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ON NORMAL FAMILIES OF HOLOMORPHIC FUNCTIONS

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Summary. Let Ω be a domain in $\mathbb{C}^n$ and $D(F)$ denote the normality domain of the family $F \subset H(\Omega)$. We prove that the set $\Omega \setminus D(F)$ cannot be a compact subset of Ω which does not separate Ω. We also prove that the property of the family F to be normal in a neighborhood of the holomorphic surfaces S_j is also enjoyed by for a limit set of these surfaces.

1 INTRODUCTION

Let $H(\Omega)$ be a class of holomorphic functions defined for a domain Ω in $\mathbb{C}^n$, $n \geq 1$. We recall that a family $F \subset H(\Omega)$ is normal in Ω if every sequence in F has a subsequence which converges uniformly on compact subsets of Ω either to a holomorphic function or to ∞.

It is well known that a family F of holomorphic functions is a normal family if and only if it is locally a normal family, that is, it is a normal family on a neighborhood of each point of Ω.

We define the chordal distance $d(z,w)$ between two points $z, w \in \overline{\mathbb{C}}$ to be the length of the straight line segment joining the points P and Q on the unit sphere $\Sigma = \{x \in \mathbb{R}^3 : |x| = 1\}$ whose stereographic projections are z and w respectively.

Recall that the chordal metric d induced on $\mathbb{C}$ by the Euclidean metric of the sphere Σ via the stereographic projection is given explicitly by

$$d(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}.$$

The infinitesimal form of this metric is $2|dz|/\left(1+|z|^2\right)$. The spherical length

$$s(\gamma) = 2\int_\gamma \frac{|dz|}{1+|z|^2} = 2\int_\gamma \frac{|\gamma'(t)|}{1+|\gamma(t)|^2}dt$$

of a curve γ in $\overline{\mathbb{C}}$ induces a metric in the following manner. Given distinct points z, w on the Riemann sphere, define

$$s(z,w) = \inf\{s(\gamma)\}$$

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where the infimum is taken over all piecewise C^1 curves on $\overline{\mathbb{C}}$ which join z with w. Then defines a metric on the sphere known as the spherical metric.

Indeed, $d(z_1,z_2) \le s(z_1,z_2) \le \frac{\pi}{2} d(z_1,z_2)$, so that the two metrics are uniformly equivalent and generate the same open sets on Σ. Thus, from a topological point of view, the metrics d and s can be treated as one and the same.

As a consequence of Marty's and Ascoli-Arzela's Theorems, we have the following theorem. On normality of a family of continuous functions, the theorem of Ascoli-Arzela is well known. Especially, if we include the case of compact divergence, for a family of holomorphic functions, normality follows from the condition of equicontinuity at each point.

Theorem (1.1). *A family of holomorphic functions F is normal on Ω if and only if it is spherically equicontinuous on Ω.*

The family F is said to be normal at a point $a \in \Omega$ if it is normal in some open ball $B(a,r) = \{z \in \mathbb{C}^n : |z-a| < r\}$, $B(a,r) \subset \Omega$, with center a and radius r.

Theorem (1.2). *If a family of holomorphic functions F is normal in a domain $n \ge 1$, $\Omega \subset \mathbb{C}^n$, then about each point $a \in \Omega$ there is a ball $B(a,r) \subset \Omega$, $r = r(a)$, such that either*

$$|f(z)| < 2 \text{ or } |f(z)| > \frac{1}{2}$$

holds for all $z \in B(a,r)$, $f \in F$ and conversely.

The partitioning of the family F into two normal families works equally well if the constant 2 in the above is replaced by an arbitrary positive constant.

Corollary (1.3). *If F is not normal at some point $a \in \Omega$ then exists a sequence $\{f_j\} \subset F$ and points $z^j = z^j(f_j) \to a$, $w^j = w^j(f_j) \to a$ as $j \to \infty$, such that*

$$|f_j(z^j)| > j \text{ or } |f_j(w^j)| < \frac{1}{j}.$$

For the proof of Theorem (1.2) and Corollary (1.3) see [1].

2 ON EXCEPTIONAL SETS

In [3], P. Montel defined the notion of a quasi-normal family of meromorphic functions and obtained several results relating to this. Let F be a family of analytic (meromorphic) functions on a domain Ω. in $\mathbb{C}^1$. Then F is quasi-normal on Ω if every sequence of functions $\{f_j\} \subset F$ contains a subsequence which converges uniformly (spherically uniformly) on compact subsets of $\Omega \setminus Q$ where Q is a (possibly empty) finite set of points of Ω. This set Q of exceptional points may vary with the particular sequence and constitutes the set of irregular points.

Let Ω be a bounded open set in $\mathbb{C}^n$ and let Ω_0 be a relatively open subset; the difference $\Omega \setminus \Omega_0$ will be denoted by E. The set E is bounded and it is only relatively closed in Ω. At first we do not qualify it at all. It may very well have an interior.

Suppose that the family $F \subset H(\Omega)$ is normal in Ω_0. We will look upon E as an "exceptional set" of the family F ask the following question. Under what conditions on E the family F is normal in all Ω? If the latter situation arises we say that the set E is removable for F.

The following theorem shows that in several variables the exceptional set of a normal family cannot be a compact set which separate the domain.

Theorem (2.1). *Let K be a compact subset of a domain Ω in $\mathbb{C}^n$ $n > 1$, (n>1) such that K does not separate the domain (i.e., such that $\Omega \setminus K$ is connected). If a family $F \subset H(\Omega)$ is normal in $\Omega \setminus K$, then F is the normal family in Ω.*

Proof. Consider a sequence $\{f_j\} \subset F$. There are two cases to consider.

(i) Suppose that $\{f_j\}$ converges uniformly on $\Omega \setminus K$ to a holomorphic function f. In $\Omega \setminus K$ we can choose smooth $(2n-1)$-dimensional surface S, such that it bound domain G with connected complements, such that $K \subset\subset \overline{G}$ and such that $G \subset\subset D$.

Since $f_j \to f$ normally on $\Omega \setminus K$, and $S \subset \Omega \setminus K$ there exists j_0 such that all $j > j_0$

$$\max_S \left| f_j - f \right| < \varepsilon.$$

By the maximal modulus principle for bounded domains

$$\max_G \left| f_j - f \right| < \max_S \left| f_j - f \right| < \varepsilon,$$

and so $f_j \to f$ uniformly on every compact subset of G.

(ii) Suppose that $\{f_j\}$ converges uniformly to ∞ on $\Omega \setminus K$. In $\Omega \setminus K$ we choose smooth $(2n-1)$-dimensional surfaces S_1 and S_2 such that they respectively bound domains G_1 and G_2 with connected complements, such that $K \subset\subset G_1 \subset\subset G_2$ and such that the layer $G = G_2 \setminus G_1 \subset\subset \Omega \setminus \mathrm{K}$.

Since $\{f_j\}$ converges uniformly to ∞ on $\Omega \setminus K$. Then a subsequence $\{f_{j_k}\}$ can be extracted from $\{f_j\}$ such that

$$(2.2) \qquad \left| f_{j_k}(z) \right| > 1 \quad \textit{for all } k \textit{ and all } z \in \overline{G}.$$

It follows that any function f_{j_k} is zero-free in G_1. Assume the converse. Then $f_{j_k}(z_0) = 0$ for some $z_0 \in G_1$.

By the removal of compact singularities theorem a zero set of a holomorphic function cannot be compact, hence the zero set $\{z \in G_2 : f_{j_k}(z) = 0\}$ must go to the surface S_2. This contradicts (2.2).

Hence, $\{1/f_{j_k}\}$ converges uniformly to 0 on S_2. The remainder of the proof can be carried out as in the case (i) and we obtain that $\{1/f_{j_k}\}$ converges uniformly to 0 on compact subsets of G_2, and whence f_{j_k}, converges uniformly to ∞ on compact subsets of G_2. We conclude that the family F is a normal family on Ω. □

Remark (2.3). This result cannot occur in the theory of one variable. Consider the family $F = \{j \cdot z : j = 1,2,3,...\}$. Then $f_j(0) \to 0$, but $f_j(z) \to \infty$ as $j \to \infty$ for $z \neq 0$. Hence F cannot be normal in any domain containing the origin.

Theorem 2.1 shows the fundamental difference between the cases $n = 1$ and $n \geq 2$.

Remark (2.4). Theorem 2.1 demonstrates, incidentally, that in high dimensions a set of singular points of a family F must go to the boundary.

The normality domain of the family $F \subset H(\Omega)$ is the union $D(F)$ of the open connected subsets $U \subset \Omega$ such that the restriction $F|_U$ is normal. The normality domain is the largest open subset of Ω having this property.

Theorem (2.5) . *Let* Ω *be a domain in* $\mathbb{C}^n$, $n > 1$, *and* A *is an analytic subset codimension at least* 2. *Let* $F \subset H(\Omega)$ *be a family such that* $f(\Omega \setminus A) \subset \mathbb{C} \setminus \{0,1\}$. *Then* $D(F) \equiv \Omega$.

Proof. Any function $f \in F$ can take only those values at points of $\Omega \setminus A$ that f takes in Ω.

Suppose on the contrary that a function $f \in F$ takes some value c in A that it does not take in $\Omega \setminus A$. Then the function

$$g(z) = \frac{1}{f(z) - c}$$

is obviously holomorphic in $\Omega \setminus A$ but cannot be extended analytically into Ω, since at some point of A it goes to infinity. This contradicts Theorem 4 \cite[p.176]{[BS]}. □

A set $M \subset \mathbb{C}^n$ is called thin if for every point $p \in M$ there are a ball $B(p,r)$ and a nonconstant function $\varphi \in H(B(p,r))$ such that $\varphi(z) = 0$ for $z \in M \cap B(p,r)$.

As the following example shows, the singular set $E = \Omega \setminus D(F)$ is the thin subset of Ω.

Example (2.6). Let $g \in H(B(0,1))$, $n \geq 1$. Set $A = \{z \in B(0,1) : g(z) = 0\}$. Consider the family $F = \{f_j = j \cdot g : j = 1,2,3,...\}$. Then F is normal in $B(0,1) \setminus A$, but fails to be normal in $B(0,1)$.

Indeed, let L be any compact subset of $B(0,1) \setminus A$, then $f_j(z) \geq j \cdot m$, where $m = \min\{|g(z)|, z \in L\}$. Since $L \cap A = \varnothing$ we have $m > 0$, it follows that $f_j(z) \to \infty$ uniformly on L. Let K be any compact subset of $B(0,1)$ which contains the point $q \in A$ and at least one point $p \notin A$. No subsequence of $\{f_j\}$ can converge uniformly on K (since $f_j(p) \to \infty$), nor can any subsequence of $\{f_j\}$ diverge to infinity (as $f_j(q) = 0$ for all j).

In [4], H. Rutishauser generalized some of Montel's results to the case of meromorphic functions of several variables. By definition, a quasi-normal family of meromorphic functions on a domain Ω in $\mathbb{C}^n$ is a family F such that any sequence in F has a subsequence which converges compactly outside a thin subset of Ω.

Theorem (2.7). *Let Ω be a domain in $\mathbb{C}^n$, $n > 1$, and A is an analytic subset. Let a sequence $\{f_j\} \subset H(\Omega)$ converge uniformly on compact subsets of $\Omega \setminus A$ to $f \in H(\Omega)$. Then $\{f_j\}$ converges uniformly on compact subsets of Ω to f.*

Proof. Let $(z_1,...,z_n)$ be the standard coordinate system in $\mathbb{C}^n$. Since normality is a local property. It suffices to prove that $\{f_j\}$ converges uniformly on a neighborhood at an arbitrary point $a \in A$, and we may assume that $a = 0$. Let A be given by holomorphic functions $g_1,...,g_k$ in a connected neighborhood U of a point 0 of it, and let, say, g_1 is not identically 0 in U. Set $g \equiv g_1$. Then $A \cap U$ belongs to the principal analytic set $A_1 = \{z \in U : g(z) = 0\}$. We may also assume that the function g satisfies the condition $g('0, z_n) \neq 0$.

Then there is a sufficiently small r such that $g('0, z_n) \neq 0$ on $\Delta(r)^{n-1} \times \partial\Delta(r)$, where $\Delta(r) = \{z \in \mathbb{C} : |z| < 1\}$, and $\partial\Delta(r) = \{z \in \mathbb{C} : |z| = 1\}$. Hence the skeleton $(\partial\Delta(r))^n$ of the polydisc $(\Delta(r))^n$ lies in $\Omega \setminus A$. In general, let h be a continuous function on $\left(\overline{\Delta(r)}\right)^n$ which is holomorphic on $(\Delta(r))^n$. Then we have the following maximum principle:

$$\max\left\{h(z); z \in \left(\overline{\Delta(r)}\right)^n\right\} = \max\left\{h(z); z \in (\partial\Delta(r))^n\right\}.$$

Take the restrictions $f_j|_{(\partial\Delta(r))^n}$ of f_j and $f|_{(\partial\Delta(r))^n}$ of f over $(\partial\Delta(r))^n$. Since $\left\{f_j|_{(\partial\Delta(r))^n}\right\}$ converges uniformly to $f|_{(\partial\Delta(r))^n}$, the above maximum principle implies the uniform convergence of $\left\{f_j|_{(\Delta(r))^n}\right\}$ with limit $f|_{(\partial\Delta(r))^n}$. □

As an immediate consequence of Theorem (2.7), we have the following result.

Corollary (2.8). Let the notation be as in Theorem (2.7). Suppose that a sequence $\{f_j\}$ uniformly on every compact subset of $\Omega\setminus A$, but not on Ω. Then the limit function of the sequence must be ∞ on $\Omega\setminus A$.

3 JULIA'S THEOREM

In this section we shall find condition under which a family F normal in Ω extends to a family which is normal in some boundary points of Ω. The result of this section one can consider as the extension F outside Ω.

We define a holomorphic m-dimensional surface S in $\mathbb{C}^n$ to be the image of some bounded domain $D\subset\mathbb{C}^m$ $(m<n)$ under a non-degenerate holomorphic mapping

(3.1) $$\varphi: D\to\mathbb{C}^n.$$

In particular, for $m=1$ S is a holomorphic curve, and if $D\subset\mathbb{C}$ is a unit disc Δ and φ is continuous in $\overline{\Delta}$, then $S=\varphi(\Delta)$ is called a holomorphic disc. Recall that a mapping (3.1) is said to be non-degenerate if the rank of the Jacobi matrix $(\partial\varphi_k/\partial z_l)$ is equal to m at all points of D. The surface (3.1) is said to be bounded if the set $S=\varphi(D)$ is bounded in $\mathbb{C}^n$.

In 1934 Behnke and Sommer presented his continuity principle. Relatively speaking, it states that the property of a function f to be holomorphic in a neighborhood of the holomorphic surfaces $S_j\subset\Omega$ is also enjoyed by for the limit set S of these surfaces.

For bounded holomorphic surfaces the following maximum modulus principle is valid.

If f is a function that is holomorphic in some open set $U\subset\mathbb{C}^n$ that contains a bounded holomorphic surface S and if f is continuous in its closure S (in the topology of $\mathbb{C}^n$), then

(3.2) $$\|f\|_S<\|f\|_{\partial S},$$

where $\partial S=\overline{S}\setminus S$. (Here and what follows we use the notation $\|f\|_A=\sup\{|f(z)|, z\in A\}$.)

We shall say that a sequence of sets M_j converges to a set M (notation: $M_j \to M$) if, for any $\varepsilon > 0,$ there is a j_0 such that for all $j > j_0$ we have

$$M_j \subset M^{(\varepsilon)} \text{ and } M \subset M_j^{(\varepsilon)},$$

where $M^{(\varepsilon)}$ and $M_j^{(\varepsilon)}$ denote the ε-dilations of the sets M and M_j respectively (i.e., the union of all balls $B(z,\varepsilon)$ with centers at points of the sets).

A relatively compact subspace subset A of $\mathbb{C}^n$ is a subset whose closure is compact. If A and B are subsets of $\mathbb{C}^n$ we write $A \subset\subset B$ if A is relatively compact in $B.$ As usual, A^o is the interior, $\overline{A}$ the closure of $A,$ $\partial A = A \setminus A^o$ the boundary of $A.$

As an application of Theorem (1.2) we obtain the following theorem.

Theorem (3.3). *Let S_j be a sequence of bounded holomorphic m-dimension surfaces which, together with the boundaries ∂S_j, are contained in a bounded domain $\Omega \subset \mathbb{C}^n$. If S_j converges to some set S, ∂S_j converges to a set T and $T \subset\subset \Omega$, then any normal family $F \subset H(\Omega)$ is extends to a family F which is normal in any point of S.*

Proof. Since $T \subset\subset \Omega,$ then there exists a relatively compact subdomain $G \subset\subset \Omega,$ such that $T \subset G;$ we let $\rho(G,\Omega) = r.$ Because of the convergence $\partial S_j \to T$ there is a j_0 such that $\partial S_j \subset G$ for $j > j_0.$

As we see from the proof of the Behnke-Sommer theorem [5, p.189] any $f \in F$ can be extended to a function f holomorphic in $S^{(r/2)} \cap \Omega.$

Assuming $F = \{f : f \in F\}$ is not normal at some point of S, we shall derive a contradiction. If F is not normal at point $z^0 \in S$ then from Corollary (1.3) we see that for every natural number l there exists a sequence $f_j \subset F$ and points $z^j = z^j(f_j),$ $w^j = w^j(f_j),$ $z^j, w^j \in B(z^0, r),$ with $z^j \to z^0,$ $w^j \to z^0$ as $j \to \infty,$ such that

$$|f_j(z^j)| > l \text{ and } |f_j(w^j)| < \frac{1}{l}. \tag{3.4}$$

Let $S_k \subset S^{(r/8)}$ and $z_j^0 \in S_k$ such that $|z^j - z_j^0| < r/8.$ Set $S_j = S_k + z_j - z_j^0$ for all $j \geq 1.$ It is easy to see that $S_j \subset S^{(r/2)}$ and $\partial S_j \subset G^{(r/2)}$ for all sufficiently large $j.$ For any f_j and any point $z \in S_j,$ by the maximum modulus principle we have

$$\|f_j\| < \|f_j\|_{\partial S_j}. \tag{3.5}$$

Since $F|_\Omega \equiv F$ and F is normal in Ω we can select from the sequence $\{f_j\}$ a subsequence $\{f_{j_k}\}$ that converges uniformly on compact subset of Ω to an holomorphic function, or converges uniformly to ∞.

(a) Consider first the case the sequence $\{f_{j_k}\}$ converges uniformly on $\overline{G^{(r/2)}}$ to holomorphic function f. It follows that $\|f_{j_k}\|_{G^{(r/2)}} < 2\|f\|_{\partial G^{(r/2)}}$ for sufficiently large j. Since $\partial S_j \subset \overline{G^{(r/2)}}$ and $\|f_{j_k}\|_{\partial S_j} \le \|f_{j_k}\|_{\overline{G^{(r/2)}}}$ for all sufficiently large k. It follows from (3.2) that

$$\|f_{j_k}\|_{\partial S_j} \le 2\|f\|_{\overline{G^{(r/2)}}} < \infty \text{ for all sufficiently large } k.$$

This combined with (3.4) yields

$$l < 2\|f\|_{\overline{G^{(r/2)}}} < \infty.$$

But this is impossible since l is arbitrary natural number.

(b) Next, we consider the case $\{f_{j_k}\}$ converges uniformly on $\overline{G}$ to ∞. We first show that $\{f_{j_k}\}$ is zero-free in $S^{(r/2)} \cap \Omega$. To this end, suppose that f_j has a zero $z^j \in S^{(r/8)}$. Let $S_k \subset S^{(r/8)}$ and $z^0 \in S_k$ such that $|z^j - z_j^0| < r/4$. Set $S_j = S_k + z^j - z_j^0$ for all $j \ge 1$. It is easy to see that $S_j \subset S^{(r/2)}$ and $\partial S_j \subset \overline{G^{(r/2)}}$ for all sufficiently large j. From Osgood's theorem follows that the zeros of holomorphic functions must go beyond the boundary of the domain of definition. It follows that exists $w'_{j_k} \in \partial S_{j_k}$ such that $f_{j_k}(w'_{j_k}) = 0$. Since $\{f_{j_k}\}$ converges uniformly on compact subsets of Ω to ∞ we must have $|f_{j_k}(w'_{j_k})| > 1$ for all k sufficiently large. This contradiction proves our claim that f_{j_k} is zero-free in $S^{(r/2)} \cap \Omega$.

It follows that the sequence of holomorphic functions $\{1/f_{j_k}\}$ converges uniformly to 0 on $\overline{G}$. The remainder of the proof can be carried out as in the case (a) and we obtain that $\{1/f_{j_k}\}$, and whence $\{f_{j_k}\}$, is normal in a neighborhood of the point z^0. This completes the proof. □

Let us agree to say that a domain Ω at a boundary point a can be tangent from the interior by a family of holomorphic surfaces if there is a family of holomorphic surfaces $S_t \subset \Omega$, $0 < t < t_0$, converging as $t \to 0$ to a S such that $S_t \to S$, $\partial S_t \to \partial S$, where $\partial S \subset\subset \Omega$, and S contains the point a.

A domain $\Omega \subset C^n$, $n \ge 2$, is said to be pseudoconvex at a boundary point a, if at a it cannot be tangent from the interior by a family of holomorphic surfaces.

Corollary (Julia [2]). The normality domain of $F \subset H(\Omega)$ is pseudoconvex.

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