameters, i.e. q, R and the number of rays.

One can see that for small R but not very large number of rays the noise is smaller for $q < 1$ then for $q = 1$ i.e. for the standard method.

5 CONCLUSION

In previous work [16] we developed a method of gathering of photons in scattering medium which uses several integration spheres stochastically distributed over a camera ray segment. The resulting disjoint integration volume is intermediate between usual integration sphere (one per segment) and integration cylinder. In our method the control parameter q allows to vary integration volume from a single integration sphere to practically a cylinder which is a limit of the union of a large number of spheres.

We obtained analytical estimates of noise in the bi-directional MCRT and applied them to the particular case of calculation of luminance of turbid medium, suggested in [16]. It happened possible to derive a closed-form analytical expressions for the noise and its dependence on the method parameters. These can be used to find the optimal parameters and/or choose between single and multiple integration spheres or cylinders.

One must realize that the method (or parameter) that provides the lowest noise for the given *number of rays* is not always really the best. One reason is that if integration volumes occupy most of the medium domain, interaction of them with an FMCRT ray is slow and this seriously decelerate ray tracing, thus the number of rays traced in *the same time* of calculation drops and this increases noise. Second, the own luminance of the medium is not the only image component but there is also the luminance of objects "seen through" the medium. Usually both are calculated from with the same rays thus it may happen that although because of better integration volumes the noise of the "own" luminance of medium is still decreased in spite of the lower number of traced rays, the noise in the rest part of image increases because there is no such "compensation" for the rays count.

APPENDIX A. INTEGRALS OF CORRELATIONS

Let us calculate approximation for $R \to 0$ of

$$
\int_0^{\infty}\int_0^{\infty} \langle \mathscr{C}(s)\mathscr{C}(s+t)\rangle \sigma_{ext}^2 e^{-\sigma_{ext}(t+s)} dt ds \tag{23}
$$

and

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \langle \mathscr{C}(s)\mathscr{C}(t)\rangle \sigma_{ext}^2 e^{-\sigma_{ext}(s+t)} ds dt \\
= \quad & 2\int_0^\infty \int_0^\infty \langle \mathscr{C}(s)\mathscr{C}(s+t)\rangle \sigma_{ext}^2 e^{-\sigma_{ext}(2s+t)} ds dt \qquad (24)
\end{aligned}$$

where $\langle \mathscr{C}(s)\mathscr{C}(s+t)\rangle$ is given by (21) and (22).

Both integrals are much similar and we begin with (23); as (24) differs from it only by the extra factor $e^{-\sigma_{ext}s}$ these calculations will apply to it as well. We thus begin with (23).

Let us subdivide interval of integration in ϑ into 3 parts: the "main" $[\Theta, \pi-\Theta]$ and two short intervals near the poles: $[0,\Theta]$ and $[\pi-\Theta,\pi]$. Then

$$\int_0^\infty \int_0^\infty \langle \mathscr{C}(s)\mathscr{C}(s+t)\rangle \sigma_{ext}^2 e^{-\sigma_{ext}(t+s)} dt ds = \frac{\mathcal{F}^2}{\pi^2 R^2}\left(I_1 + I_{2a} + I_{2b}\right)$$

where

$$\begin{aligned}
I_1 &\equiv \int_0^\infty \left(\int_0^\infty \left(\int_0^{2\pi}\int_\Theta^{\pi-\Theta} A\left(\frac{t\sin\vartheta}{R}\right) e^{-\sigma_{ext}t|\cos\vartheta|} h(\vartheta,\varphi,p)\sin\vartheta d\vartheta d\varphi\right) dt\right) ds \\
I_{2a} &\equiv \int_0^\infty \left(\int_0^\infty \left(\int_0^{2\pi}\int_0^{\Theta} A\left(\frac{t\sin\vartheta}{R}\right) e^{-\sigma_{ext}t|\cos\vartheta|} h(\vartheta,\varphi,p)\sin\vartheta d\vartheta d\varphi\right) dt\right) ds \\
I_{2b} &\equiv \int_0^\infty \left(\int_0^\infty \left(\int_0^{2\pi}\int_{\pi-\Theta}^{\pi} A\left(\frac{t\sin\vartheta}{R}\right) e^{-\sigma_{ext}t|\cos\vartheta|} h(\vartheta,\varphi,p)\sin\vartheta d\vartheta d\varphi\right) dt\right) ds
\end{aligned}$$

The idea is to take such small Θ that in the "main" part I_1 the integration area in t is very narrow, so $p \approx s$ thus integration over t can be done analytically and after some tedious transformations we obtain a simple approximation to I_1. For the rest parts near the poles, the range of t is wide, and the point p is s in I_{2a} and $s+t$ in I_{2b}. The range of integration in Θ is small which allows to estimate the integrals from above and it happens that for $R \to 0$ they are negligible as compared to I_1, so the sought-for integral of correlations is close to I_1 for which we have a simple approximation. We shall see that the small angle Θ must be chosen so that it as $R \to 0$ it goes to 0 but *slower* that R so that $\Theta \to 0$ but $R/\Theta \to 0$.

Now let us apply the above intentions quantitatively.

As the integrand vanishes for $\frac{t\sin\vartheta}{R} > 1$, in I_1 the $t \le R/\Theta \to 0$. Therefore the point $p \approx s$ even for $\cos\vartheta \le 0$. Changing then the order of integration so that the one in t to be the first, we obtain

$$\begin{aligned}
I_1 &\approx 2R\int_0^\infty \int_0^{2\pi}\int_\Theta^{\pi-\Theta} \left(\int_0^1 A(y)dy\right) h(\vartheta,\varphi,s)\sigma_{ext}^2 e^{-\sigma_{ext}s} d\vartheta d\varphi ds \\
&= \frac{4R}{3}\int_0^\infty \int_0^{2\pi}\int_\Theta^{\pi-\Theta} h(\vartheta,\varphi,s)\sigma_{ext}^2 e^{-\sigma_{ext}s} d\vartheta d\varphi ds \\
&\approx \frac{4R}{3}\int_0^\infty \left(\int_0^{2\pi}\int_0^{\pi} h(\vartheta,\varphi,s) d\vartheta d\varphi\right)\sigma_{ext}^2 e^{-\sigma_{ext}s} ds
\end{aligned}$$

Now let us come to the integrals in the near-pole areas. Assuming illumination and phase function are not singular near the view direction, we can take h exactly at the pole direction. Then, we can replace $\sin\vartheta$ with ϑ or $\pi-\vartheta$:

$$\begin{aligned}
I_{2a} &\approx 8\pi\int_0^\infty\left(\int_0^\infty\left(\frac{R^2}{t^2}\int_0^{\min(\frac{t\Theta}{2R},1)}A(y)ydy\right)e^{-2\sigma_{ext}t}\sigma_{ext}^2dt\right)e^{-\sigma_{ext}s}h(0,0,s)ds\\
I_{2b} &\approx 8\pi\int_0^\infty\left(\int_0^\infty\left(\frac{R^2}{t^2}\int_0^{\min(\frac{t\Theta}{2R},1)}A(y)ydy\right)e^{-2\sigma_{ext}t}\sigma_{ext}^2h(\pi,0,s+t)dt\right)e^{-\sigma_{ext}s}ds
\end{aligned}$$

Since

$$\int_0^X A(y)ydy=\frac{\frac{\pi}{2}-(1-4X^2)\arccos X-X\left(1+2X^2\right)\sqrt{1-X^2}}{8}$$

with very good accuracy is limited by $\pi X^2/4$, we have

$$\begin{aligned}
I_{2a} &\le \frac{\pi^2}{2}\int_0^\infty\left(\int_0^\infty\left(\min(\frac{2R}{t},\Theta)\right)^2e^{-2\sigma_{ext}t}\sigma_{ext}^2dt\right)e^{-\sigma_{ext}s}h(0,0,s)ds\\
&\le 2\pi^2\Theta\sigma_{ext}R\int_0^\infty\sigma_{ext}e^{-\sigma_{ext}s}h(0,0,s)ds
\end{aligned}$$

where we used the inequality $\int_X^\infty u^{-2}e^{-u}du=X^{-1}\left(e^{-X}-X\mathrm{Ei}\left(1,X\right)\right)\le X^{-1}$.
Similarly,

$$\begin{aligned}
I_{2b} \le \frac{\pi^2}{2}\int_0^\infty&\left(\Theta^2\int_0^{\frac{2R}{\Theta}}e^{-2\sigma_{ext}t}\sigma_{ext}^2h(\pi,0,s+t)dt\right.\\
&\left.+4R^2\int_{\frac{2R}{\Theta}}^\infty t^{-2}e^{-2\sigma_{ext}t}\sigma_{ext}^2h(\pi,0,s+t)dt\right)e^{-\sigma_{ext}s}ds
\end{aligned}$$

The two its terms can be estimated as

$$\begin{aligned}
&\int_0^\infty\left(\int_0^{\frac{2R}{\Theta}}e^{-2\sigma_{ext}t}\sigma_{ext}^2h(\pi,0,s+t)dt\right)e^{-\sigma_{ext}s}ds\\
&\le\left(\int_0^\infty h(\pi,0,s)\sigma_{ext}e^{-\sigma_{ext}s}ds\right)\left(1-e^{-\frac{2\sigma_{ext}R}{\Theta}}\right)\\
&\approx\frac{2\sigma_{ext}R}{\Theta}\left(\int_0^\infty h(\pi,0,s)\sigma_{ext}e^{-\sigma_{ext}s}ds\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty \left(\int_{\frac{2R}{\Theta}}^\infty t^{-2} e^{-2\sigma_{ext} t} \sigma_{ext}^2 h(\pi, 0, s+t) dt \right) e^{-\sigma_{ext} s} ds \\
&\quad \le \sigma_{ext}^2 \left(\int_{\frac{2\sigma_{ext} R}{\Theta}}^\infty t^{-2} e^{-t} dt \right) \left(\int_0^\infty h(\pi, 0, s) \sigma_{ext} e^{-\sigma_{ext} s} ds \right) \\
&\quad \le \sigma_{ext} \frac{\Theta}{2R} \left(\int_0^\infty h(\pi, 0, s) \sigma_{ext} e^{-\sigma_{ext} s} ds \right)
\end{aligned}$$

so

$$I_{2b} \;\; \le \;\; 2\pi^2 \Theta \sigma_{ext} R \left(\int_0^\infty h(\pi, 0, s) \sigma_{ext} e^{-\sigma_{ext} s} ds \right)$$

Therefore for $R \to 0$ we can neglect $I_{2a} + I_{2b}$ as compared to I_1, and obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \langle \mathscr{C}(s) \mathscr{C}(s+t) \rangle \sigma_{ext}^2 e^{-\sigma_{ext}(t+s)} dt ds \\
&\quad \approx \frac{4\sigma_{ext} \mathcal{F}^2}{3\pi^2 R} \int_0^\infty \left(\int_0^{2\pi} \int_0^\pi h(\vartheta, \varphi, p) d\vartheta d\varphi \right) \sigma_{ext} e^{-\sigma_{ext} s} ds
\end{aligned} \tag{25}$$

The second integral, (24) differs only by the extra factor $e^{-\sigma_{ext} s}$. Therefore the calculation done above applies to it as well and give in this case

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \langle \mathscr{C}(s) \mathscr{C}(t) \rangle \sigma_{ext}^2 e^{-\sigma_{ext}(s+t)} ds dt \\
&\quad \approx \frac{8\sigma_{ext} \mathcal{F}^2}{3\pi^2 R} \int_0^\infty \left(\int_0^{2\pi} \int_0^\pi h(\vartheta, \varphi, p) d\vartheta d\varphi \right) \sigma_{ext} e^{-2\sigma_{ext} s} ds
\end{aligned} \tag{26}$$

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