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A NOTE ON THE PERPETUAL AMERICAN STRADDLE

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Summary. In this paper we a derive a single parameter characterization of the value and the optimal exercise time of the perpetual American straddle (the portfolio consisting of a put and a call option on the same underlying asset with the same price) in the classical Black-Scholes-Samuelson model. The parameter is the unique solution of a single non-linear equation with one unknown variable; this is the first time that the single equation characterization has been obtained for the perpetual American straddle. The equation is derived after multiple transformations of the defining optimal stopping problem in continuous time using a combination of classical techniques: Hamilton-Jacobi-Bellman equation, reduction to a Cauchy-Euler first order differential equation, smooth pasting conditions, and, finally, verification theorem for optimal stopping problems.

1 INTRODUCTION

Pricing of derivatives in the classical Black-Scholes-Merton model of a financial market is a classical topic in financial mathematics. Given that the stock price is modeled by a geometric Brownian motion, pricing problems can often be formulated as problems of optimal stopping in continuous time. Arguably the best known models involve pricing of perpetual American options: options without expiration date. Although the financial derivatives of this kind are not actively traded they represent an important theoretical concept and, due to their diminishing value, a valuable first approximation of the value of American derivatives with expiration dates. When considering these kind of problems of interest are the value function of the problem, which gives information about the price of the derivative, as well as the optimal stopping time, which gives information about the optimal exercise time of the derivative under consideration.

In this paper we consider the perpetual *American straddle*: a classical portfolio consisting of a put option and a call option on the same underlying asset with the same strike price. The pricing of the perpetual American straddle has been studied using different approaches and tools: in [1] by applying the theory of Laplace transforms, in [5] by transforming the problem to a "generalized parking problem", in [6] by exploiting "an analogy with asymmetric rebates of double knock-out barrier options", in [7] "by means of the Esscher transform and the optional sampling theorem", and, more recently, by using a combination of several optimization techniques [3] and [4]. The characterizations obtained in these papers are often cumbersome: indeed, in all of these papers the value function and the optimal exercise time are characterized by a solution of a non-linear system of equations consisting of (at least) two equations.

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In this note, we show that the value function and the optimal exercise time of the perpetual American straddle can be characterized via a unique solution of a *single* one-variable equation; the solution lies in the interval $(0,1)$. We do so by using one of the classical optimal stopping theory approaches: the Hamilton-Jacobi-Bellman (HJB) equation and the smooth-fit principle in combination with a verification theorem. In particular, we begin with an optimal stopping problem in continuous time and assume that it's optimal stopping time is the first exit time of a bounded interval. The HJB equation for the value function of the problem on the continuation region is a partial differential equation that can be reduced to an ordinary differential equation (the Cauchy-Euler equation), as is customary for the problems of this nature. After solving the equation and exploiting the assumption about the continuation region we are able to explicitly write down the form of the value function: it is a piecewise function with several unknown parameters. As the value function is expected to be continuous and differentiable we are able to apply what is known as smooth pasting conditions to obtain a nonlinear system of equations the solution of which will give us the unknown parameters. Finally, after applying the verification theorem and some theoretical considerations about the uniqueness of the solution of the system, we reduce the system to a single equation and prove that it's solution is unique and in the (0,1) interval. To the best of our knowledge this is the first time that such one-equation characterization of the value and the optimal exercise time of the perpetual American straddle is obtained. In the next section we present our result in full detail.

2 RESULT

Let the price process S_t be a a geometric Brownian motion,

$$dS_t = \alpha S_t dt + \sigma S_t dB_t,$$

where $\alpha \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ are known constants. The American straddle yields a payoff

$$f(t,S_t) = e^{-rt}|S_t - I|$$

when exercised at time t, where $I > 0$ is the strike price and $r \leq \alpha$ is a given discount rate (the inequality $r \leq \alpha$ is a standard assumption; see for example [10]).

The value of the perpetual American straddle at time t is given by

$$V_t = \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} E[e^{-r\tau}|S_\tau - I|], \tag{1}$$

where $\mathscr{T}_t$ is a set of all stopping times $\tau \geq t$. Our goal is to find a value function $v(t,x)$ such that $v(t,S_t) = V_t$ and an optimal stopping time τ^* such that $V_{\tau^*} = E[e^{-r\tau^*}|S_{\tau^*} - I|]$. Hamilton-Jacobi-Bellman (HJB) equation related to this problem is:

$$\max_{(t,x)\in[0,+\infty]\times\mathbb{R}} \{f(t,x) - v(t,x), v_t(t,x) + \mathscr{L}v(t,x)\} = 0, \tag{2}$$

where $\mathscr{L} := \alpha\partial_x + \frac{1}{2}\sigma^2\partial_{xx}$ is a differential operator related to Ito's lemma (see e.g. [8, ch.11]).

A well known approach when dealing with time-discounted optimal stopping problems is to assume that the value function is of the form

$$v(t,x) = e^{-rt}\varphi(x);$$

this will later be confirmed using a verification theorem. The equality

$$v_t(t,x) + \mathscr{L}v(t,x) = 0$$

holds on the continuation region (due to the HJB equation). After canceling out e^{-rt} this gives:

$$r\varphi(x) - \alpha x\varphi'(x) - \frac{1}{2}\sigma^2x^2\varphi''(x) = 0.$$

The last equation is a well known Cauchy-Euler ordinary differential equation and its solution is

$$\varphi(x) = Ax^\lambda + Bx^\mu,$$

where A and B are two unknown constants and λ and μ are the solutions of the characteristic equation

$$r - \alpha m - \frac{1}{2}\sigma^2 m(m-1) = 0.$$

It can be easily verified that inequalities $\lambda > 1$ and $\mu < 0$ hold.

It is known that the optimal stopping time will be the first exit time from the interval $(x_1, x_2) \ni I$: it is optimal to exercise the put (call) option when the value of S_t goes beneath x_1 (above x_2). Furthermore, on the stopping region, the HJB equation implies that $f = v$. Thus, we assume that the function v should be of the form:

$$v(t,x) = \begin{cases} e^{-rt}(I-x), & 0 < x < x_1 \\ e^{-rt}(Ax^\lambda + Bx^\mu), & x_1 \leq x \leq x_2 \\ e^{-rt}(x-I), & x > x_2 \end{cases} \tag{3}$$

where A, B, x_1, x_2 are constants chosen in a way that makes the function v differentiable (smooth pasting conditions). In particular, we require continuity and differentiability in x_1 and x_2.

It is already clear that, should we find such constants, the above function $v(t,x)$ will be a value function. Indeed, conditions of any of the well known verification theorems for the optimal stopping of diffusions (e.g. ch. 3 in [9] or ch. 10 in [8]) are easily satisfied for functions that coincide, piecewise, with (discounted) linear combinations of power functions. Furthermore, since the functions v and f coincide outside the interval (x_1, x_2), if v is indeed the value function, then the optimal stopping time is:

$$\tau^* = \inf\{t \geq 0 | S_t \notin (x_1, x_2)\}.$$

Smooth pasting conditions lead to a highly non-linear system of equations. We show that it can be reduced to a single equation:

Theorem 1. *The value process of the perpetual American Straddle V_t defined in (1) satisfies the equality $V_t = v(t, S_t)$ for the function v as defined in (3) where*

$$A = \frac{1}{\mu-\lambda}((1-\mu)x_1^{1-\lambda} + \mu x_1^{-\lambda}); \qquad x_2 = \frac{\mu I}{\mu-1}\frac{1+\gamma^{-\lambda}}{1+\gamma^{1-\lambda}}$$
$$B = \frac{1}{\lambda-\mu}((1-\lambda)x_1^{1-\mu} + \lambda x_1^{-\mu}); \qquad x_1 = \gamma x_2$$

and $\gamma \in (0,1)$ is the unique number satisfying:

$$\frac{\mu}{\mu-1}\frac{1+\gamma^{-\lambda}}{1+\gamma^{1-\lambda}} - \frac{\lambda}{\lambda-1}\frac{1+\gamma^{-\mu}}{1+\gamma^{1-\mu}} = 0. \tag{4}$$

Proof. Smooth pasting conditions, after cancelling out e^{-rt}, can be written as:

$$\begin{aligned} I - x_1 &= Ax_1^{\lambda} + Bx_1^{\mu}, & -x_1 &= A\lambda x_1^{\lambda} + B\mu x_1^{\mu}, \\ x_2 - I &= Ax_2^{\lambda} + Bx_2^{\mu}, & x_2 &= A\lambda x_2^{\lambda} + B\mu x_2^{\mu}. \end{aligned} \tag{5}$$

In order to prove the theorem it is, by construction of the value function v, sufficient to prove that unique solution of the system (5) is the one given in the formulation of the theorem. The proof consists of reducing the system to equation (4), and proving that the solution of the latter is unique on the interval $(0,1)$.

First we comment on the uniqueness of the solution of the system of equations (5). Due to the uniqueness of the value function of the optimal stopping problems the solution of the system above must be unique. Indeed, two different solutions of the system (5) would lead to two functions v_1 and v_2 both of which would satisfy the verification theorem and the equation $v_1(t, S_t) = v_2(t, S_t)$ would holds almost surely, which is clearly impossible.

We now turn to proving the existence. We can eliminate variables A and B in the two equations containing x_1 by treating them as a two dimensional linear system. Since determinant of that system is $D = x_1^{\lambda+\mu}(\mu - \lambda) \neq 0$, A and B are uniquely determined by it. We can do the same for the two equations containing x_2. If we introduce, for notational purposes, the function $Q(x; \mu, \lambda) = (\mu - \lambda)^{-1}(1-\mu)x^{1-\lambda} + \mu x^{-\lambda}$, we can write the solutions of those two systems as:

$$A = Q(x_1; \mu, \lambda);\ B = Q(x_1; \lambda, \mu);\ A = -Q(x_2; \mu, \lambda);\ B = -Q(x_2; \lambda, \mu).$$

Equating the expressions for A and B we obtain the following nonlinear system with two equations and two variables, x_1 and x_2:

$$Q(x_1; \mu, \lambda) + Q(x_2; \mu, \lambda) = 0 \qquad Q(x_1; \lambda, \mu) + Q(x_2; \lambda, \mu) = 0 \tag{6}$$

Due to the nice form of the above system, we immediately see that if (x_1, x_2) is its solution so is (x_2, x_1). This means that there is a unique solution pair satisfying $x_1 < x_2$, and it will be the unique solution that we are looking for. We introduce a variable γ such that $x_1 = x_2\gamma$; since inequality $0 < x_1 < x_2$ holds, we have $\gamma \in (0,1)$. The right hand side of the first equation of the system (6) can now, after some simple calculations, be written as:

$$Q(x_2\gamma;\mu,\lambda) + Q(x_2;\mu,\lambda) = (1-\mu)x_2^{1-\lambda}(1+\gamma^{1-\lambda}) + \mu x_2^{-\lambda}(1+\gamma^{-\lambda}).$$

from which we obtain:

$$x_2 = \frac{\mu I}{\mu - 1}\frac{1+\gamma^{-\lambda}}{1+\gamma^{1-\lambda}}$$

Similarly, by changing $x_1 = x_2\gamma$ in $Q(x_1;\lambda,\mu) + Q(x_2;\lambda,\mu) = 0$ after multiplication with $x_1^{-\mu}$ we obtain:

$$x_2 = \frac{\lambda I}{\lambda - 1}\frac{1+\gamma^{-\mu}}{1+\gamma^{1-\mu}}.$$

Equating the two obtained expressions for x_2, after rearanging and cancelling out parameter I, we obtain the one-dimensional equation (4), stated in the formulation of the theorem.

It remains to prove that there exists a unique solution of equation (4) in the interval $(0,1)$. Let us denote the left hand side of the equation with $h(\gamma)$. It is obvious that function h is continuous on $(0,1)$ and, since $\lambda > 1$ and $\mu < 0$, it is easy to check that $h(1) < 0$ and $\lim_{\gamma \to 0+} h(\gamma) = +\infty$. We can thus conclude that a solution exists on the interval $(0,1)$, and its uniqueness is a consequence of the argument from the beginning of the proof.

□

3 CONCLUSION

We have demonstrated that the perpetual American straddle, a classical and well studied portfolio of options, can be priced and fully characterized using a unique solution of a single non-linear equation on the unit interval. Our contribution is technical, and it's value lies in it's elegance as well as the fact that the solution itself gives a direct relation between two exercise boundaries of the American straddle. The result represents an represents a rare advancement in a well studied field, showing that even in classical literature on derivative pricing there can be space for contributions to the theory; the contributions are likely to be of the technical kind, as demonstrated in this material.

As the result we presented belongs to a wide field of financial mathematics, we conclude the paper with two comments that hopefully address the relative significance of the results within finance and mathematics, respectively.

Curiously, the equation the solution of which is the single parameter that characterizes the optimal exercise time and the price of the perpetual American straddle does not depend on the strike price I. This technical curiosity might have a deeper economic interpretation within the field of finance; this however lies beyond the mathematical aspects studied in this work.

From the perspective of the mathematics of the theory of optimal stopping in continuous time, the result we presented gives rise to a natural question for future research: which optimal stopping problems with bounded continuation regions can be characterized by a single parameter? The solution we presented exploited the ”symmetry” of the function f and one can naturally assume that one needs to formalize this condition in the most general terms in order to prove the general version of the result presented here.

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